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Presupposition, Admittance and Karttunen Calculus

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Abstract: Classic works define presuppositions of a sentence *S* as conclusions that follow from both *S* and its negation. Other studies focus on the necessary conditions for admitting *S* as true or false, assuming that those conditions converge with *S*'s presuppositions. Here we study this assumption in three systems: asymmetric Kleene truth tables, Heim's admittance-based theory, and a new propositional calculus inspired by Karttunen's entailment-based approach. Common versions of the Kleene and Heim systems are known to be semantically congruent, and we show that they identify presuppositions with admittance conditions. By contrast, it is proved that the proposed *Karttunen calculus* distinguishes the two notions. This aspect of the Karttunen calculus avoids the "proviso problem" for the Kleene/Heim approaches: the generation of presuppositions that appear to be too weak.

Keywords: presupposition, admittance, propositional logic, Kleene truth tables, three valued logic, proviso problem

1 Introduction

Presuppositions may disappear when the expression that triggers them is embedded in a complex sentence. For instance, the term "*the king of France*" famously presupposes that France is a monarchy, but the sentence "if France has a king, *the king of France* must be living at the Élysée Palace" does not. In such cases, we say that the presupposition "France is a monarchy" does not *project*. Karttunen (1973, 1974) analyzed presupposition projection and the lack thereof using rules that draw on entailment relations between logical forms. Peters (1979) suggested to emulate Karttunen's proposals using a truth-functional analysis that employs an asymmetric version of the Strong

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Kleene tables. In this three-valued semantics, a *presupposition* of a sentence S is classically defined as a proposition that follows from both S and its negation (van Fraassen, 1971). The Kleene-Peters analysis has been opposed to the "dynamic" approach in (Heim, 1983; Stalnaker, 1978), which defines a presupposition of a sentence S as a proposition that is entailed by all contexts that *admit* S, i.e. make S true or false.

This paper first shows that the Heim-Stalnaker account derives the same consequence relation as the Kleene-Peters system. In both systems, a close relation is rendered between classic presuppositions and admittance conditions: a proposition is a logically *strongest* presupposition of a sentence S if and only if it is a *weakest* admittance condition of S. This property has linguistically undesirable ramifications, known as the *proviso problem* (Winter, 2019 and references therein). For example, in the sentence "if Sue is busy, her spouse is away", the Kleene-Peters and Heim-Stalnaker analyses expect an unintuitive presupposition: "Sue is married if she is busy". We propose a solution of the proviso problem that generalizes Karttunen's rules into a so-called *Karttunen calculus*. This calculus derives the same admittance conditions as in the Kleene/Heim system. However, the presuppositions that are derived in the Karttunen calculus may be stronger than those admittance conditions, which allows the system to avoid the proviso problem.

The Karttunen calculus is similar to the Kleene-Peters system in relying on *left-determinant* values of binary operators for defining presupposition projection. Like the Heim-Stalnaker system, it uses local contexts for satisfying presuppositions. However, unlike the Kleene/Heim systems, the calculus relies on entailment between propositional formulas as in Karttunen's work, rather than on implication or set inclusion between their denotations. Contrary to Peters' claims, this makes context a non-redundant element of the Karttunen calculus, indeed of Karttunen's (1974) original proposal.

Section 2 introduces the notions of *left-determinant value* and *projection calculus* and illustrates their use for presenting the Kleene-Peters tables. Section 3 shows that the Heim-Stalnaker semantics leads to the same equivalence and entailment relations as those tables. Section 4 shows that the Kleene/Heim system conflates strongest presuppositions with weakest admittance conditions. It is conjectured that this conflation is inadequate for describing natural language and leads to the proviso problem. Section 5 introduces the *Karttunen calculus* and shows that it distinguishes a sentence's strongest presuppositions from its, possibly weaker, weakest admittance conditions, thus avoiding the proviso problem. Section 6 concludes.

2 Kleene truth tables and projection calculi

The Strong Kleene truth tables are one of the earliest logical treatments of presupposition. While these tables are symmetric, presupposition projection is often not (Mandelkern, Zehr, Romoli, & Schwarz, 2020). In view of this fact, Peters (1979) proposed the tables in figure 1, where '1', '0' and '$*$' stand for *true*, *false* and *undefined*, respectively. These trivalent *Kleene-Peters* (KP) tables asymmetrically extend the standard bivalent tables. A bivalent conjunction (disjunction/implication) is false (true/true) when the lefthand operand is false (true/false, respectively). This property is preserved in the KP truth tables, also when the righthand operand is undefined. However, when the lefthand operand is undefined, the result is undefined with no respect to the value of the righthand operand.

α	$\neg\alpha$
0	1
1	0
$*$	$*$

$\alpha \wedge \beta$	0	1	$*$
0	0	0	0
1	0	1	$*$
$*$	$*$	$*$	$*$

$\alpha \vee \beta$	0	1	$*$
0	0	1	$*$
1	1	1	1
$*$	$*$	$*$	$*$

$\alpha \rightarrow \beta$	0	1	$*$
0	1	1	1
1	0	1	$*$
$*$	$*$	$*$	$*$

Figure 1: The Kleene-Peters (KP) truth tables

Presuppositional and assertive elements of English sentences are analyzed as *bivalent*, and are expressed using a standard *propositional language*: a closure of a non-empty set of constants C under the propositional operators $\neg$, $\wedge$, $\vee$ and $\rightarrow$. When the constants in C are arbitrary we assume that they are assigned a bivalent interpretation, and refer to the propositional language over C as 'L_2'. English sentences are analyzed as simple *trivalent* propositions, which are represented as pairs of formulas from L_2: a presuppositional part and an assertive content. Such pairs are denoted $(\alpha\!:\!\beta)$ and are interpreted in $\{0, 1, *\}$ using Blamey's (1986) *transplication* operator, which is defined below:

Definition 1 (transplication) *For any bivalent interpretation $[\![\cdot]\!]^{bi}$ of L_2, we extend the interpretation $[\![\cdot]\!]^{bi}$ of L_2 into an interpretation of $L_2 \times L_2$ by defining, for any $\alpha, \beta \in L_2$:*

$$[\![(\alpha\!:\!\beta)]\!]^{bi} = \begin{cases} [\![\beta]\!]^{bi} & [\![\alpha]\!]^{bi} = 1 \\ * & [\![\alpha]\!]^{bi} = 0 \end{cases}$$

Complex trivalent formulas are obtained using definition 2 below:

Definition 2 (L_3) *Given a propositional language L_2 over arbitrary constants, the language L_3 is a propositional language over $L_2 \times L_2$.*

One way to analyze presupposition projection is by defining the trivalent denotation of complex L_3 formulas for any bivalent interpretation of L_2. Definition 3 below uses the KP truth tables to extend the trivalent interpretation of $L_2 \times L_2$ in definition 1 into a trivalent interpretation of L_3:

Definition 3 (KP interpretation of L_3) *Let $[\![\cdot]\!]^{bi}$ be a bivalent interpretation of L_2, which is extended to $L_2 \times L_2$ as in definition 1. For any formula $\kappa \in L_3$, the* KP interpretation *of κ is denoted $[\![\kappa]\!]^{KP}$ and is defined as follows:*

$$[\![\kappa]\!]^{KP} = \begin{cases} [\![\kappa]\!]^{bi} & \kappa \in L_2 \times L_2 \\ [\![\neg]\!]^{KP}([\![\varphi]\!]^{KP}) & \kappa = \neg\varphi \\ [\![\mathsf{op}]\!]^{KP}([\![\varphi]\!]^{KP}, [\![\psi]\!]^{KP}) & \kappa = \varphi \,\mathsf{op}\, \psi \end{cases}$$

where $[\![\varphi]\!]^{KP}$ and $[\![\psi]\!]^{KP}$ are inductively defined, and negation and the binary operator 'op' are interpreted using the KP tables (fig. 1)

It is useful to note that the corresponding equivalence relation ($\overset{\text{KP}}{\equiv}$) over L_3 satisfies the following standard equivalences, for any $\varphi, \psi \in L_3$:

Fact 1 $\quad \varphi \vee \psi \overset{KP}{\equiv} \neg((\neg\varphi) \wedge \neg\psi) \qquad \varphi \to \psi \overset{KP}{\equiv} (\neg\varphi) \vee \psi$

The following example illustrates how KP semantics is used for analyzing presupposition projection:

Example 1 Sentences S1 and S2 below are represented as L_3 formulas:

S1 = *if Sue is married her spouse is away* = $(\top{:}\alpha_1) \to (\beta{:}\gamma)$

S2 = *if Sue is busy her spouse is away* = $(\top{:}\alpha_2) \to (\beta{:}\gamma)$

where α_1="Sue is married", α_2="Sue is busy", β="Sue has a spouse", and γ="Sue has a spouse who is away" are bivalent propositions. We now observe the following KP equivalence:

$$(\top{:}\alpha) \to (\beta{:}\gamma) \quad \equiv \quad (\alpha \to \beta{:}\alpha \to \gamma)$$

While $\alpha_1 \to \beta$ is tautological, $\alpha_2 \to \beta$ is not. Thus, according to the KP semantics, the presupposition of sentence S1 is expected to be patently true, in agreement with linguistic judgements, where S1 shows no presupposition. By contrast, S2 is analyzed as presupposing "if Sue is busy, she has a spouse", which is weaker than the presupposition that ordinary speakers report ("Sue has a spouse"). This incongruence between theory and speaker judgements illustrates the *proviso problem* (Karttunen, 1973, p.188; Geurts, 1996).

