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Presupposition Projection and Repair Strategies in Trivalent Semantics

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Abstract

In binary propositional constructions S_1 *con* S_2, the Strong Kleene connectives explain filtering of S_1's and S_2's presuppositions depending on their logical relations with their non-presuppositional content. However, the presuppositions derived by the Strong Kleene connectives are weak conditional presuppositions, which raise the "proviso problem" in cases where no filtering is motivated. Weak Kleene connectives do not face this problem, but only because their presuppositions are often too strong, and hence do not account for filtering phenomena altogether. While various mechanisms have been proposed to allow filtering without the proviso problem, their relations with the standard trivalent Kleene systems have remained unclear. This paper shows that by sacrificing truth-functionality, we uncover a rich domain of possibilities in trivalent semantics in between the Weak Kleene and Strong Kleene connectives. These systems derive presupposition filtering while avoiding the proviso problem. The Kleene-style operators studied are generalized to arbitrary binary functions, which further clarifies the connection between their different "repair" strategies and presupposition projection.

1 Introduction

Logical theories of natural language semantics and pragmatics treat presupposition (Beaver and Geurts, 2014) as a special sort of inference, which in the following examples we denote '$\rightsquigarrow$':

(1) Sue stopped smoking
$\rightsquigarrow$ Sue used to smoke

The king of France is bald
$\rightsquigarrow$ There is a (unique) king of France

Dan regretted visiting LA
$\rightsquigarrow$ Dan visited LA

It was Zoe who stole my car
$\rightsquigarrow$ Someone stole my car

What makes presuppositions semantically distinguished from other entailments is their special projection properties: presuppositions are preserved under various operators that make other entailments disappear. For instance, the complex sentences in (2)-(4) below embed the sentence *Sue stopped smoking*, whose non-presuppositional entailment *Sue doesn't smoke* is not entailed by any of these complex sentences. By contrast, the statement *Sue used to smoke* is also a presupposition of sentences (2)-(4). In semantic jargon we say that presuppositions are "projected" from conditionals, negation and epistemic modals.[1]

(2) If *Sue stopped smoking* then Dan is happy now.

(3) It is not the case that *Sue stopped smoking*.

(4) Possibly, *Sue stopped smoking*.

We treat such basic projections in trivalent semantics, where natural language sentences are represented using propositions that denote 1 (true), 0 (false) or $*$ (presupposition failure). We say that a proposition φ is *bivalent* if $[\![\varphi]\!]^M \neq *$ for any model M. Basic entailment, presupposition and bivalent-presupposition relations are defined as follows (Keenan, 1973; Beaver, 1997):

Definition 1.1. *For propositions φ and ψ:*

[1] When it comes to complex sentences like (2)-(4), the literature is divided on the status of projected presuppositions like *Sue used to smoke*: semantic theories treat projected presuppositions as logically entailed from one reading of the embedding sentence; pragmatic theories (Gazdar, 1979; Chierchia and McConnel-Ginet, 1990) see them as instances of defeasible reasoning. With the rest of the literature on semantic presupposition, we here assume a logical entailment account, embracing a systematic ambiguity of sentences like (2)-(4). Under one reading the presupposition is *suspended*, e.g. using Bochvar's *assertion operator* (Beaver and Krahmer, 2001) or Heim's strategy of *local accommodation* (Heim, 1983). This suspension is highlighted in contexts *it is not the case that Sue stopped smoking, since she never smoked*, which were the center of the Russell-Strawson debate.

$\varphi \Rightarrow \psi \quad \equiv \quad \varphi$ **entails** ψ *if*

for every model M*: if* $[\![\varphi]\!]^M = 1$ *then* $[\![\psi]\!]^M = 1$

$\varphi \rightsquigarrow \psi \quad \equiv \quad \varphi$ **presupposes** ψ *if*

for every model M*: if* $[\![\varphi]\!]^M \neq *$ *then* $[\![\psi]\!]^M = 1$

$\psi = MBP(\varphi) \quad \equiv \quad \psi$ *is the* **maximal bivalent presupposition** *(MBP) of* φ *if*

for every model M*:* $[\![\varphi]\!]^M \neq *$ *iff* $[\![\psi]\!]^M = 1$

To construct elementary presuppositional propositions from bivalent propositions, we employ Blamey's *transplication* operation on bivalent propositions (Blamey, 1986; Beaver and Krahmer, 2001). For bivalent propositions φ_1 and φ_2, the transplication $(\varphi_1 : \varphi_2)$ is defined by:

$$[\![(\varphi_1 : \varphi_2)]\!]^M = \begin{cases} [\![\varphi_2]\!]^M & [\![\varphi_1]\!]^M = 1 \\ * & [\![\varphi_1]\!]^M = 0 \end{cases}$$

For instance, using the bivalent propositions *US* and *S*, we employ the following treatments of simple natural language sentences:

(5) a. Sue used to smoke $\quad US$

b. Sue doesn't smoke $\quad \neg S$

c. Sue stopped smoking $\quad (US : \neg S)$

In this analysis, (5c) entails but doesn't presuppose (5b), and presupposes (hence entails) (5a), as intuitively required. Furthermore, (5a) is the maximal bivalent presupposition of (5c).

For any bivalent φ_1, φ_2 and ψ, the implication operator in the Weak Kleene system (WK, Table 1) satisfies the following:

(WK$_1$) $\mathrm{MBP}((\varphi_1 : \varphi_2) \rightarrow \psi) \; = \; \varphi_1$

For instance, in sentence (2), property (WK$_1$) correctly accounts for the projection of the presupposition *Sue used to smoke*. Implication in the Strong Kleene system (SK, Table 2) supports a weaker presupposition:

(SK$_1$) $\mathrm{MBP}((\varphi_1 : \varphi_2) \rightarrow \psi) \; = \; \varphi_1 \vee \psi$

This property means that SK implication expects sentence (2) to presuppose *Sue used to smoke or Dan is happy*, invoking the intuitively irrelevant disjunct *Dan is happy*. This kind of derivation of irrelevant disjuncts in presuppositions is referred to as the "proviso problem" (Geurts, 1996), and appears with all SK connectives (see e.g. (10) below).

¬	
0	1
1	0
*	*

∧	0	1	*
0	0	0	*
1	0	1	*
*	*	*	*

∨	0	1	*
0	0	1	*
1	1	1	*
*	*	*	*

→	0	1	*
0	1	1	*
1	0	1	*
*	*	*	*

Table 1: Weak Kleene (WK) connectives

¬	
0	1
1	0
*	*

∧	0	1	*
0	0	0	0
1	0	1	*
*	0	*	*

∨	0	1	*
0	0	1	*
1	1	1	1
*	*	1	*

→	0	1	*
0	1	1	1
1	0	1	*
*	*	1	*

Table 2: Strong Kleene (SK) connectives

However, as has been often observed (Peters, 1979; Beaver, 1997), other cases of presupposition projection reveal substantial advantages to SK connectives over WK connectives in terms of their linguistic adequacy. Let us consider for example the following sentence:

(6) If Sue used to smoke, she stopped smoking.

Sentence (6), unlike (2), is not felt to presuppose that Sue used to smoke, and similarly sentence (7) below:

(7) If Sue used to smoke Marlboros, she stopped smoking.

In semantic jargon, we say that sentences (6) and (7) are cases of presupposition *filtering*. In these sentences, the antecedent *Sue used to smoke* (*Marlboros*) entails the presupposition *Sue used to smoke* of the consequent. As a result, that presupposition gets "filtered out" and is not projected as an entailment of the conditional sentence. Such linguistic facts about filtering are accounted for by SK connectives but not by WK connectives. For WK and SK implication, this is exemplified by the following facts for any bivalent φ, ψ_1 and ψ_2:

(WK$_2$) $\mathrm{MBP}(\varphi \rightarrow (\psi_1 : \psi_2)) \; = \; \psi_1$

(SK$_2$) $\mathrm{MBP}(\varphi \rightarrow (\psi_1 : \psi_2)) \; = \; \varphi \rightarrow \psi_1$

Thus, for sentences (6) and (7), WK implication counter-intuitively expect the MBP to be *Sue used to smoke*. By contrast, (SK_2) correctly expects the MBPs of these sentences to be tautological, which accounts for presupposition filtering.

Conditional MBPs as in the SK system have been argued to also be intuitively correct in cases that do not involve simple filtering (Karttunen and Peters, 1979; Heim, 1983; Beaver, 2001). For instance, let us consider sentence (8) below:

(8) If Sue used to smoke, she stopped smoking Marlboros.

In this conditional sentence, the antecedent is asymmetrically entailed by the consequent. Fact (WK_2) above means that the WK implication operator expects the MBP of (8) to be *Sue used to smoke Marlboros*. This is incorrect, for such an MBP would entail the antecedent in sentence (8), with would counter-intuitively treat the sentence as equivalent to the non-conditional sentence *Sue stopped smoking Marlboros*. To avoid this problem, Karttunen and Peters (1979) and others proposed treatments where the MBP of sentence (8) is as paraphrased below:

(9) If Sue used to smoke, she used to smoke Marlboros.

When analyzed as a material implication, this conditional statement is also what fact (SK_2) about Strong Kleene implication expects as the MBP of sentence (8).

To summarize, while both the WK implication and the SK implication deal with basic projection problems, they are facing complementary difficulties. The WK implication often "projects too much", failing to filter out presuppositions in the consequent, or at least conditionalize them. In other cases, however, WK implication is advantageous to the conditional presuppositions derived by the SK implication. These SK-based presuppositions are often too weak, and lead to the so-called "proviso problem" for SK implication.

Similar puzzles appear with the other binary propositional connectives in the Kleene truth tables. For instance, the WK and SK conjunction connectives satisfy the following:

$$(\mathrm{WK}_3)\quad \mathrm{MBP}(\varphi \wedge (\psi_1 : \psi_2)) \;=\; \psi_1$$

$$(\mathrm{SK}_3)\quad \mathrm{MBP}(\varphi \wedge (\psi_1 : \psi_2)) \;=\; \varphi \rightarrow \psi_1$$

For instance, with *EX*, *S*/*SM* and *US*/*USM* for "Sue exercises", "Sue smokes (Marlboros)" and "Sue used to smoke (Marlboros)", respectively, this leads to the following analyses of the sentences below:

(10) Sue exercises a lot and stopped smoking
$\rightsquigarrow$ Sue used to smoke

by (WK_3): $\mathrm{MBP}(EX \wedge (US : \neg S)) \;=\; US$
✓ (no proviso problem)

by (SK_3): $\mathrm{MBP}(EX \wedge (US : \neg S)) \;=\; EX \rightarrow US$
✗ (proviso problem)

(11) Sue used to smoke Marlboros and stopped smoking
$\not\rightsquigarrow$ Sue used to smoke

by (WK_3): $\mathrm{MBP}(USM \wedge (US : \neg S)) \;=\; US$
✗ (no filtering)

by (SK_3): $\mathrm{MBP}(USM \wedge (US : \neg S)) \;=\; USM \rightarrow US$
✓ (tautological presupposition, hence filtering)

(12) Sue used to smoke and stopped smoking Marlboros
$\not\rightsquigarrow$ Sue used to smoke Marlboros[2]

by (WK_3): $\mathrm{MBP}(US \wedge (USM : \neg SM)) \;=\; USM$
✗ (no filtering or conditional presupposition)

by (SK_3): $\mathrm{MBP}(US \wedge (USM : \neg SM)) \;=\; US \rightarrow USM$
✓ (conditional presupposition)

Table 3 summarizes the theoretical puzzle for the WK and SK connectives. To solve this puzzle, we should like a trivalent account of presupposition projection to avoid proviso-like presuppositions while allowing presupposition filtering and conditional presuppositions. This paper proposes such an account, by considering the possibilities that open up once we renounce the truth-functionality of the WK and SK connectives.

	Filtering	Conditional presuppositions	No Proviso
WK	✗	✗	✓
SK	✓	✓	✗

Table 3: Kleene systems and presupposition projection

[2]To see that *Sue used to smoke Marlboros* is not simply presupposed by the sentence in (12), we should look at sentences where such conjunctions are embedded, as in: *if Sue used to smoke and stopped smoking Marlboros, then she might be smoking now other brands of cigarettes*. This makes it clear that the statement *Sue used to smoke Marlboros* is not projected as a presupposition of the conditional, hence, reasonably, also not of the conjunction that it contains.

2 Kleene systems of intermediate strength

This section develops two trivalent systems that on the one hand account for presupposition filtering and conditional presuppositions, and on the other hand avoid the unnecessarily weak presuppositions that lead to the "proviso problem" with SK connectives. This account involves employing inferential relations between arguments of binary propositional operators. While such inferences between operands have occasionally been employed under various assumptions (Beaver, 1999; Lassiter, 2012; Mandelkern, 2016), we aim here to employ them within a purely trivalent semantics that allows a better insight the role of Kleene connectives in natural language semantics, in search for a linguistically adequate "intermediate" trivalent semantics in between WK and SK.[3]

2.1 Entailment relations and presupposition filtering

Let us first reconsider the contrast between sentences (10) and (11), which are restated below:

(10') Sue exercises a lot and stopped smoking.

(11') Sue used to smoke Marlboros and stopped smoking.

As we saw, sentence (10) intuitively presupposes that *Sue stopped smoking*, whereas sentence (11) does not. This kind of difference in filtering is often analyzed in terms of whether the presupposition of the second conjunct is entailed by the first conjunct (Mandelkern, 2016). In example (11) the first conjunct *Mary used to smoke Marlboros* entails the MBP *Sue used to smoke* of the second conjunct. Such an entailment is missing in (10). These facts are used as the source of filtering in (11) and the lack thereof in (10).

Formalizing this filtering principle in trivalent semantics, we get the following restriction on the interpretation of $\varphi \wedge \psi$:

(13) Left-to-right filtering in conjunctions $\varphi \wedge \psi$:
If $\varphi \Rightarrow \mathrm{MBP}(\psi)$, then $\mathrm{MBP}(\varphi) \Rightarrow \mathrm{MBP}(\varphi \wedge \psi)$

In words: if the left-hand conjunct φ in a conjunction $\varphi \wedge \psi$ entails the maximal presupposition of the right-hand conjunct ψ, then that presupposition gets "filtered out", i.e. all presuppositions of $\varphi \wedge \psi$ are inherited from φ. In example (11), the left-hand conjunct (*Sue used to smoke*) is bivalent, hence (13) correctly expects the conjunction to also be bivalent. This accounts for the "filtering" of the presupposition in the right-hand conjunct. At the same time, (13) on its own does not expect filtering in (10), where the entailment $\varphi \Rightarrow \mathrm{MBP}(\psi)$ does not hold.