An alternative way of analyzing presupposition projection is by rewriting any L_3 formula κ into a formula $(\kappa_1 : \kappa_2)$ in $L_2 \times L_2$, where the bivalent formula κ_1 is viewed as κ's strongest presupposition and κ_2 is viewed as κ's assertive context. We refer to this technique as a *projection calculus*.

The *Weak Kleene* (WK) tables let a propositional formula be interpreted as '$\star$' if any of its sub-formulas is interpreted as '$\star$'. Thus, the WK tables trivially "project" all presuppositions of κ's sub-formulas by letting κ_1 be their conjunction. This is modelled by the following projection calculus:

Definition 4 (WK calculus) *For any formula κ in L_3, let $WK(\kappa)$ be the formula in $L_2 \times L_2$ that is inductively defined as follows:*

$$WK(\kappa) = \begin{cases} \kappa & \kappa = (\kappa_1 : \kappa_2) \\ (\varphi_1 : \neg\varphi_2) & \kappa = \neg\varphi \\ (\varphi_1 \wedge \psi_1 : \varphi_2 \,\mathsf{op}^{bi}\, \psi_2) & \kappa = \varphi \,\mathsf{op}\, \psi \end{cases}$$

where: - op^{bi} *is the bivalent propositional operator corresponding to* op
- $\kappa_1, \kappa_2 \in L_2$*, and inductively:* $(\varphi_1 : \varphi_2) = WK(\varphi)$ *and* $(\psi_1 : \psi_2) = WK(\psi)$

A similar rewriting technique describes the KP tables (figure 1). We first assign a unary operator '$\textsc{ldv}_{op}$' (*left determinant value*) to any bivalent binary operator op. This is defined below:

Definition 5 (left determinant value) *For any binary operator* op*, the corresponding unary operator specifying the* left determinant value*(*s*) of* op *is defined as follows for any $\alpha \in L_2$:*

$$\textsc{ldv}_{op}(\alpha) = (\alpha \,\mathsf{op} \perp \leftrightarrow \alpha \,\mathsf{op} \top).$$

Thus, we have: $\textsc{ldv}_{\wedge}(\alpha) = \textsc{ldv}_{\rightarrow}(\alpha) \equiv \neg\alpha$ and $\textsc{ldv}_{\vee}(\alpha) \equiv \alpha$.

Using the $\textsc{ldv}$ operator, we define the *KP calculus* as follows:

Definition 6 (KP calculus) *For any formula κ in L_3, let $KP(\kappa)$ be the formula in $L_2 \times L_2$ that is inductively defined as follows:*

$$KP(\kappa) = \begin{cases} \kappa & \kappa = (\kappa_1 : \kappa_2) \\ (\varphi_1 : \neg\varphi_2) & \kappa = \neg\varphi \\ WK((\varphi_1 : \varphi_2) \,\mathsf{op}\, ((\psi_1 \vee \textsc{ldv}_{op}(\varphi_2)) : \psi_2)) & \kappa = \varphi \,\mathsf{op}\, \psi \end{cases}$$

where $\kappa_1, \kappa_2 \in L_2$*, and inductively:* $(\varphi_1 : \varphi_2) = KP(\varphi)$ *and* $(\psi_1 : \psi_2) = KP(\psi)$

By definitions 4, 5 and 6 we have:

$$KP(\varphi \wedge \psi) \equiv (\varphi_1 \wedge (\psi_1 \vee \neg\varphi_2) : \varphi_2 \wedge \psi_2)$$
$$KP(\varphi \vee \psi) \equiv (\varphi_1 \wedge (\psi_1 \vee \varphi_2) : \varphi_2 \vee \psi_2)$$
$$KP(\varphi \rightarrow \psi) \equiv (\varphi_1 \wedge (\psi_1 \vee \neg\varphi_2) : \varphi_2 \rightarrow \psi_2)$$

Definition 6 of the KP calculus is *sound* with respect to KP interpretations:

Fact 2 *For any formula $\kappa \in L_3$ and KP interpretation:* $[[KP(\kappa)]]^{KP} = [[\kappa]]^{KP}$.

Example 2 $KP((\top : \alpha) \rightarrow (\beta : \gamma)) \equiv (\top \wedge (\beta \vee \neg\alpha) : \alpha \rightarrow \gamma)$, which is equivalent to $(\alpha \rightarrow \beta : \alpha \rightarrow \gamma)$ as in example 1.

3 Heim-Stalnaker semantics

Heim (1983) analyzes presupposition projection in terms of a sentence's admittance by a given context. Following Stalnaker (1978), Heim defines a context as a set of possible worlds, which *admits* a sentence S if it is contained in the set of possible worlds where S's presuppositions hold. A sentence S is analyzed using a pair $\langle A, B \rangle$, where A and B are the sets of possible worlds denoted by S's presupposition and S's assertive content, respectively. Such pairs are used to update the context. Propositional connectives modify the updates induced by their operand(s). This view of presupposition projection seems quite different from the Kleene tables in traditional three-valued logic. However, following Peters (1979), this section shows that in terms of the entailment and equivalence relations they describe over formulas in L_3, the *Heim-Stalnaker* (HS) semantics and the KP truth tables are congruent.

3.1 Heim-Stalnaker semantics – language and interpretation

When representing a sentence's semantic import as its *context change potential* (CCP), it is convenient to use the following propositional language:

Definition 7 $L_{CCP} \overset{def}{=} L_2 \cup \{ \chi[\kappa] : \chi \in L_{CCP}$ *and* $\kappa \in L_3 \}$

Thus, any L_{CCP} formula is made of a context formula in L_2 and a (possibly empty) sequence of formulas in L_3.

Example 3 Given $C, \alpha, \beta, \gamma \in L_2$, the following are all L_{CCP} formulas –
$C,\ C[(\alpha{:}\beta)],\ (C[(\alpha{:}\beta)])[(\top{:}\gamma) \vee (\alpha{:}\beta)]$.

Adding disjunction to Heim's system, we get the following canonical semantics of L_{CCP} (Nouwen, Brasoveanu, van Eijck, & Visser, 2016; Rothschild, 2011):

Definition 8 (HS interpretation of L_{CCP}) *Let $W \neq \varnothing$ be an arbitrary set of possible worlds, and let $[\![\cdot]\!]^M_W$ (in short: '$[\![\cdot]\!]^M$') be a* modal *interpretation of L_2, which assigns any constant $p \in L_2$ a set $[\![p]\!] \subseteq W$, and interprets any complex L_2 formula using the set-theoretical operators corresponding to the propositional connectives. An* HS interpretation *over W is a function $[\![\cdot]\!]^{HS}_W$ (in short: '$[\![\cdot]\!]^{HS}$' or '$[\![\cdot]\!]$') from L_{CCP} to $\wp(W) \cup \{*\}$ that inductively extends $[\![\cdot]\!]^M_W$ to any formula $\chi \in L_{CCP}$. This is defined as follows:*

- *For any $\chi \in L_2$, we define: $[\![\chi]\!] = [\![\chi]\!]^M$.*
- *For any $\chi = \mu[\kappa] \in L_{CCP} \smallsetminus L_2$:*

(a) If $[\![\mu]\!] = *$, we define: $[\![\mu[\kappa]]\!] = *$.

(b) If $[\![\mu]\!] \neq *$ and $\kappa = (\kappa_1 : \kappa_2) \in L_2 \times L_2$, we define:

$$[\![\mu[(\kappa_1 : \kappa_2)]]\!] = \begin{cases} [\![\mu]\!] \cap [\![\kappa_2]\!] & \text{if } [\![\mu]\!] \subseteq [\![\kappa_1]\!] \\ * & \text{otherwise} \end{cases}$$

(c) If $[\![\mu]\!] \neq *$ and $\kappa \in L_3 \smallsetminus (L_2 \times L_2)$, we define inductively:

$$[\![\,\mu[\neg\varphi]\,]\!] = \begin{cases} [\![\mu]\!] \smallsetminus [\![\mu[\varphi]]\!] & \text{if } [\![\mu[\varphi]]\!] \neq * \\ * & \text{otherwise} \end{cases}$$

$$[\![\,\mu[\varphi \wedge \psi]\,]\!] = [\![(\mu[\varphi])[\psi]]\!]$$

$$[\![\,\mu[\varphi \vee \psi]\,]\!] = \begin{cases} [\![\mu[\varphi]]\!] \cup [\![(\mu[\neg\varphi])[\psi]]\!] & \text{if } [\![\mu[\varphi]]\!] \neq * \\ & \text{and } [\![(\mu[\neg\varphi])[\psi]]\!] \neq * \\ * & \text{otherwise} \end{cases}$$

$$[\![\,\mu[\varphi \to \psi]\,]\!] = \begin{cases} [\![\mu[\neg\varphi]]\!] \cup [\![(\mu[\varphi])[\psi]]\!] & \text{if } [\![\mu[\neg\varphi]]\!] \neq * \\ & \text{and } [\![(\mu[\varphi])[\psi]]\!] \neq * \\ * & \text{otherwise} \end{cases}$$

Similarly to KP connectives (fact 1), the corresponding *HS equivalence* relation ($\overset{\text{HS}}{\equiv}$) over L_{CCP} satisfies, for any $\chi \in L_{CCP}$ and $\varphi, \psi \in L_3$:

Fact 3 $\quad \chi[\varphi \vee \psi] \overset{HS}{\equiv} \chi[\neg((\neg\varphi) \wedge \neg\psi)] \qquad \chi[\varphi \to \psi] \overset{HS}{\equiv} \chi[(\neg\varphi) \vee \psi]$

For the *proof* of fact 3 see Appendix A .