As illustrated by the WK analysis of sentence (11) (section 1), WK conjunction does not satisfy the condition in (13), hence its failure to account for filtering phenomena in such sentences. By contrast, the following fact about SK conjunction makes it clear that it does satisfy (13):

(14) $[[\mathrm{MBP}(\varphi \wedge_{\mathrm{SK}} \psi)]]^M =$

$$\begin{cases} 1 & ([[\mathrm{MBP}(\varphi)]]^M = 1 \text{ and } [[\mathrm{MBP}(\psi)]]^M = 1) \\ & \text{or } [[\varphi]]^M = 0 \text{ or } [[\psi]]^M = 0 \\ 0 & \text{otherwise} \end{cases}$$

Fact (14) about SK conjunction leads to its desirable filtering property, but it also leads to proviso problems as in (10), for there are cases where the entailment $\varphi \Rightarrow \mathrm{MBP}(\psi)$ does *not* hold, but SK conjunction admits models where $[[\varphi \wedge \psi]]^M$ is bivalent although $[[\psi]]^M = *$ – namely, the models M where $[[\varphi]]^M = 0$.

To address these problems of the WK and SK systems, it is useful to first observe their take on the following question:

(Q) Let op_2 be a *bivalent* binary propositional operator, and let op_3 be the corresponding *trivalent* operator. Which formulas φ, ψ and models M admit a bivalent value for $[[\varphi\, op_3\, \psi]]^M$ when $[[\varphi]]^M = *$ or $[[\psi]]^M = *$?

The WK system treats the value '$*$' as "nonsense", and accordingly, its answer on (Q) is "no formulas, and no models".

The SK system treats the value '$*$' as "unknown", and uses the fact that certain values of an argument of a binary function may determine the result of the function regardless of the value

[3] The present modification of Kleene systems is orthogonal to the familiar proposal in (Peters, 1979), where the "intermediate" Kleene system is aimed to treat left-right asymmetries of presupposition projection with binary connectives (Mandelkern et al., 2017). This kind of asymmetry does not concern "proviso problems" that result from presuppositions of the righthand operand, which are the focus of the present paper. Peters's asymmetries are introduced in the current non-truth-functional proposal (see note 6 below), but if needed they can also be removed.

of the other argument. For the standard bivalent connectives, these "decisive values" are 0 for both operands of conjunction, 1 for both operands of disjunction, and 0/1 respectively for the left-hand/righthand operand of implication.[4] The answer of the SK system to (Q) may then be expressed as follows:

(A_1) **SK's answer on (Q)**: All formulas, and any model where the value $[[\varphi]]^M$ or $[[\psi]]^M$ determines the result of op_2.
(*Motivation*: extract as much information as possible from known values)

The proviso problem demonstrates that for natural language, the answer in (A_1) is too liberal. The problem lies in the fact that the SK answer allows "saving" a formula $\varphi\, op_3\, \psi$ from having a '$*$' value in some model, with no respect to whether the formula can also be "saved" in the same way in other models. Thus, supposing that the second conjunct in sentence (10) involves a presupposition failure, we see that SK incorrectly "saves" the conjunction from failure if the first conjunct is false. At the same time, SK correctly treats such a conjunction as a failure in models where the first conjunct is true. We consider this "global instability" of the way failures are handled in SK as the source of the proviso problem. Instead of (A_1), we propose a "globally stable" variant of SK's answer to (Q). Since this answer minimally strengthened WK, we refer to the system on which it is based as 'WK^+'. The "WK^+ answer" is informally stated below:

(A_2) **WK^+ answer on (Q)**: Only formulas where a failure of one operand guarantees that the other operand also fails, or else has a value that determines the result of op_2.
(*Motivation*: extract as much information as possible from known values in formulas that can be globally saved from failure)

This answer, put informally here, summarizes a common linguistic intuition about the contrast between sentences (10) and (11). In sentence (11) it is guaranteed than whenever the second conjunct fails, the first conjunct is *false*. This is the value that determines the result of bivalent conjunction, hence can "save" the formula from failing.[5] There is no such guarantee for sentence (10). Thus, answer (A_2) employs the general SK reasoning, but only for "saving", or "repairing", some of the presupposition failures that SK addresses: those failures that can be globally saved from failing the formula (or, using another metaphor: can be globally "repaired").

Following this reasoning, in (15) below we define a conjunction operator that, like SK conjunction, satisfies the condition in (13), but without the general property (14). The operator in (15) "strengthens" WK conjunction to satisfy property (13), hence we refer to it as a *strengthened WK* (WK^+) conjunction operator, which is denoted '$\wedge_{\mathrm{WK}}$':

(15) **Conjunction in WK^+**:
For propositional formulas φ and ψ, with $\mathcal{M}$ a class of models and $M \in \mathcal{M}$ s.t. $[[\varphi]]^M$ and $[[\psi]]^M$ are inductively specified, we define:

$$[[\varphi \wedge_{\mathrm{WK}} \psi]]^M = \begin{cases} [[\varphi \wedge \psi]]^M & [[\varphi]]^M \neq * \text{ and } [[\psi]]^M \neq * \\ [[\varphi]]^M & [[\psi]]^M = * \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [[\psi]]^{M'} = * \text{ then } [[\varphi]]^{M'} \neq 1 \\ * & \text{otherwise} \end{cases}$$

The first clause in definition (15) standardly retains bivalent conjunction. The second clause makes sure to respect the condition (13). [6] An advantage of the $\wedge_{\mathrm{WK}}$ operator over SK conjunction is the avoidance of proviso problems as in (10): falsity of φ entails falsity of $\varphi \wedge_{\mathrm{WK}} \psi$ only if the condition in the second clause of definition (15) holds, which is not the case in (10). Formally, for any bivalent propositions φ, ψ_1 and ψ_2, we observe the following fact on the WK, WK^+ and SK conjunction operators:

[4] Some bivalent connectives, specifically exclusive disjunction, do not have decisive values. For such connectives, the "WK and SK answers" above give identical results, as they do for formulas like $(\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$.

[5] Cases where both conjuncts fail are discussed below.

[6] Note that this clause is asymmetric: it makes WK^+ conjunction respect the condition in (13), but not the symmetric condition (if $\psi \Rightarrow \mathrm{MBP}(\varphi)$, then $\mathrm{MBP}(\psi) \Rightarrow \mathrm{MBP}(\varphi \wedge \psi)$). Modifying (15) into a symmetric version is straightforward, but it is questionable if such left-right symmetry (similar to that of the SK connectives) would be empirically motivated, as it would expect right-to-left filtering, which is empirically dubious (Peters, 1979; Mandelkern et al., 2017). Thus, to simplify the presentation in this paper, we here only define operators that derive asymmetric filtering and conditional presuppositions.