In HS semantics, the analysis of presupposition projection in sentences S1 and S2 from example 1 goes as follows:

Example 4 Sentences S1 and S2 below are represented as L_{CCP} formulas:

S1 = *if Sue is married her spouse is away* = $C[(\top{:}\alpha_1)\to(\beta{:}\gamma)]$

S2 = *if Sue is busy her spouse is away* = $C[(\top{:}\alpha_2)\to(\beta{:}\gamma)]$

where C is arbitrary, and $\alpha_1, \alpha_2, \beta$ and γ are as in example 1. We observe that under HS interpretations:

$$C[(\top{:}\alpha)\to(\beta{:}\gamma)] \quad \equiv \quad C[(\alpha\to\beta{:}\alpha\to\gamma)]$$

Thus, for any context C and interpretation $[\![\cdot]\!]^{HS}$, the formula $\kappa = (\top{:}\alpha)\to(\beta{:}\gamma)$ is well-defined relative to C (i.e. has a non-'$*$' interpretation) *iff* the set $[\![C]\!]^M$ is contained in $[\![\alpha\to\beta]\!]^M$. Accordingly, and similarly to KP semantics (example 1), the proposition $\alpha\to\beta$ is viewed as κ's presupposition.

3.2 HS semantics and KP semantics

Any HS interpretation over a set of possible worlds $W \neq \varnothing$ has a modal interpretation of L_2 at its basis. Such a modal interpretation corresponds with a family F of bivalent interpretations of L_2 that is indexed by W. Thus, a modal interpretation of L_2 gives rise to a family of KP interpretations of L_3. In this section we show that any HS interpretation can be represented as such a family of KP interpretations. First, for any family of bivalent interpretations of L_2 we define an alternative semantics of L_{CCP} that directly employs the KP semantics of L_3. Definition 9 specifies this *KP-based interpretation*:

Definition 9 *Given a set $W \neq \varnothing$, let $F = [\![\cdot]\!]^{bi}_i|_{i\in W}$ be a family of bivalent interpretations of L_2. For any $i \in W$, let $[\![\cdot]\!]^{KP}_i$ be the KP interpretation of L_3 corresponding to $[\![\cdot]\!]^{bi}_i$. A* KP-based interpretation of L_{CCP} relative to F *is a function $[\![\cdot]\!]^{KP}_W$ from L_{CCP} to $\wp(W)\cup\{*\}$ that is inductively defined as follows for any $\chi\in L_{CCP}$:*

<u>*If $\chi = \alpha$*</u>*, s.t. $\alpha\in L_2$:*

$$[\![\alpha]\!]^{KP}_W = \{i \in W : [\![\alpha]\!]^{bi}_i = 1\}$$

<u>*If $\chi = \mu[\kappa]$*</u>*, s.t. $\mu\in L_{CCP}$ and $\kappa\in L_3$:*

$$[\![\mu[\kappa]]\!]^{KP}_W = \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\kappa]\!]^{KP}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } \\ & \quad [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\kappa]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

In the standard definition 8 of HS interpretations, a complex formula $\mu[\kappa]$ is interpreted by updating the context μ inductively using sub-formulas of κ. By contrast, in KP-based interpretations, $\mu[\kappa]$ is interpreted using the KP semantics of κ in L_3, through the given family F of bivalent interpretations. Theorem 1 shows that this way of interpreting L_{CCP} using the KP tables covers all HS interpretations. As summarized in Appendix B , this theorem makes the same point as the main property proved in Peters (1979).

Theorem 1 *Let* $[\![\cdot]\!]^{HS}_W$ *be an HS interpretation of* L_{CCP} *for some* $W \neq \varnothing$*. For any* $i \in W$*, let* $[\![\cdot]\!]^{bi}_i$ *be the bivalent interpretation of* L_2 *s.t. for any* $\alpha \in L_2$*:* $[\![\alpha]\!]^{bi}_i = 1$ *iff* $i \in [\![\alpha]\!]^{HS}$*. Let* $[\![\cdot]\!]^{KP}_W$ *be the KP-based interpretation of* L_{CCP} *relative to the family* $F = [\![\cdot]\!]^{bi}_i\,|_{i \in W}$*. Then for any* $\chi \in L_{CCP}$ *we have:*

$$[\![\chi]\!]^{HS}_W = [\![\chi]\!]^{KP}_W.$$

The *proof* of theorem 1 in Appendix C is by induction on the structure of χ for the subset of L_{CCP} involving only negation and conjunction. This proof is directly applicable to disjunction and implication due to the standard facts 1 and 3 under KP and HS interpretations.

Using theorem 1, we now show that HS semantics is congruent with KP semantics in two senses. First, we show the soundness of a so-called *HS calculus*, which uses the KP calculus to rewrite any formula $\chi \in L_{CCP}$ as a maximally simple formula in L_{CCP}. Second, we use the HS calculus to show that entailment and equivalence relations over L_3 that are naturally induced by the HS semantics are identical to those induced by KP semantics.

3.3 HS calculus

Relying on theorem 1, we first show that the KP calculus can be used to simplify any L_{CCP} formula while preserving its HS semantics:

Corollary 1 *For any* $\chi \in L_{CCP}$ *and* $\kappa \in L_3$*:* $\chi[\kappa] \overset{HS}{\equiv} \chi[KP(\kappa)]$.

For the *proof* see Appendix D .

On the basis of corollary 1, we show that the KP calculus extends into a sound method of rewriting L_{CCP} formulas into equivalent, maximally simple CCP formulas. Rewriting in this *HS calculus* is defined below:

Definition 10 (HS calculus) *Let* $min(L_{CCP})$ *be the following set of* minimal L_{CCP} *formulas:*

$$min(L_{CCP}) = L_2 \cup \{ C[(\kappa_1 : \kappa_2)] : C, \kappa_1, \kappa_2 \in L_2 \}.$$

For any formula χ *in* L_{CCP}*, we define* HS(χ) *as the formula in* $min(L_{CCP})$

that is inductively defined as follows:

For any $\chi = C \in L_2$*:*

$HS(C) = C.$

For any $\chi = C[\kappa] \in L_{CCP}$ *where* $C \in L_2$ *and* $\kappa \in L_3$*:*

$HS(C[\kappa]) = C[KP(\kappa)].$

For any $\chi = (\mu[\varphi])[\kappa]$ *where* $\mu \in L_{CCP}$ *and* $\varphi, \kappa \in L_3$*, inductively:*

$HS((\mu[\varphi])[\kappa]) = HS(\mu[\varphi \wedge \kappa]).$

This calculus maps any simple L_2 formula in L_{CCP} to itself. Any formula $C[\kappa]$ where $C \in L_2$ is mapped to $C[KP(\kappa)]$, where $KP(\kappa)$ is the $L_2 \times L_2$ formula obtained from κ in the KP calculus. More complex L_{CCP} formulas are of the form $(\ldots((C[\varphi_1])[\varphi_2])\ldots)[\varphi_n]$, where $\varphi_1, \varphi_2, \ldots, \varphi_n \in L_3$. Such formulas are "flattened" to the form $C[\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n]$, which is inductively simplified using the KP calculus.

Example 5 $HS(C[(\varphi_1{:}\varphi_2)])[(\psi_1{:}\psi_2)]) = HS(C[(\varphi_1{:}\varphi_2)\wedge(\psi_1{:}\psi_2)])$
$= C[KP((\varphi_1{:}\varphi_2)\wedge(\psi_1{:}\psi_2))] = C[(\varphi_1\wedge(\psi_1\vee\neg\varphi_2){:}\varphi_2\wedge\psi_2)]$

This HS calculus is *sound* with respect to HS interpretations of L_{CCP} formulas:

Corollary 2 *For any formula* $\chi \in L_{CCP}$ *and HS interpretation:*

$[\![HS(\chi)]\!]^{HS} = [\![\chi]\!]^{HS}.$

The *proof* for formulas $C[\kappa]$ where $C \in L_2$ follows from corollary 1. For other formulas of the form $\mu[\kappa]$, corollary 2 is proved inductively, relying on its proof for μ. See Appendix E for details.

3.4 KP/HS entailment and KP/HS equivalence

KP interpretations naturally specify an equivalence relation ($\overset{\text{KP}}{\equiv}$) over L_3. As for *entailment* over L_3, we standardly employ the following "Tarskian" definition in trivalent semantics (van Fraassen, 1971):

Definition 11 (trivalent entailment) *Let* $\mathcal{C}$ *be a class of trivalent interpretations mapping a language* L *to* $\{0, 1, *\}$*. For any two formulas* $\varphi, \psi \in L$*, we denote* $\varphi \overset{\mathcal{C}}{\Rightarrow} \psi$ *if for every interpretation* $[\![\cdot]\!] \in \mathcal{C}$*:*

if $[\![\varphi]\!] = 1$ *then* $[\![\psi]\!] = 1$.

When $\mathcal{C}$ in definition 11 is the class of KP interpretations of L_3, we obtain a relation of *KP-entailment* ($\overset{KP}{\Rightarrow}$). We note that definition 11 distinguishes bidirectional entailment from equivalence: trivalent propositions may agree on the interpretations that make them *true* without necessarily agreeing on the interpretations that make them *false*.

HS semantics is defined over the language L_{CCP}, hence specifies an equivalence relation ($\overset{\text{HS}}{\equiv}$) over that language. To allow comparing HS and KP semantics, this relation is extended to an equivalence relation over L_3:

Definition 12 (HS equivalence over L_3) *For any two formulas $\varphi, \psi \in L_3$, we denote $\varphi \overset{HS}{\equiv} \psi$ iff for every $\chi \in L_{CCP}$: $\chi[\varphi] \overset{HS}{\equiv} \chi[\psi]$.*

For any two L_3 formulas φ and ψ, we also define *HS entailment*, by requiring that whenever φ leaves a context intact, so does ψ. Formally:

Definition 13 (HS entailment over L_3) *For any two formulas $\varphi, \psi \in L_3$, we denote $\varphi \overset{HS}{\Rightarrow} \psi$ iff for every $\chi \in L_{CCP}$ and HS interpretation:*

if $[\![\chi[\varphi]]\!]^{HS} = [\![\chi]\!]^{HS}$ then $[\![\chi[\psi]]\!]^{HS} = [\![\chi]\!]^{HS}$.

The claim below follows from the soundness of HS calculus (corollary 2):

Corollary 3 *For any two formulas $\varphi, \psi \in L_3$:*

(i) $\varphi \overset{KP}{\equiv} \psi$ iff $\varphi \overset{HS}{\equiv} \psi$ *(ii) $\varphi \overset{KP}{\Rightarrow} \psi$ iff $\varphi \overset{HS}{\Rightarrow} \psi$*

The *proof* of corollary 3 is in Appendix F .

The KP/HS entailment relation is *monotonic*, in the following sense:

Fact 4 *For all $\varphi, \psi, \kappa \in L_3$: if $\varphi \overset{KP}{\Rightarrow} \psi$ then $\kappa \wedge \varphi \overset{KP}{\Rightarrow} \psi$.*

This fact is related to the proviso problem, discussed in the following section.

4 Admittance vs. Presupposition

In the HS semantics of L_{CCP}, *admittance* of a proposition by a context is defined as follows:

Definition 14 (HS-admittance) *We say that a context $C \in L_2$* HS-admits *a formula $\kappa \in L_3$ if $[\![C[\kappa]]\!]^{HS} \neq \star$ for all HS interpretations $[\![\cdot]\!]^{HS}$.*

A parallel notion is defined over L_3 using the KP semantics. We first introduce the following general notation:

Notation. Given a projection calculus Ω *mapping* L_3 *to* $L_2 \times L_2$*, for any formula* $\kappa \in L_3$ *we denote:*

$\Omega(\kappa) = (\alpha^{\Omega}_{\kappa} : \beta^{\Omega}_{\kappa})$*, where* $\alpha^{\Omega}_{\kappa}, \beta^{\Omega}_{\kappa} \in L_2$.

In KP semantics we define admittance by first observing the following fact:

Fact 5 *For any* $C \in L_2$ *and* $\kappa \in L_3$*: C HS-admits* κ *iff* $\alpha^{KP}_{(\top:C)\wedge\kappa} \equiv \top$.

Thus, C admits κ in HS semantics *iff* the KP calculus rewrites the conjunction $(\top : C) \wedge \kappa$ into a pair $(\alpha : \beta)$ where α is a tautology. By soundness of KP calculus, this means that no KP interpretation makes the formula $(\top : C) \wedge \kappa$ *undefined* ('$*$'). When this condition holds we say that *C KP-admits* κ.

Presuppositions are standardly defined using entailment:

Definition 15 (presupposition) *Given an entailment relation* $\overset{\mathcal{C}}{\Rightarrow}$ *over* L_3*, we say that* $\kappa \in L_3$ $\mathcal{C}$-presupposes $\beta \in L_2$ *if* $\kappa \overset{\mathcal{C}}{\Rightarrow} (\top : \beta)$ *and* $\neg\kappa \overset{\mathcal{C}}{\Rightarrow} (\top : \beta)$.

Due to the convergence of the entailment relations in KP and HS semantics (corollary 3), KP-presupposition and HS-presupposition converge as well.

Furthermore, in KP/HS semantics, the logically *weakest admitting context* and *strongest presupposition* converge for any formula $\kappa \in L_3$. This is shown by the following theorem:

Theorem 2 *For* $\kappa \in L_3$*, let C be a weakest formula in* L_2 *that KP-admits* κ*, and let* β *be a strongest KP-presupposition of* κ *in* L_2*. Then* $C \equiv \beta \equiv \alpha^{KP}_{\kappa}$.

Standardly, we here say that $\alpha \in L_2$ is a weakest (strongest) formula in L_2 with a property Π if any $\alpha' \in L_2$ that has the property Π and satisfies $\alpha \Rightarrow \alpha'$ (respectively: $\alpha' \Rightarrow \alpha$) satisfies $\alpha' \equiv \alpha$. The *proof* of theorem 2 is in Appendix G .

Theorem 2 is closely related to the *proviso problem* for KP/HS semantics (example 1). To highlight this, we propose the following empirical conjecture about English, which stands in opposition to theorem 2:

Conjecture 1 *There exists an English sentence S that is admitted by a context C such that C is logically weaker than any strongest presupposition of S.*

Example 6 Sentences S3 and S4 below are represented as L_3 formulas:

S3 = *if Sue visited Dan, his beard annoyed her* = $(\top : \alpha) \rightarrow (\beta : \gamma)$

S4 = *if Sue visited Dan, he had grown a beard before she arrived*
= $(\top : \alpha) \rightarrow (\top : \beta')$

Where α="Sue visited Dan", β= "Dan had a beard", β'= "Dan had grown a beard before Sue arrived" and γ= "Dan had a beard that annoyed Sue".

Substantiating conjecture 1, we make the following empirical claims:

(a) Sentence *S3* presupposes that Dan had a beard.

(b) The conjunction *S4 and S3* does not presuppose that Dan had a beard. Furthermore, *S4 and S3* has no non-tautological presupposition.

Claim (b) is consistent with the expectation of KP/HS-semantics that the weakest admittance condition of S3 is S5 below, which is entailed by S4:

S5 = *if Sue visited Dan, he had a beard* = $(\top{:}\alpha)\rightarrow(\top{:}\beta)$

However, claim (a) is inconsistent with the expectation of KP/HS-semantics that S5 is also the strongest presupposition of S3.

5 The Karttunen Calculus

Conjecture 1 as illustrated in example 6 suggests that theorem 2 is problematic for using KP/HS semantics as a model of presupposition projection in English. To solve this problem, we propose an alternative projection calculus called the *Karttunen* (K) *calculus*. Like the KP calculus, the K-calculus maps any L_3 formula to a formula in $L_2 \times L_2$. However, unlike the KP calculus, the K-calculus does not emerge from any straightforward trivalent semantics. Rather, as in (Karttunen, 1973, 1974), the K-calculus takes *entailment* between bivalent formulas (or "logical forms") as the key to admitting a sentence by way of satisfying its presuppositions.

At the basis of the mechanism lie two assumptions: (i) a context $C \in L_2$ *admits* a simple L_3 formula $(\kappa_1 : \kappa_2)$ iff C *entails* κ_1 in bivalent logic; (ii) in binary constructions $\varphi\,\mathsf{op}\,\psi$, the assertive content of φ updates the context of ψ's evaluation using the LDV operator. The reliance on entailment in (i) prevents a direct interpretation of L_3 according to the K-calculus. Rather, L_3 formulas need to first be transformed into formulas in $L_2 \times L_2$ before they can be semantically interpreted. This representational analysis of presupposition projection follows Karttunen's reliance on logical forms, but it squarely aligns with the truth-functional practice of involving left-determinant values as the key to presupposition projection and admittance, as in Kleene-Peters semantics (Winter, 2019). Unlike HS semantics, where contexts are arguably redundant due to the operational equivalence with KP semantics (see Peters 1979 and corollary 3 above), the K-calculus uses L_2 formulas non-redundantly for recording *local contexts*. These local contexts are not denotations like sets of possible worlds as in HS semantics but L_2 formulas (or "logical forms") as in (Karttunen, 1974).

Formally, the K-calculus maps any L_3 formula to a formula in $L_2 \times L_2$ using a bivalent *context* $C \in L_2$, which is assumed to be tautological in the base case. This is specified in definition 16 below:

Definition 16 (K-calculus) *For any formula $C[\kappa]$ in L_{CCP} where $C \in L_2$, let $K(C[\kappa])$ be the formula in L_3 that is inductively defined as follows:*

If $\kappa = (\kappa_1 : \kappa_2) \in L_2 \times L_2$:

$$K(C[(\kappa_1 : \kappa_2)]) = \begin{cases} (\top : \kappa_2) & C \Rightarrow \kappa_1 \\ (\kappa_1 : \kappa_2) & \textit{otherwise} \end{cases}$$

If $\kappa \in L_3 \smallsetminus (L_2 \times L_2)$:

$$K(C[\kappa]) = \begin{cases} (\varphi_1 : \neg\varphi_2) & \kappa = \neg\varphi \\ WK(\, (\varphi_1 : \varphi_2) \,\mathsf{op}\, K((C \wedge \varphi_1 \wedge \neg \text{LDV}_{op}(\varphi_2))[\psi])\,) & \kappa = \varphi \,\mathsf{op}\, \psi \end{cases}$$

$$\textit{where inductively: } (\varphi_1 : \varphi_2) = K(C[\varphi])$$

For any $\kappa \in L_3$ (without any given C), we abbreviate:

$K(\kappa) = K(\top[\kappa])$.