(16) Assuming that $\varphi \Rightarrow \psi_1$, we have:

WK✗ $\mathrm{MBP}(\varphi \wedge_{\mathrm{WK}} (\psi_1 : \psi_2)) = \psi_1$

WK+✓ $\mathrm{MBP}(\varphi \,\dot{\wedge}_{\mathrm{WK}} (\psi_1 : \psi_2)) = \top$

(17) Assuming that $\varphi \not\Rightarrow \psi_1$, we have:

SK✗ $\mathrm{MBP}(\varphi \wedge_{\mathrm{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$

WK+✓ $\mathrm{MBP}(\varphi \,\dot{\wedge}_{\mathrm{WK}} (\psi_1 : \psi_2)) = \psi_1$

The $\top$ symbol standardly refers to the univalent proposition denoting 1 in all models. In (16), the '✓'/'✗' symbols mark the correct/incorrect treatment of filtering in sentences like (11). In (17) they mark the avoidance/retainment of the proviso problem in sentences like (10).

A disadvantage of WK+ conjunction over SK conjunction is that the second clause in definition (15) makes WK+ conjunction non-truth-functional, as it relies on logical relations within the whole class of models $\mathcal{M}$.[7]

The $\dot{\wedge}_{\mathrm{WK}}$ operator follows the general "repair" strategy of SK conjunction. When the second clause in (15) is met, the assignment of the interpretation of φ to $[\![\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi]\!]^M$ is motivated by the wish to preserve the following classical property of bivalent conjunction:

$$[\![\varphi]\!]^M = 0 \;\Rightarrow\; [\![\varphi \wedge \psi]\!]^M = 0.$$

Similarly, the following classical property of material implication motivates the treatment of filtering with conditionals as in (7):

$$[\![\varphi]\!]^M = 0 \;\Rightarrow\; [\![\varphi \rightarrow \psi]\!]^M = 1.$$

With disjunction the motivation is to preserve the following property:

$$[\![\varphi]\!]^M = 1 \;\Rightarrow\; [\![\varphi \vee \psi]\!]^M = 1.$$

This motivation is geared by filtering as in the following disjunctive example, which does not presuppose that *Sue used to smoke*:

(18) Either Sue never smoked Marlboros, or she stopped smoking.

These considerations about filtering with conditionals and disjunction lead to the following definitions of the respective WK+ operators, in analogy to (15) above:[8]

(19) Implication and disjunction in WK+:

$$[\![\varphi \,\dot{\rightarrow}_{\mathrm{WK}}\, \psi]\!]^M \;=$$

$$\begin{cases} [\![\varphi \rightarrow \psi]\!]^M & [\![\varphi]\!]^M \neq * \text{ and } [\![\psi]\!]^M \neq * \\ [\![\neg\varphi]\!]^M & [\![\psi]\!]^M = * \text{ and } \forall M' \in \mathcal{M}\text{:} \\ & \text{if } [\![\psi]\!]^{M'} = * \text{ then } [\![\varphi]\!]^{M'} \neq 1 \\ * & \text{otherwise} \end{cases}$$

$$[\![\varphi \,\dot{\vee}_{\mathrm{WK}}\, \psi]\!]^M \;=$$

$$\begin{cases} [\![\varphi \vee \psi]\!]^M & [\![\varphi]\!]^M \neq * \text{ and } [\![\psi]\!]^M \neq * \\ [\![\varphi]\!]^M & [\![\psi]\!]^M = * \text{ and } \forall M' \in \mathcal{M}\text{:} \\ & \text{if } [\![\psi]\!]^{M'} = * \text{ then } [\![\varphi]\!]^{M'} \neq 0 \\ * & \text{otherwise} \end{cases}$$

Based on the"WK+ answer" in (A_2) above, the reasoning behind all these definitions is general, as is further explored in section 3.

The WK+ conjunction operator, as well as WK+ disjunction and implication, also deals with cases where the first conjunct is not bivalent, as in the following example:

(20) [Sue stopped drinking but continued to smoke two Marlboro packs a day], and [now she has finally also stopped smoking].

Intuitively, sentence (20) presupposes that Sue used to drink. This requirement holds independently of Sue's smoking habits. Definition (15) makes sure that the presupposition *Sue used to drink* of the first conjunct in (20) gets projected, despite the filtering of the presupposition *Sue used to smoke* of the second conjunct. In general: for any proposition φ' that is presupposed by φ, we have φ' presupposed by $\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi$ as well.[9]

[7]This drawback of WK+ conjunction is shared with other previous "globalist" accounts of the proviso problem (Lassiter, 2012; Mandelkern, 2016). If desirable, it might be removed by couching definition (15) within a possible world semantics, replacing quantification over models by quantification over indices in a given model.

[8]Since the 0 value in the left argument similarly determines the result of both conjunction and material implication, the general principle underlying left-to-right filtering with implication is the same as for conjunction in (13). By contrast, with disjunction, the 1 value determines the result, hence the general principle analogous (13) is: if $\neg\varphi \Rightarrow \mathrm{MBP}(\psi)$, then $\mathrm{MBP}(\varphi) \Rightarrow \mathrm{MBP}(\varphi \vee \psi)$.

[9]*Proof*: Let M be a model where $[\![\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi]\!]^M \neq *$. Thus, either clause 1 or 2 of definition (15) is satisfied. Clause 1 trivially entails $[\![\varphi]\!]^M \neq *$. Clause 2 entails $[\![\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi]\!]^M = [\![\varphi]\!]^M$, from which we also conclude $[\![\varphi]\!]^M \neq *$. If $\varphi \rightsquigarrow \varphi'$, then in every model M s.t. $[\![\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi]\!]^M \neq *$, we have $[\![\varphi']\!]^M = 1$. We conclude that $(\varphi \,\dot{\wedge}_{\mathrm{WK}}\, \psi) \rightsquigarrow \varphi'$.

Definitions (15) and (19) quantify over models in a way that accounts for filtering phenomena as in the following example (Beaver, 1999) :

(21) If Jane takes a bath, Bill will be annoyed that there is no more hot water.

As Beaver notes, while the relation between taking a bath and lack of hot water is by no means logical, in normal conversations the presupposition *there is no more hot water* of the consequent in (21) gets filtered out. In general, this filtering is on a par with the filtering phenomena discussed above, where the relations between conjuncts are logical. However, there is one empirical caveat: an entailment $\varphi \Rightarrow \mathrm{MBP}(\psi)$ in (13) which is not logical but restricted to a designated class of models can be explicitly denied in conversation. For instance, when a given context explicitly denies the relation between taking a bath and lack of hot water, filtering in (21) disappears:

(22) The hot water supply in Bill's place uses gas heating, so that no single person could possibly take a bath that would stop the hot water supply. At present there's some problem with Bill's heating system. Not knowing that, Bill suggests Jane, who is staying at his place, to take a bath whenever she pleases. *If Jane takes a bath, Bill will be annoyed (to hear from her) that there is no more hot water.*

Unlike the use of sentence (21) in an out-of-the-blue context, in the context of (22) sentence (21) does presuppose that *there in no more hot water*. Thus, due to the explicit denial in (22) of any causal relation between Jane's bath and the lack of hot water, filtering does not take place. Using a given class of models $\mathcal{M}$ in definitions (15) and (19), rather than all possible models, allows the filtering mechanism to take into account implicit epistemic assumptions, without getting into the separate question of how these assumptions should be modeled. A similar point is made in (Mandelkern, 2016) in relation to a framework of context-change potentials.

2.2 Conditional presuppositions

The WK$^+$ operators defined above do not expect conditional presuppositions, which were exemplified in sentence (8), restated below:

(8') If Sue used to smoke, she stopped smoking Marlboros.

In this case the MBP of the consequent *Sue used to smoke Marlboros* asymmetrically entails the antecedent. The second clause in the definition of the WK$^+$ implication operator in (19) does not hold in such cases. Accordingly, this operator incorrectly expects the presupposition *Sue used to smoke Marlboros* to be projected in (8). Formally, for any bivalent propositions φ, ψ_1 and ψ_2, we observe the following fact on the SK and WK$^+$ implication operators:

(23) Assuming that $\psi_1 \Rightarrow \varphi$ and $\varphi \not\Rightarrow \psi_1$, we have:

SK✓ $\mathrm{MBP}(\varphi \rightarrow_{\mathrm{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$

WK$^+$✗ $\mathrm{MBP}(\varphi \dot{\rightarrow}_{\mathrm{WK}} (\psi_1 : \psi_2)) = \psi_1$

The '✓'/'✗' symbols mark here the correct/incorrect modelling of conditional presuppositions in sentences like (8).

Treating this kind of problem has led previous work to assume that the MBP of sentences like (8) should be expressed by the following disjunction:

(24) Either Sue never smoked or she used to smoke Marlboros.

Within a trivalent system, this treatment of (8) is generalized using the following condition:

(25) Left-to-right conditional presuppositions in implications $\varphi \rightarrow \psi$:

If $\mathrm{MBP}(\psi) \Rightarrow \varphi$, then:

$\neg\varphi \vee (\mathrm{MBP}(\varphi) \wedge \mathrm{MBP}(\psi)) \Rightarrow \mathrm{MBP}(\varphi \rightarrow \psi)$.

In words: when the MBP of the consequent ψ in $\varphi \rightarrow \psi$ entails the antecedent φ, the negation of φ satisfies the MBP of $\varphi \rightarrow \psi$, as a possible alternative to the straightforward WK-based presupposition $\mathrm{MBP}(\varphi) \wedge \mathrm{MBP}(\psi)$. Principle (25) correctly makes the disjunction in (24) entail the MBP of sentence (8), as expected by the Strong Kleene system. Indeed, SK implication satisfies (25). However, as in relation to presupposition filtering, this treatment of conditional presuppositions comes at the cost of leading to the proviso problem.

A simple trivalent extension of WK$^+$ derives some of the most typical conditional presuppositions that were addressed in the literature.[10] We

[10]There are empirical questions on whether conditional presuppositions are needed at all (Mandelkern, 2016). On the other hand, there are also empirical questions on whether principle (25) can cover all conditional presuppositions (Schlenker, 2011). For space and time limitations I ignore

refer to this extension as *weakened SK* (SK$^-$), and base its behavior on the following answer to question (Q) above regarding the formulas that allow a repair of a presupposition failure:

(A$_3$) **SK$^-$ answer on (Q)**: Only formulas as in WK$^+$ (cf. (A$_2$)) as well as formulas where if one operand has a value that determines the result of op_2, the other operand fails.
(*Motivation*: as in (A$_2$), plus the additional motivation to extract information from a single known value only when this is globally required in order to save a formula from a failure)

Minimal strengthening of the '$\dot\rightarrow_{\text{WK}}$' operator using this principle leads to the following operator, which we denote '$\dot\rightarrow_{\text{SK}}$':

(26) Implication in SK$^-$:
For propositional formulas φ and ψ, with $\mathcal{M}$ a class of models and $M \in \mathcal{M}$ s.t. $[\![\varphi]\!]^M$ and $[\![\psi]\!]^M$ are inductively specified, we define:

$$[\![\varphi \dot\rightarrow_{\text{SK}} \psi]\!]^M =$$
$$\begin{cases} [\![\varphi \rightarrow \psi]\!]^M & [\![\varphi]\!]^M \neq * \text{ and } [\![\psi]\!]^M \neq * \\ [\![\neg\varphi]\!]^M & ([\![\psi]\!]^M = * \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\psi]\!]^{M'} = * \text{ then } [\![\varphi]\!]^{M'} \neq 1) \\ & \text{or} \\ & ([\![\varphi]\!]^{M'} \neq 1 \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\varphi]\!]^{M'} \neq 1 \text{ then } [\![\psi]\!]^{M'} = *) \\ * & \text{otherwise} \end{cases}$$

This definition of SK$^-$ implication agrees with SK implication on conditional presuppositions for sentences like (8), but, like WK$^+$ and unlike SK, does not generate proviso problems. Formally, for any bivalent propositions φ, ψ_1 and ψ_2, we have:

(27) a. Filtering – if $\varphi \Rightarrow \psi_1$:
SK✓: $\text{MBP}(\varphi \rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = \top$
SK$^-$✓: $\text{MBP}(\varphi \dot\rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = \top$

b. Cond.Pres. – if $\psi_1 \Rightarrow \varphi$:
SK✓: $\text{MBP}(\varphi \rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$
SK$^-$✓: $\text{MBP}(\varphi \dot\rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$

c. No Proviso – if $\varphi \not\Rightarrow \psi_1$ and $\psi_1 \not\Rightarrow \varphi$:
SK✗: $\text{MBP}(\varphi \rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$
SK$^-$✓: $\text{MBP}(\varphi \dot\rightarrow_{\text{SK}} (\psi_1 : \psi_2)) = \psi_1$

This establishes that in cases like (8), SK$^-$ implication shows the desirable properties of SK implication, without the undesirable proviso problem.

The way in which definition (26) quantifies over models accounts for conditional presuppositions that are not triggered by logical entailment, but only due to contextually salient inferential relations, similarly to filtering in sentence (21). For instance, according to (Schlenker, 2011), sentence (28) below has the presupposition in (29):

(28) If John visits his parents for Christmas, his sister too will give them hard time.

(29) If John visits his parents for Christmas, someone (namely John) will give them hard time.

This presupposition is treated here by assuming a contextual entailment from *John visits his parents for Christmas* to *someone will give John's parents a hard time*, which is of course far from being a logical entailment.