By definition of the WK calculus and the LDV_{op} operator we now have:

$$K(C[\varphi \wedge \psi]) = WK((\varphi_1 : \varphi_2) \wedge K((C \wedge \varphi_1 \wedge \varphi_2)[\psi]))$$
$$K(C[\varphi \vee \psi]) = WK((\varphi_1 : \varphi_2) \vee K((C \wedge \varphi_1 \wedge \neg\varphi_2)[\psi]))$$
$$K(C[\varphi \to \psi]) = WK((\varphi_1 : \varphi_2) \to K((C \wedge \varphi_1 \wedge \varphi_2)[\psi]))$$

According to definition 16, in binary constructions both the presuppositional content and the (negation of) the assertive content of the lefthand operand are accommodated into the context of the righthand operand. This is useful in sentences like *if Sue stopped smoking, then Dan knows that Sue stopped smoking*. This sentence inherits the presupposition of the antecedent ("Sue used to smoke"), but not the presupposition of the consequent ("Sue used to smoke and doesn't smoke now"). According to the K-calculus, this happens due to the accommodation of the whole antecedent (both presupposition and assertive content) into the context of the consequent. This local context of the consequent entails its presupposition, hence that presupposition is not projected.

Let us now consider the application of the K-calculus to the analysis of sentences S3 and S5 from example 6:

Example 7 For S3 and S5 from example 6, we denote, respectively: $\eta = (\top : \alpha) \to (\beta : \gamma)$, and $\theta = (\top : \alpha) \to (\top : \beta')$.

Since $K(\top[(\top{:}\alpha)] = (\top{:}\alpha))$, and since $\alpha \not\Rightarrow \beta$, we conclude:
$K(\eta) = K((\top{:}\alpha) \to (\beta{:}\gamma)) = K(\top[(\top{:}\alpha) \to (\beta{:}\gamma)])$
$\quad = WK((\top{:}\alpha) \to K((\top \wedge \top \wedge \alpha)[(\beta{:}\gamma)])) = WK((\top{:}\alpha) \to (\beta{:}\gamma)) = (\beta{:}\alpha \to \gamma)$
$K(\theta \wedge \eta) = \ldots = WK((\top{:}\alpha{\to}\beta') \wedge K((\alpha{\to}\beta')[(\top{:}\alpha) \to (\beta{:}\gamma)]))$
$\quad = WK((\top{:}\alpha{\to}\beta') \wedge WK((\top{:}\alpha) \to K(((\alpha{\to}\beta') \wedge \alpha)[(\beta{:}\gamma)])))$, since $\beta' \Rightarrow \beta$:
$\quad = WK((\top{:}\alpha{\to}\beta') \wedge WK((\top{:}\alpha) \to (\top{:}\gamma))) = \ldots = (\top{:}\alpha{\to}(\beta' \wedge \gamma))$

Unlike the KP/HS semantics, these derivations are consistent with claims (a) and (b) in example 6. They show that β is the strongest K-presupposition of η, but the bivalent proposition $\alpha \to \beta'$ (=$K(\theta)$'s assertive content) K-admits η although it does not logically entail that presupposition. Thus, K-admittance and KP/HS-admittance converge in this case, although the K-presupposition is stronger than its KP/HS correlate.

More generally, we claim that weakest admittance conditions in the K-calculus are the same as in the KP/HS-calculus, for all L_3 formulas. By contrast, presuppositions in the K-calculus are at least as strong as those of the KP/HS-calculus, but they may also be properly stronger as in example 7. For this comparison between calculi, we first define the necessary semantic notions in the K-calculus. Definition 17 below *K-interprets* any $\kappa \in L_3$ by rewriting it into $K(\kappa)$ – an $L_2 \times L_2$ formula interpreted by transplication under any bivalent interpretation (definition 1):

Definition 17 (K-interpretation of L_3) *Let* $[\![\cdot]\!]^{bi}$ *be a bivalent interpretation of* L_2*, and let* κ *be a* L_3 *formula. The* Karttunen *(K)* interpretation *of* κ *is defined by* $[\![\kappa]\!]^{K} = [\![K(\kappa)]\!]^{bi}$.

Using K-interpretations, we define *K-equivalence* ($\overset{K}{\equiv}$), *K-entailment* ($\overset{K}{\Rightarrow}$) and *K-presupposition*, similarly to KP semantics. It is useful to note that similarly to KP/HS semantics (facts 1 and 3), K-interpretations satisfy the following standard equivalences, for any $\varphi, \psi \in L_3$:

Fact 6 $\quad \varphi \vee \psi \overset{K}{\equiv} \neg((\neg\varphi) \wedge \neg\psi) \qquad \varphi \to \psi \overset{K}{\equiv} (\neg\varphi) \vee \psi$

The *proof* in Appendix H simply applies the K-calculus.

Unlike entailment in KP/HS semantics (fact 4), K-entailment is *not monotonic*. This is illustrated by example 7, where η K-entails $(\top{:}\beta)$ but $\theta \wedge \eta$ does not.

K-admittance of $\kappa \in L_3$ by a context $C \in L_2$ is defined, similarly to KP-admittance, as follows:

Definition 18 (K-admittance) *We say that a context* $C \in L_2$ K-admits *a formula* $\kappa \in L_3$ *if* $\alpha^{K}_{(\top:C)\wedge\kappa} \equiv \top$.

By definition 17, this boils down to requiring that no K-interpretation assigns the formula $(\top : C) \wedge \kappa$ an *undefined* value ('$*$').

We now observe the following general fact about the K-calculus:

Theorem 3 *For any $\kappa \in L_3$, let C be a weakest formula in L_2 that K-admits κ, and let α be a strongest K-presupposition of κ in L_2. Then we have:*

$\alpha \equiv \alpha_\kappa^K$, $\alpha_\kappa^K \Rightarrow C$, *and* $C \equiv \alpha_\kappa^{KP}$.

In words: the strongest *K-presupposition* of κ is directly obtained in the K-calculus as α_κ^K. This K-presupposition entails any weakest context that *K-admits* κ, although it is not necessarily entailed by it (wit. example 7). Rather, any weakest context that admits κ in the K-calculus is equivalent to any weakest context that KP-admits κ. See Appendix I for a *proof* of theorem 3.

6 Conclusions

The Kleene-Peters and the Heim-Stalnaker systems are at the basis of many on-going attempts to describe the linguistic behavior of presupposition projection. The proviso problem threatens these attempts. Following Peters (1979), this paper has argued that the Kleene-Peters and the Heim-Stalnaker systems are logically congruent. However, contrary to Peters' claim that his system adequately mimics (Karttunen, 1974), we have proposed the K-calculus, maintaining Karttunen's aim of avoiding the proviso problem and distinguishing presuppositions from admittance conditions. The proviso problem for the Kleene-Peters/Heim-Stalnaker semantics is argued to result from these systems' conflation of *strongest presuppositions* with *weakest admittance conditions*. Both systems rely on a truth-functional account, where the semantic value of a sentence's presupposition is fully determined by the base language's bivalent interpretation. By contrast, the K-calculus relies, following Karttunen, on bivalent *entailment* as the basis for presupposition projection. This system distinguishes presupposition from admittance conditions, and is conjectured to be empirically more adequate than the Kleene-Peters/Heim-Stalnaker semantics. Notwithstanding, similarly to the Kleene-Peters tables, the Karttunen calculus relies on *left determinant values*, and like the Heim-Stalnaker semantics, it uses local contexts operationally in its account of presupposition projection. Furthermore, the admittance conditions that the Karttunen calculus derives are the same as in those two systems.

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Appendices

A Proof of fact 3

Fact 3 $\chi[\varphi \vee \psi] \overset{HS}{\equiv} \chi[\neg((\neg\varphi) \wedge \neg\psi)] \qquad \chi[\varphi \rightarrow \psi] \overset{HS}{\equiv} \chi[(\neg\varphi) \vee \psi]$

Proof. The property $\chi[\varphi \vee \psi] \equiv \chi[\neg((\neg\varphi) \wedge \neg\psi)]$ follows, since for all $\chi \in L_{CCP}$ and $\varphi, \psi \in L_3$, under the assumption $[\![\chi[\neg\varphi \wedge \neg\psi]]\!] \neq *$:

$[\![\chi[\neg(\neg\varphi \wedge \neg\psi)]]\!]$
$= [\![\chi]\!] \smallsetminus [\![\chi[\neg\varphi \wedge \neg\psi]]\!]$
$= [\![\chi]\!] \smallsetminus [\![(\chi[\neg\varphi])[\neg\psi]]\!]$
$= [\![\chi]\!] \smallsetminus ([\![\chi[\neg\varphi]]\!] \smallsetminus [\![(\chi[\neg\varphi])[\psi]]\!])$
$= [\![\chi]\!] \smallsetminus (([\![\chi]\!] \smallsetminus [\![\chi[\varphi]]\!]) \smallsetminus [\![(\chi[\neg\varphi])[\psi]]\!])$
$= [\![\chi]\!] \smallsetminus ([\![\chi]\!] \smallsetminus ([\![\chi[\varphi]]\!] \cup [\![(\chi[\neg\varphi])[\psi]]\!]))$
$= [\![\chi[\varphi]]\!] \cup [\![(\chi[\neg\varphi])[\psi]]\!]$
$= [\![\chi[\varphi \vee \psi]]\!]$

And by definition: the assumption $[\![\chi[\neg\varphi \wedge \neg\psi]]\!] \neq *$ holds
iff $[\![\chi[\neg\varphi][\neg\psi]]\!] \neq *$ holds
iff $[\![\chi[\neg\varphi][\psi]]\!] \neq *$ and $[\![\chi[\varphi]]\!] \neq *$ hold,
as required by the definition of disjunction.

The property $\chi[\varphi \rightarrow \psi] \equiv \chi[(\neg\varphi) \vee \psi]$ follows directly from the definitions of implication and disjunction. □

B Peters (1979)

Peters (1979) introduced a modal semantics based on the KP tables, and proved that it describes the same admittance relation that can be obtained by a modal interpretation of the rules in (Karttunen, 1974). This admittance relation was later used as the basis for the HS semantics developed in Heim (1983). Thus, the property that Peters proved is essentially the same property proved in theorem 1 of section 3. To help observing that, table 1 summarizes the main differences between in the statement of Peters' result and theorem 1.

	Peters (1979)	**Theorem 1**
propositional language	fragment with sentential conjunction, disjunction and implication	L_3
proposition type	pair of sets of possible worlds $\langle [\![S]\!]_{\mathrm{T}}, [\![S]\!]_{\mathrm{F}} \rangle$ (p.302)	equivalently – function $[\![\cdot]\!]^{KP}_{W}$ from possible worlds to $\{0, 1, *\}$ (def. 9)
propositional semantics	KP (footnotes 3 and 4)	KP
treatment of context change	admittance and update defined model-theoretically (theorem, p.311)	admittance and update defined in KP-based semantics of L_{CCP} (def. 9)
admittance	set of possible worlds Γ is subset of $[\![S]\!]_{\mathrm{T}} \cup [\![S]\!]_{\mathrm{F}}$	similar
update	intersection of sets of possible worlds	similar
negation	the denotation of S's negation is defined indirectly by $[\![S]\!]_{\mathrm{F}}$	part of L_3 (and L_{CCP})

Table 1: differences between Peters (1979) and Theorem 1

Note that contrary to Peters's claim, his purely denotational semantics of logical forms does not fully model Karttunen's (1974) proposal, which uses entailments between logical forms to avoid the proviso problem.

C Proof of theorem 1

Theorem 1 *Let $[\![\cdot]\!]^{HS}_W$ be an HS interpretation of L_{CCP} for some $W \neq \varnothing$. For any $i \in W$, let $[\![\cdot]\!]^{bi}_i$ be the bivalent interpretation of L_2 s.t. for any $\alpha \in L_2$: $[\![\alpha]\!]^{bi}_i = 1$ iff $i \in [\![\alpha]\!]^{HS}$. Let $[\![\cdot]\!]^{KP}_W$ be the KP-based interpretation of L_{CCP} relative to the family $F = [\![\cdot]\!]^{bi}_i |_{i \in W}$. Then for any $\chi \in L_{CCP}$ we have:*

$[\![\chi]\!]^{HS}_W = [\![\chi]\!]^{KP}_W$.

Proof. Let L'_3 be the closure of formulas in $L_2 \times L_2$ under negation and conjunction. The claim of theorem 1 is proved for the subset of L_{CCP} corresponding to L'_3, which is referred to as 'L'_{CCP}' and defined by:

$$L'_{CCP} \stackrel{def}{=} L_2 \cup \{\chi[\kappa] : \chi \in L'_{CCP} \text{ and } \kappa \in L'_3\}$$

Theorem 1 follows directly from this proof, together with the standard facts 1 and 3 on disjunction and implication under the KP and HS interpretations.

As a structural induction relation over L'_{CCP}, we introduce a *precedence* relation PRE, which is defined as follows:

For any L'_{CCP} formula $\alpha \in L_2$:

$$\text{PRE}(\alpha) = \varnothing$$

For complex L'_{CCP} formulas of the form $\mu[\kappa]$ where $\mu \in L'_{CCP}$ and $\kappa \in L'_3$:

$$\begin{array}{lcll} \text{PRE}(\mu[(\kappa_1{:}\kappa_2)]) & = & \{\mu, \kappa_1, \kappa_2\} & \text{where } \kappa_1, \kappa_2 \in L_2 \\ \text{PRE}(\mu[\neg\varphi]) & = & \{\mu, \mu[\varphi]\} & \text{where } \varphi \in L'_3 \\ \text{PRE}(\mu[\varphi \wedge \psi]) & = & \{(\mu[\varphi])[\psi]\} & \text{where } \varphi, \psi \in L'_3 \end{array}$$

Thus, for any formula $\chi \in L'_{CCP}$, the set $\text{PRE}(\chi)$ includes those formulas on whose interpretation the definition of $[\![\chi]\!]^{HS}$ immediately rests.

The proof of the identity $[\![\chi]\!]^{HS}_W = [\![\chi]\!]^{KP}_W$ is by induction on the structure of χ according to the PRE relation.

<u>If $\text{PRE}(\chi) = \varnothing$</u>, then χ is by definition a *simple* CCP formula $\alpha \in L_2 \subset L_{CCP}$, and we have $[\![\alpha]\!]^{HS}_W = [\![\alpha]\!]^{KP}_W$ by construction.

<u>If $\text{PRE}(\chi) \neq \varnothing$</u>, then we assume by induction $[\![\chi']\!]^{HS}_W = [\![\chi']\!]^{KP}_W$ all $\chi' \in \text{PRE}(\chi)$, and prove $[\![\chi]\!]^{HS}_W = [\![\chi]\!]^{KP}_W$ by induction on the structure of χ, as follows.

<u>If $\chi = \mu[(\kappa_1{:}\kappa_2)]$</u>:

By definition of transplication, we have for any bivalent interpretation $[\![\cdot]\!]^{bi}$:

$[\![(\kappa_1{:}\kappa_2)]\!]^{KP} = 1$ iff $[\![\kappa_1]\!]^{bi} = [\![\kappa_2]\!]^{bi} = 1$;

$[\![(\kappa_1{:}\kappa_2)]\!]^{KP} \neq *$ iff $[\![\kappa_1]\!]^{bi} = 1$.

By substituting these identities in the definition of $[\![\mu[(\kappa_1 : \kappa_2)]]\!]^{KP}_W$, we get:

$[\![\mu[(\kappa_1 : \kappa_2)]]\!]^{KP}_W$

$$= \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\kappa_1]\!]^{bi}_i = [\![\kappa_2]\!]^{bi}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\kappa_1]\!]^{bi}_i = 1\} \\ * & \text{otherwise} \end{cases}$$

By the condition in this piecewise definition, in the first clause we have $[\![\kappa_1]\!]^{bi}_i = 1$ for any $i \in [\![\mu]\!]^{KP}_W$, hence:

$$= \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\kappa_2]\!]^{bi}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\kappa_1]\!]^{bi}_i = 1\} \\ * & \text{otherwise} \end{cases}$$

$$= \begin{cases} [\![\mu]\!]^{KP}_W \cap [\![k_2]\!]^{KP}_W & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq [\![k_1]\!]^{KP}_W \\ * & \text{otherwise} \end{cases}$$

By definition, μ, κ_1 and κ_2 are in $\textsc{pre}(\mu[(\kappa_1 : \kappa_2)])$, hence by induction:

$$= \begin{cases} [\![\mu]\!]^{HS}_W \cap [\![k_2]\!]^{HS}_W & [\![\mu]\!]^{HS}_W \neq * \text{ and } [\![\mu]\!]^{HS}_W \subseteq [\![k_1]\!]^{HS}_W \\ * & \text{otherwise} \end{cases}$$

$$\overset{(def)}{=} [\![\mu[(\kappa_1 : \kappa_2)]]\!]^{HS}_W$$

<u>If $\chi = \mu[\neg\varphi]$</u>:

$[\![\mu[\neg\varphi]]\!]^{KP}_W$

$$\overset{(def)}{=} \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\neg\varphi]\!]^{KP}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\neg\varphi]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

$$\overset{(def)}{=} \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\varphi]\!]^{KP}_i = 0\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\varphi]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

By the piecewise definition:[2]

$$= \begin{cases} [\![\mu]\!]^{KP}_W \setminus \{i \in W : [\![\varphi]\!]^{KP}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\varphi]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

By definition of KP-based interpretations, $[\![\mu[\varphi]]\!]^{KP}_W = [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\varphi]\!]^{KP}_i = 1\}$ whenever $[\![\mu]\!]^{KP}_W \neq *$ and $[\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\varphi]\!]^{KP}_i \neq *\}$. Thus:

$$[\![\mu[\neg\varphi]]\!]^{KP}_W$$

$$\stackrel{(def)}{=} \begin{cases} [\![\mu]\!]^{KP}_W \smallsetminus [\![\mu[\varphi]]\!]^{KP}_W & [\![\mu]\!]^{KP}_W \neq * \text{ and } [\![\mu[\varphi]]\!]^{KP}_W \neq * \\ * & \text{otherwise} \end{cases}$$

By definition, μ and φ are in $\text{PRE}(\mu[\neg\varphi])$, hence by induction:

$$[\![\mu[\neg\varphi]]\!]^{KP}_W$$

$$= \begin{cases} [\![\mu]\!]^{HS}_W \smallsetminus [\![\mu[\varphi]]\!]^{HS}_W & [\![\mu]\!]^{HS}_W \neq * \text{ and } [\![\mu[\varphi]]\!]^{HS}_W \neq * \\ * & \text{otherwise} \end{cases}$$

$$\stackrel{(def)}{=} [\![\mu[\neg\varphi]]\!]^{HS}_W$$

<u>If $\chi = \mu[\varphi \wedge \psi]$:</u>

$$[\![\mu[\varphi \wedge \psi]]\!]^{KP}_W$$

$$\stackrel{(def)}{=} \begin{cases} [\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\varphi \wedge \psi]\!]^{KP}_i = 1\} & [\![\mu]\!]^{KP}_W \neq * \text{ and} \\ & \quad [\![\mu]\!]^{KP}_W \subseteq \{i \in W : [\![\varphi \wedge \psi]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