The reasoning behind the definition of SK$^-$ implication is also used in the following definitions of conjunction and disjunction:

(30) Conjunction and disjunction in SK$^-$:

$$[\![\varphi \dot\wedge_{\text{SK}} \psi]\!]^M =$$
$$\begin{cases} [\![\varphi \wedge \psi]\!]^M & [\![\varphi]\!]^M \neq * \text{ and } [\![\psi]\!]^M \neq * \\ [\![\varphi]\!]^M & ([\![\psi]\!]^M = * \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\psi]\!]^{M'} = * \text{ then } [\![\varphi]\!]^{M'} \neq 1) \\ & \text{or} \\ & ([\![\varphi]\!]^{M'} \neq 1 \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\varphi]\!]^{M'} \neq 1 \text{ then } [\![\psi]\!]^{M'} = *) \\ * & \text{otherwise} \end{cases}$$

these questions here. I believe that further linguistic work is needed in order to determine if conditional presuppositions, or certain types thereof, should be semantically derived. The SK$^-$ system is only presented here as one natural extension of WK$^+$, with no claims for empirical comprehensiveness.

$$[\![\varphi \vee_{\text{SK}} \psi]\!]^M =$$

$$\begin{cases} [\![\varphi \vee \psi]\!]^M & [\![\varphi]\!]^M \neq * \text{ and } [\![\psi]\!]^M \neq * \\ [\![\varphi]\!]^M & ([\![\psi]\!]^M = * \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\psi]\!]^{M'} = * \text{ then } [\![\varphi]\!]^{M'} \neq 0) \\ & \text{or} \\ & ([\![\varphi]\!]^{M'} \neq 0 \text{ and } \forall M' \in \mathcal{M}: \\ & \text{if } [\![\varphi]\!]^{M'} \neq 0 \text{ then } [\![\psi]\!]^M = *) \\ * & \text{otherwise} \end{cases}$$

Similarly to SK^- implication, these conjunction and disjunction operators admit conditional presuppositions while avoiding the proviso problem. Thus, when φ, ψ_1 and ψ_2 are bivalent, we get:

If $\psi_1 \Rightarrow \varphi$: $\text{MBP}(\varphi \wedge_{\text{SK}} (\psi_1 : \psi_2)) = (\neg\varphi) \vee \psi_1$.

If $\psi_1 \Rightarrow \neg\varphi$: $\text{MBP}(\varphi \vee_{\text{SK}} (\psi_1 : \psi_2)) = \varphi \vee \psi_1$.

2.3 Summary

We have defined two sets of binary operators, referred to as 'WK^+' and 'SK^-', which satisfy the following, for any operator *op* and trivalent propositions φ and ψ:

$$\text{MBP}(\varphi\, op_{\text{WK}}\, \psi) \Rightarrow \text{MBP}(\varphi\, op_{\text{WK}^+}\, \psi)$$
$$\Rightarrow \text{MBP}(\varphi\, op_{\text{SK}^-}\, \psi) \Rightarrow \text{MBP}(\varphi\, op_{\text{SK}}\, \psi)$$

Further, we have shown (wit. (27c), (23), (16)):

$$\text{MBP}(\varphi\, op_{\text{SK}}\, \psi) \not\Rightarrow \text{MBP}(\varphi\, op_{\text{SK}^-}\, \psi)$$
$$\not\Rightarrow \text{MBP}(\varphi\, op_{\text{WK}^+}\, \psi) \not\Rightarrow \text{MBP}(\varphi\, op_{\text{WK}}\, \psi)$$

This describes a hierarchy where SK/WK operators derive the weakest/strongest presuppositions, respectively. Equivalently, and more in line with common nomenclature, Strong Kleene operators have the strongest "failure conditions" (the negation of their MBPs) whereas the failure conditions of Weak Kleene operators are the weakest. In terms of this "strength", the WK^+ and SK^- operators are properly in between the two classical Kleene connectives.

3 Repair and value determination with general binary functions

The key to the proposal in section 2 is in the general specification of "repair" conditions for failures in propositional arguments. These are principles that specify the situations under which a presupposition failure in one of a binary function's arguments may still allow the function to return a value. Following (George, 2008, 2014), we aim to make the reasoning behind our proposal more explicit by generalizing it to arbitrary functions. Unlike George's work, we do not necessarily seek to generalize the Strong Kleene connectives, which lead to the proviso problem, but rather to avoid this problem using intermediate levels of presupposition projection as in the WK^+ and the SK^- operators. This section generalizes these operators to arbitrary binary functions.

Given a set $X \subset E$ and an element $*$ in $E - X$, we denote $X^* = X \cup \{*\}$. Following (de Groote and Lebedeva, 2010), we view presupposition failure ($*$) as an "exception", which should be optimally "handled" or "repaired". A *repair strategy* α is a method for defining $f^\alpha : (A^* \times B^*) \to C^*$ for any given binary function $f : (A \times B) \to C$.

The Weak Kleene strategy is "no repair". Thus, $f^{\text{WK}} : (A^* \times B^*) \to C^*$ is defined by:

(31) $f^{\text{WK}}(x, y) =$
$$\begin{cases} f(x,y) & x \in A \text{ and } y \in B \\ * & x = * \text{ or } y = * \end{cases}$$

By contrast, Strong Kleene is based on a "maximal repair" strategy. A function $f^{\text{SK}} : (A^* \times B^*) \to C^*$ is capable of "repairing" failures of its left/right whenever the result of f can be determined by the value of the right/left argument alone, respectively. This notion of *left/right* (L/R) *determination* is defined as follows:

Definition 3.1. *For any function $f : (A \times B) \to C$ and value $c \in C$, we say that:*

A value $a \in A$ **L-determines** *f as c if $\forall y \in B. f(a, y) = c$*

A value $b \in B$ **R-determines** *f as c if $\forall x \in A. f(x, b) = c$*

Using the notion of L-determination, we define the *L-determination function* $LD_f : A^* \to C^*$ of a function $f : (A \times B) \to C$ as follows:

(32) $LD_f(x) =$
$$\begin{cases} c & x \in A \text{ and } x \text{ L-determines } f \text{ as } c \\ * & \text{otherwise} \end{cases}$$

Symmetrically, the *R-determination function* $RD_f : B^* \to C^*$ of f is defined by:

(33) $RD_f(y) =$
$$\begin{cases} c & y \in B \text{ and } y \text{ R-determines } f \text{ as } c \\ * & \text{otherwise} \end{cases}$$

Bivalent conjunction, disjunction and implication satisfy:

$LD_\wedge(0) = RD_\wedge(0) = 0 \qquad LD_\wedge(1) = RD_\wedge(1) = *$
$LD_\vee(1) = RD_\vee(1) = 1 \qquad LD_\vee(0) = RD_\vee(0) = *$
$LD_\to(0) = RD_\to(1) = 1 \qquad LD_\to(1) = RD_\to(0) = *$

Using the *LD* and *RD* functions, we define a Strong Kleene function $f^{\text{SK}} : (A^* \times B^*) \to C^*$ for any binary function $f : (A \times B) \to C$:

(34) $f^{\text{SK}}(x, y) =$
$$\begin{cases} f(x,y) & x \in A \text{ and } y \in B \\ c & c \in C \text{ and } (LD_f(x) = c \\ & \text{or } RD_f(y) = c) \\ * & \text{otherwise} \end{cases}$$

It will be observed that the standard WK/SK connectives (tables 1 and 2) apply the respective repair strategies (31)/(34) to the bivalent connectives. Like their propositional instantiations in the Kleene tables, the more general strategies (31) and (34) are "local" in that for given values x and y, they completely determine the value $f^\alpha(x, y)$ based on f, x and y. By contrast, the WK$^+$ and SK$^-$ operators of section 2 rely on entailments between propositional formulas, hence they are not local in this sense (as mentioned above, the WK$^+$ and SK$^-$ operators are not truth-functional).