By definition of KP interpretations of L_3:

$$= \begin{cases} [\![\mu]\!]^{KP}_W \cap & [\![\mu]\!]^{KP}_W \neq * \text{ and} \\ \{i \in W : [\![\varphi]\!]^{KP}_i = [\![\psi]\!]^{KP}_i = 1\} & [\![\mu]\!]^{KP}_W \subseteq \{ i \in W : [\![\varphi]\!]^{KP}_i \neq * \text{ and} \\ & \qquad ([\![\varphi]\!]^{KP}_i = 0 \text{ or } [\![\psi]\!]^{KP}_i \neq *) \} \\ * & \text{otherwise} \end{cases}$$

We now denote:

$A^1 = \{i \in W : [\![\varphi]\!]^{KP}_i = 1\}$ $\qquad$ $B^1 = \{i \in W : [\![\psi]\!]^{KP}_i = 1\}$

$A^0 = \{i \in W : [\![\varphi]\!]^{KP}_i = 0\}$ $\qquad$ $B^0 = \{i \in W : [\![\psi]\!]^{KP}_i = 0\}$

$A^{0,1} = A^0 \dot\cup A^1 = \{i \in W : [\![\varphi]\!]^{KP}_i \neq *\}$ $\qquad$ $B^{0,1} = B^0 \dot\cup B^1 = \{i \in W : [\![\psi]\!]^{KP}_i \neq *\}$

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[2]To see that, we can denote $A = [\![\mu]\!]^{KP}_W$, $B_0 = \{i \in W : [\![\varphi]\!]^{KP}_i = 0\}$, $B_1 = \{i \in W : [\![\varphi]\!]^{KP}_i = 1\}$ and $C = \{i \in W : [\![\varphi]\!]^{KP}_i \neq *\}$. By definition $B_0 = C \smallsetminus B_1$, and by the piecewise definition $A \subseteq C$, hence $A \cap B_0 = A \smallsetminus B_1$, or $[\![\mu]\!]^{KP}_W \cap \{i \in W : [\![\varphi]\!]^{KP}_i = 0\} = [\![\mu]\!]^{KP}_W \smallsetminus \{i \in W : [\![\varphi]\!]^{KP}_i = 1\}$.

Thus, we have:

$$[\![\mu[\varphi \wedge \psi]]\!]_W^{KP} = \begin{cases} [\![\mu]\!]_W^{KP} \cap A^1 \cap B^1 & [\![\mu]\!]_W^{KP} \neq * \text{ and } [\![\mu]\!]_W^{KP} \subseteq A^{0,1} \cap (A^0 \cup B^{0,1}) \\ * & \text{otherwise} \end{cases} \qquad (i)$$

On the other hand, by definition of $[\![\cdot]\!]_W^{KP}$, we have:

$$[\![(\mu[\varphi])[\psi]]\!]_W^{KP} = \begin{cases} [\![(\mu[\varphi])]\!]_W^{KP} \cap \{i \in W : [\![\psi]\!]_i^{KP} = 1\} & [\![(\mu[\varphi])]\!]_W^{KP} \neq * \text{ and } \\ & \quad [\![(\mu[\varphi])]\!]_W^{KP} \subseteq \{i \in W : [\![\psi]\!]_i^{KP} \neq *\} \\ * & \text{otherwise} \end{cases}$$

$$= \begin{cases} [\![(\mu[\varphi])]\!]_W^{KP} \cap B^1 & [\![(\mu[\varphi])]\!]_W^{KP} \neq * \text{ and } [\![(\mu[\varphi])]\!]_W^{KP} \subseteq B^{0,1} \\ * & \text{otherwise} \end{cases} \qquad (ii)$$

And by definition of $[\![\cdot]\!]_W^{KP}$, we similarly have:

$$[\![\mu[\varphi]]\!]_W^{KP} = \begin{cases} [\![\mu]\!]_W^{KP} \cap A^1 & [\![\mu]\!]_W^{KP} \neq * \text{ and } [\![\mu]\!]_W^{KP} \subseteq A^{0,1} \\ * & \text{otherwise} \end{cases} \qquad (iii)$$

By substituting (iii) into (ii), we get:

$$[\![(\mu[\varphi])[\psi]]\!]_W^{KP} = \begin{cases} [\![\mu]\!]_W^{KP} \cap A^1 \cap B^1 & [\![\mu]\!]_W^{KP} \neq * \text{ and } [\![\mu]\!]_W^{KP} \subseteq A^{0,1} \text{ and } [\![\mu]\!]_W^{KP} \cap A^1 \subseteq B^{0,1} \\ * & \text{otherwise} \end{cases}$$

And since $A^{0,1} = A^0 \uplus A^1$, we have:

$$([\![\mu]\!]_W^{KP} \subseteq A^{0,1} \text{ and } [\![\mu]\!]_W^{KP} \cap A^1 \subseteq B^{0,1}) \text{ iff } [\![\mu]\!]_W^{KP} \subseteq A^{0,1} \cap (A^0 \cup B^{0,1})$$

Thus, we get:

$$[\![(\mu[\varphi])[\psi]]\!]_W^{KP} = \begin{cases} [\![\mu]\!]_W^{KP} \cap A^1 \cap B^1 & [\![\mu]\!]_W^{KP} \neq * \text{ and } [\![\mu]\!]_W^{KP} \subseteq A^{0,1} \cap (A^0 \cup B^{0,1}) \\ * & \text{otherwise} \end{cases}$$

This, together with (i), leads to the conclusion:

$$[\![\mu[\varphi \wedge \psi]]\!]_W^{KP} = [\![(\mu[\varphi])[\psi]]\!]_W^{KP}$$

But $(\mu[\varphi])[\psi] \in \text{PRE}(\mu[\varphi \wedge \psi])$, hence by induction $[\![(\mu[\varphi])[\psi]]\!]_W^{KP} = [\![(\mu[\varphi])[\psi]]\!]_W^{HS}$.

Thus, we conclude:

$$[\![\mu[\varphi \wedge \psi]]\!]_W^{KP} = [\![(\mu[\varphi])[\psi]]\!]_W^{HS}$$

And by definition of HS interpretations:

$[\![(\mu[\varphi])[\psi]]\!]^{HS}_{W} \quad = \quad [\![\mu[\varphi \wedge \psi]]\!]^{HS}_{W}$

Thus, we conclude:

$[\![\mu[\varphi \wedge \psi]]\!]^{KP}_{W} \quad = \quad [\![\mu[\varphi \wedge \psi]]\!]^{HS}_{W}$, as required.

□

D Proof of corollary 1

Corollary 1 *For any* $\chi \in L_{CCP}$ *and* $\kappa \in L_3$*:* $\chi[\kappa] \overset{HS}{\equiv} \chi[KP(\kappa)]$.

Proof. Let $[\![\cdot]\!]^{HS}_W$ be an HS interpretation of L_{CCP} for some $W \neq \varnothing$. For any $i \in W$, let $[\![\cdot]\!]^{bi}_i$ be the bivalent interpretation of L_2 s.t. for any $\alpha \in L_2$: $[\![\alpha]\!]^{bi}_i = 1$ iff $i \in [\![\alpha]\!]^{HS}$. Let $[\![\cdot]\!]^{KP}_W$ be the KP-based interpretation of L_{CCP} relative to the family $F = [\![\cdot]\!]^{bi}_i|_{i \in W}$.

By definition of $[\![\cdot]\!]^{KP}_W$:

$$[\![\chi[\kappa]]\!]^{KP}_W = \begin{cases} [\![\chi]\!]^{KP}_W \cap \{i \in W : [\![\kappa]\!]^{KP}_i = 1\} & [\![\chi]\!]^{KP}_W \neq * \text{ and } [\![\chi]\!]^{KP}_W \subseteq \{i \in W : [\![\kappa]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

By soundness of the KP calculus:

$$= \begin{cases} [\![\chi]\!]^{KP}_W \cap \{i \in W : [\![KP(\kappa)]\!]^{KP}_i = 1\} & [\![\chi]\!]^{KP}_W \neq * \text{ and } \\ & \quad [\![\chi]\!]^{KP}_W \subseteq \{i \in W : [\![KP(\kappa)]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

On the other hand, by definition of $[\![\cdot]\!]^{KP}_W$, we also have:

$$[\![\chi[KP(\kappa)]]\!]^{KP}_W = \begin{cases} [\![\chi]\!]^{KP}_W \cap \{i \in W : [\![KP(\kappa)]\!]^{KP}_i = 1\} & [\![\chi]\!]^{KP}_W \neq * \text{ and } \\ & \quad [\![\chi]\!]^{KP}_W \subseteq \{i \in W : [\![KP(\kappa)]\!]^{KP}_i \neq *\} \\ * & \text{otherwise} \end{cases}$$

Thus:

$$[\![\chi[\kappa]]\!]^{KP}_W = [\![\chi[KP(\kappa)]]\!]^{KP}_W$$

By theorem 1:

$$[\![\chi[\kappa]]\!]^{HS}_W = [\![\chi[\kappa]]\!]^{KP}_W \text{ and } [\![\chi[KP(\kappa)]]\!]^{HS}_W = [\![\chi[KP(\kappa)]]\!]^{KP}_W$$

We conclude:

$$[\![\chi[\kappa]]\!]^{HS}_W = [\![\chi[KP(\kappa)]]\!]^{HS}_W.$$

□

E Proof of corollary 2

Corollary 2 *For any formula $\chi \in L_{CCP}$ and HS interpretation:*
$[[HS(\chi)]]^{HS} = [[\chi]]^{HS}$.