In order to compare the WK and SK strategies (31) and (34) to global generalizations of the WK$^+$ and SK$^-$ operators, we first define global versions of the former. Let M be a model over expressions within a type system for n-place functions and products (e.g. van Benthem (1991)). For any type τ, we standardly denote D_τ^M for the domain of values of type τ in M, allowing partial function values. For any such type and model, we assume that the exceptional value $*$ is not in D_τ^M, and denote $D_\tau^{M^*} = (D_\tau^M)^*$. For an expression exp of type τ, we need to specify an element of $D_\tau^{M^*}$ as the denotation of exp. This element is denoted $[[exp]]^{M^*}$. Globalizing the WK and SK repair strategies above, we assume that F is a binary function expression of type $(\mathsf{a} \bullet \mathsf{b})\mathsf{c}$, and exp_1 and exp_2 are expressions of type a and b, respectively. We assume by induction that for every model $M \in \mathcal{M}$: $[[F]]^M \in D^M_{(\mathsf{a}\bullet\mathsf{b})\mathsf{c}}$, $[[exp_1]]^{M^*} \in D_{\mathsf{a}}^{M^*}$ and $[[exp_2]]^{M^*} \in D_{\mathsf{b}}^{M^*}$.

Definition 3.2. *The* **global WK and SK strategies** *for* $[[F(exp_1, exp_2)]]^{M^*}$ *are defined by:*

$$[[F^{\text{WK}}(exp_1, exp_2)]]^{M^*} = ([[F]]^M)^{\text{WK}}([[exp_1]]^{M^*}, [[exp_2]]^{M^*})$$

$$[[F^{\text{SK}}(exp_1, exp_2)]]^{M^*} = ([[F]]^M)^{\text{SK}}([[exp_1]]^{M^*}, [[exp_2]]^{M^*})$$

For example, let '$\mathbf{mult}$' denote a the standard binary multiplication operator over real number expressions, and let '/' denote the standard partial division operator over real number expressions. Let $[[1/r]]^{M^*}$ be inductively specified as $*$ in models M^* where $[[r]]^{M^*} = 0$. Considering the Kleene-repaired expression $\mathbf{mult}^\alpha(r, 1/r)$, we observe that any model M s.t. $[[r]]^{M^*} = 0$ satisfies $[[\mathbf{mult}^{\text{WK}}(r, 1/r)]]^{M^*} = *$ whereas $[[\mathbf{mult}^{\text{SK}}(r, 1/r)]]^{M^*} = 0$.

Generalizing the and WK$^+$ and SK$^-$ operators of section 2 involves considering the global strategies they employ. In such global strategies, we need to classify which expressions $F(exp_1, exp_2)$ are "repairable" in cases of failure of exp_2. This classification depends on the denotations of F, exp_1 and exp_2 in different models. We start out with generalizing the WK$^+$ operators. Definition 3.3 below specifies the conditions under which the WK$^+$ strategy is allowed to "repair" failures of exp_2. As in definition 3.2, we are given a binary function expression F of type $(\mathsf{a}\bullet\mathsf{b})\mathsf{c}$, and expressions exp_1 and exp_2 of type a and b, respectively, s.t. for every model $M \in \mathcal{M}$: $[[F]]^M \in D^M_{(\mathsf{a}\bullet\mathsf{b})\mathsf{c}}$, $[[exp_1]]^{M^*} \in D_{\mathsf{a}}^{M^*}$ and $[[exp_2]]^{M^*} \in D_{\mathsf{b}}^{M^*}$.

Definition 3.3. *Given a class of models* $\mathcal{M}$*, let* $c \in \bigcap_{M\in\mathcal{M}} D_{\mathsf{c}}^M$*, and suppose that every model* $M \in \mathcal{M}$ *where* $[[exp_2]]^{M^*} = *$ *and* $[[exp_1]]^{M^*} \neq *$ *satisfies:* $LD_{[[F]]^M}([[exp_1]]^{M^*}) = c$*. Then we say that the expression* $F(exp_1, exp_2)$ *is* **R-repairable as** c.

In words: an expression $F(exp_1, exp_2)$ is R-repairable when there is a value c of type c shared by all models, and for all models, c is L-determined by the value of exp_1 for F whenever exp_2 fails and exp_1 does not.[11]

Example 1: Let us again consider the expression $\mathbf{mult}(r, 1/r)$. The expression $1/r$ denotes $*$ precisely in those models where the value of r is 0, which L-determines the result of $\mathbf{mult}$ as 0. Thus, $\mathbf{mult}(r, 1/r)$ is R-repairable with respect to stan-

[11] For the sake of presentation, this condition is stronger than necessary: we might as well require the value c to only be shared by models where exp_2 fails and exp_1 does not.

dard models of the real numbers.

Example 2: Let us consider the expression $F(r, \sqrt{r})$ over the real numbers, where $F(x, y)$ is defined by 0 for $x = -3$ and by $x + y$ otherwise. In models where $r = -3$, this is the value of the left-hand argument of the expression F, which L-determines the result of the function that F denotes. Accordingly, for $r = -3$ the value of $F^{\mathrm{SK}}(r, \sqrt{r})$ is 0, which repairs the failure of $\sqrt{r}$. However, since the expression $\sqrt{r}$ fails for all negative values of r other than -3, and these values do not L-determine the value of F, the expression $F(r, \sqrt{r})$ is not generally R-repairable.

Example 2 highlights a general difference between the SK repair strategy and the WK$^+$ repair strategy employed in section 2. The SK connectives deal with failures of the right-hand value in all models where the value of the left-hand value L-determines the result. By contrast, the WK$^+$ operators only deal with failures of the right-hand argument as long as any such failure entails that the value of the left-hand argument L-determines the result. Thus, our propositional WK$^+$ operators only deal with failures of ψ formulas in *R-repairable* formulas of the form $\varphi\ op\ \psi$. To generalize this WK$^+$ strategy, we again let F be a binary function expression of type $(\mathsf{a} \bullet \mathsf{b})\mathsf{c}$, and let exp_1 and exp_2 be expressions of type a and b, respectively. We assume by induction that for every model $M \in \mathcal{M}$: $[[F]]^M \in D^M_{(\mathsf{a}\bullet\mathsf{b})\mathsf{c}}$, $[[exp_1]]^{M^*} \in D^{M^*}_{\mathsf{a}}$ and $[[exp_2]]^{M^*} \in D^{M^*}_{\mathsf{b}}$.

Definition 3.4. *The (global)* **WK$^+$ strategy** *for* $[[F(exp_1, exp_2)]]^{M^*}$ *is defined by:*

$$[[F^{\mathrm{WK}^+}(exp_1, exp_2)]]^{M^*} = \begin{cases} [[F(exp_1, exp_2)]]^M & [[exp_1]]^{M^*} \neq * \text{ and } [[exp_2]]^{M^*} \neq * \\ c & LD_{[[F]]^M}([[exp_1]]^{M^*}) = c \in \bigcap_{M \in \mathcal{M}} D^M_{\mathsf{c}} \\ & \text{and } F(exp_1, exp_2) \text{ is R-repairable as } c \\ * & \text{otherwise} \end{cases}$$

The propositional WK$^+$ operators of section 2 are instances of definition 3.4, which is also applicable to binary functions more generally. For instance, based on the facts in examples 1 and 2 above, we conclude that $\mathbf{mult}^{\mathrm{WK}^+}(r, 1/r) = 0$ when $r = 0$, but $F^{\mathrm{WK}^+}(r, \sqrt{r}) = *$ when $r = -3$, in contrast with $F^{\mathrm{SK}}(r, \sqrt{r}) = 0$. This is by virtue of the R-reparability of $\mathbf{mult}(r, 1/r)$ and the non-R-reparability of $F(r, \sqrt{r})$.

The generalization of the SK$^-$ operators of section 2 is similarly obtained, by reversing the direction of the implication in the definition of R-reparability. We call this notion *anti-R-reparability* and define it as follows:

Definition 3.5. *Given a class of models $\mathcal{M}$, let $c \in \bigcap_{M \in \mathcal{M}} D^M_{\mathsf{c}}$, and suppose that every model where $LD_{[[F]]^M}([[exp_1]]^{M^*}) = c$, we have $[[exp_2]]^{M^*} = *$. Then we say that the expression $F(exp_1, exp_2)$ is* **anti-R-repairable as** *c.*

In words: an expression $F(exp_1, exp_2)$ is anti-R-repairable when there is a value c of type c shared by all models, and in all models where c is L-determined by exp_1 for F, the evaluation of exp_2 fails.

The expression $F(r, \sqrt{r})$ of example 2 above is an instance of an anti-R-repairable expression, for the only value of r that L-determines the value of this expression, namely $r = -3$, fails the right-hand argument.

In the following definition we use the notion of anti-R-reparability to generalize the SK$^-$ strategy.

Definition 3.6. *The (global)* **SK$^-$ strategy** *for* $[[F(exp_1, exp_2)]]^{M^*}$ *is defined by:*

$$[[F^{\mathrm{SK}^-}(exp_1, exp_2)]]^{M^*} = \begin{cases} [[F(exp_1, exp_2)]]^M & [[exp_1]]^{M^*} \neq * \text{ and } [[exp_2]]^{M^*} \neq * \\ c & LD_{[[F]]^M}([[exp_1]]^{M^*}) = c \in \bigcap_{M \in \mathcal{M}} D^M_{\mathsf{c}} \\ & \text{and } F(exp_1, exp_2) \text{ is R-repairable as } c \\ & \text{or anti-R-repairable as } c \\ * & \text{otherwise} \end{cases}$$

Definition 3.6 adds to definition 3.4 the possibility that the expression $F(exp_1, exp_2)$ is anti-R-repealable. The propositional SK$^-$ operators of section 2 are instances of definition 3.6, which is also applicable to binary functions more generally. For instance, the expression $F(r, \sqrt{r})$ in example 2 satisfies $F^{\mathrm{SK}^-}(r, \sqrt{r}) = 0$ when $r = -3$. Still, in terms of its repair potential, the SK$^-$ strategy is weaker than the SK strategy. The following example illustrates that with a non-propositional expression.

Example 3: Let us consider the expression $G(r, \sqrt{s})$, where $G(x, y)$ is defined by 0 for $x < 0$ and by $x + y$ otherwise. In models where $r = -5$ and $s = -3$, the value of the left-hand argument

(-5) L-determines the value of G. Accordingly, when $r = -5$ and $s = -3$, the value of $G^{\mathrm{SK}}(r, \sqrt{s})$ is 0 despite the failure of $\sqrt{s}$.[12] By contrast, there are models where the expression $\sqrt{s}$ fails and r is positive, hence does not L-determine the value of G. Conversely, there are also models where the expression $\sqrt{s}$ does not fail and r is negative, hence L-determines the value of G. This means that the expression $G(r, \sqrt{s})$ is neither R-repairable nor anti-R-repairable. As a result, in models where $r = -5$ and $s = -3$, the expression $G^{\mathrm{SK}^-}(r, \sqrt{s})$ fails, unlike its SK parallel.

From the definitions above and examples 1-3 we conclude that the more general repair strategies for binary functions show the same hierarchy that we pointed out for the propositional connectives: the SK strategy is the most general repair strategy, WK allows no repair, whereas the repair strategies of WK$^+$ and SK$^-$ are properly in between these two extremes.

4 Concluding remarks

This paper proposed new binary operators on truth-value denoting expressions, which, unlike the Weak Kleene connectives, allow filtering and conditional presuppositions, and unlike the Strong Kleene connectives, do not face the "proviso" problem. We defined asymmetric operators that allow left-to-right filtering (the "Weak Kleene plus" operators) as well as conditional presuppositions (the "Strong Kleene minus" operators) without proviso-like problems. These operators were generalized for arbitrary binary function expressions, which reveals the centrality of values that left/right-determine the result of a function for the treatment of presupposition projection in trivalent semantics.

One last note is in place about the special status of Strong Kleene (SK) connectives in the treatment of the third truth-value $*$. As has been previously observed (Muskens, 1995; Beaver and Krahmer, 2001), SK conjunction and disjunction are greatest lower bound and least upper bound operators, respectively, with respect to the less-or-equal partial order, where '1' and '0' are treated numerically and '$*$' is treated numerically as ½. This gives the following lattice structure in SK trivalent logic (see Fitting (1991) for generalizations):

$$\begin{tikzcd} 1 \\ * \arrow[u] \\ 0 \arrow[u] \end{tikzcd}$$

When it comes to theories of presupposition projection, this formal elegance has an empirical price: the SK-based partial order is a proper subset of the order determined by the Tarskian notion of entailment (Keenan, 1973; Beaver, 1997) in definition 1.1. Tarskian entailment in trivalent semantics supports the following preorder:

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr, leftrightarrow] & & * \arrow[ul] \end{tikzcd}$$

Importantly, when it comes to entailment in natural language there is reason to prefer the Tarskian preorder to the SK partial order. For instance, the sentence *Sue has stopped shouting* intuitively entails the sentence *Sue is not shouting*, but the former can denote '$*$' while the latter denotes '0' (e.g. in situations where Sue has just started shouting). This indicates that the preorder determined by the Tarskian condition is advantageous to the SK-based partial order as a basis for a trivalent semantics of presuppositions. The 0 and $*$ values are distinguished by their projection but no by their support of entailment relations. This is expected by the Tarskian preorder and not by the SK partial order. Thus, although the SK truth tables are logically natural, and indeed have led to interesting logical results, their modelling of the $*$ value as "unknown" or "in between 0 and 1" is the source of their proviso problem when used for meanings of natural language operators. A linguistically more adequate view ensues from treating the $*$ value as a "failure" or an "exception" as in the Weak Kleene connectives or (de Groote and Lebedeva, 2010). This requires further inquiries into intermediate systems like the WK$^+$ or SK$^-$ operators that were studied above. These operators sacrifice truth-functionality – or, more generally, locality or at least extensionality – but model filtering and conditional presuppositions similarly to the Strong Kleene connectives, without running into their well-known problems.

[12]We may consider this kind of case as an illustration of a "proviso problem" for non-propositional binary functions: L-determination by the left-hand argument guarantees a successful evaluation of the function in cases when its right-hand argument fails, even if that failure is logically unrelated to L-determination.

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