Proof. Let χ be any formula in L_{CCP}.

If $\chi = C \in L_2$:

By definition of the *HS* calculus: $HS(C) = C$, hence $[[HS(C)]]^{HS} = [[C]]^{HS}$.

Otherwise, $\chi = \mu[\kappa]$, where $\mu \in L_{CCP}$ and $\kappa \in L_3$:

We need to show $[[HS(\mu[\kappa])]]^{HS} = [[\mu[\kappa]]]^{HS}$ for any $\mu \in L_{CCP}$ and $\kappa \in L_3$. We do that by induction on the structure of μ. The predecessor function for the induction is defined – more straightforwardly than in Theorem 2 – by:

$$\text{PRE}(\mu) = \begin{cases} \varnothing & \mu = C \in L_2 \\ \{\mu'\} & \mu = \mu'[\varphi] \text{ where } \mu' \in L_{CCP} \text{ and } \varphi \in L_3 \end{cases}$$

Base: $\mu = C \in L_2$.

By definition of the HS calculus: $HS(C[\kappa]) = C[KP(\kappa)]$.

By Corollary 1: $[[C[KP(\kappa)]]]^{HS} = [[C[\kappa]]]^{HS}$.

Thus: $[[HS(C[\kappa])]]^{HS} = [[C[\kappa]]]^{HS}$.

Induction hypothesis: For $\mu = \mu'[\varphi]$, where $\mu' \in L_{CCP}$ and $\varphi \in L_3$ we assume by induction:

$[[HS(\mu'[\kappa'])]]^{HS} = [[\mu'[\kappa']]]^{HS}$ for any $\kappa' \in L_3$.

In particular: $[[HS(\mu'[\varphi \wedge \kappa])]]^{HS} = [[\mu'[\varphi \wedge \kappa]]]^{HS}$ $\qquad (i)$

Induction: We need to show $[[HS((\mu'[\varphi])[\kappa])]]^{HS} = [[(\mu'[\varphi])[\kappa]]]^{HS}$.

This follows directly from (i) and the definitions of the HS calculus *(cal)* and HS interpretations *(int)*:

$$[[HS((\mu'[\varphi])[\kappa])]]^{HS} \overset{(cal)}{=} [[HS(\mu'[\varphi \wedge \kappa])]]^{HS} \overset{(i)}{=} [[\mu'[\varphi \wedge \kappa]]]^{HS} \overset{(int)}{=} [[(\mu'[\varphi])[\kappa]]]^{HS}.$$

□

F Proof of corollary 3

Corollary 3 *For any two formulas $\varphi, \psi \in L_3$:*

(i) $\varphi \overset{KP}{\equiv} \psi$ *iff* $\varphi \overset{HS}{\equiv} \psi$ *(ii)* $\varphi \overset{KP}{\Rightarrow} \psi$ *iff* $\varphi \overset{HS}{\Rightarrow} \psi$

Proof. For any $\varphi, \psi \in L_3$, we denote $KP(\varphi) = (\varphi_1 : \varphi_2)$ and $KP(\psi) = (\psi_1 : \psi_2)$.

First, we show that $\varphi \overset{KP}{\equiv} \psi$ iff $\varphi \overset{HS}{\equiv} \psi$:

For all $\varphi, \psi \in L_3$, by soundness of KP rewriting:

$\varphi \overset{KP}{\equiv} \psi$

iff $(\varphi_1 : \varphi_2) \overset{KP}{\equiv} (\psi_1 : \psi_2)$

By definition of transplication, where '$\overset{bi}{\equiv}$' is bivalent equivalences:

iff $\varphi_1 \overset{bi}{\equiv} \psi_1$ and $\varphi_1 \wedge \varphi_2 \overset{bi}{\equiv} \psi_1 \wedge \psi_2$

By definition of HS semantics:

iff for any $[\![\cdot]\!]^{HS}$: $[\![\varphi_1]\!] = [\![\psi_1]\!]$ and $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\psi_1 \wedge \psi_2]\!]$

iff for any $[\![\cdot]\!]^{HS}$: $[\![\varphi_1]\!] = [\![\psi_1]\!]$ and $[\![\varphi_1]\!] \cap [\![\varphi_2]\!] = [\![\psi_1]\!] \cap [\![\psi_2]\!]$

iff for any $[\![\cdot]\!]^{HS}_W$, for any $A \subseteq W$:

$A \subseteq [\![\varphi_1]\!]$ iff $A \subseteq [\![\psi_1]\!]$ and $A \cap [\![\varphi_1]\!] \cap [\![\varphi_2]\!] = A \cap [\![\psi_1]\!] \cap [\![\psi_2]\!]$

iff for any $[\![\cdot]\!]^{HS}$, for any $C \in L_2$:

$[\![C]\!] \subseteq [\![\varphi_1]\!]$ iff $[\![C]\!] \subseteq [\![\psi_1]\!]$ and $[\![C]\!] \cap [\![\varphi_1]\!] \cap [\![\varphi_2]\!] = [\![C]\!] \cap [\![\psi_1]\!] \cap [\![\psi_2]\!]$

iff for any $[\![\cdot]\!]^{HS}$, for any $C \in L_2$:

$[\![C[(\varphi_1 : \varphi_2)]]\!] = [\![C[(\psi_1 : \psi_2)]]\!]$, since by definition:

$$[\![C[(\varphi_1 : \varphi_2)]]\!] \overset{(def)}{=} \begin{cases} [\![C]\!] \cap [\![\varphi_2]\!] & [\![C]\!] \subseteq [\![\varphi_1]\!] \\ * & \text{otherwise} \end{cases}$$

$$[\![C[(\psi_1 : \psi_2)]]\!] \overset{(def)}{=} \begin{cases} [\![C]\!] \cap [\![\psi_2]\!] & [\![C]\!] \subseteq [\![\psi_1]\!] \\ * & \text{otherwise} \end{cases}$$

By definition of HS equivalence:

iff for any $C \in L_2$: $C[(\varphi_1 : \varphi_2)] \overset{HS}{\equiv} C[(\psi_1 : \psi_2)]$

By our notation:

iff for any $C \in L_2$: $C[KP(\varphi)] \overset{HS}{\equiv} C[KP(\psi)]$

By definition of the HC calculus:

iff for any $C \in L_2$: $HS(C[\varphi]) \overset{HS}{\equiv} HS(C[\psi])$

We have established:

for all $\varphi, \psi \in L_3$: $\varphi \overset{KP}{\equiv} \psi$ *iff* for every $C \in L_2$: $HS(C[\varphi]) \overset{HS}{\equiv} HS(C[\psi])$ *(i)*

We need to show:

for all $\varphi, \psi \in L_3$: $\varphi \overset{KP}{\equiv} \psi$ *iff* for every $\chi \in L_{CCP}$: $HS(\chi[\varphi]) \overset{HS}{\equiv} HS(\chi[\psi])$ *(ii)*

The "*if*" direction of *(ii)* follows from *(i)*, thus it is left to show.

for all $\varphi, \psi \in L_3$: $\varphi \overset{KP}{\equiv} \psi$ *only if* for every $\chi \in L_{CCP}$: $HS(\chi[\varphi]) \overset{HS}{\equiv} HS(\chi[\psi])$

Or:

for every $\chi \in L_{CCP}$: for all $\varphi, \psi \in L_3$: $HS(\chi[\varphi]) \overset{HS}{\equiv} HS(\chi[\psi])$ *if* $\varphi \overset{KP}{\equiv} \psi$ *(iii)*

We show (iii) by induction on the structure of χ, for any $\chi \in L'_{CCP}$:

- For any $\chi = C \in L_2 \subset L'_{CCP}$, we have by (i):

  for all $\varphi, \psi \in L_3$: $HS(C[\varphi]) \overset{HS}{\equiv} HS(C[\psi])$ *if* $\varphi \overset{KP}{\equiv} \psi$

- For any $\chi \in L'_{CCP} \setminus L_2$ we denote $\chi = \mu[\kappa]$ where $\mu \in L'_{CCP}$ and $\kappa \in L'_3$.

  By induction on χ's structure:

  for all $\varphi', \psi' \in L_3$: $HS(\mu[\varphi']) \overset{HS}{\equiv} HS(\mu[\psi'])$ *if* $\varphi' \overset{KP}{\equiv} \psi'$

  Thus, in particular:

  for all $\varphi, \psi \in L_3$: $HS(\mu[\kappa \wedge \varphi]) \overset{HS}{\equiv} HS(\mu[\kappa \wedge \psi])$ *if* $\kappa \wedge \varphi \overset{KP}{\equiv} \kappa \wedge \psi$

  Thus:

  for all $\varphi, \psi \in L_3$: $HS(\mu[\kappa \wedge \varphi]) \overset{HS}{\equiv} HS(\mu[\kappa \wedge \psi])$ *if* $\varphi \overset{KP}{\equiv} \psi$

  But by definition of HS semantics:

  $\mu[\kappa \wedge \varphi] \overset{HS}{\equiv} (\mu[\kappa])[\varphi]$ and $\mu[\kappa \wedge \psi] \overset{HS}{\equiv} (\mu[\kappa])[\psi]$.

  And by soundness of HS calculus we conclude:

  $(\mu[\kappa])[\varphi] \overset{HS}{\equiv} (\mu[\kappa])[\psi]$, or: $\chi[\varphi] \overset{HS}{\equiv} \chi[\psi]$.

We have proved property (iii) for any $\chi \in L'_{CCP}$, hence by facts 1 and 3, property (iii) holds of any $\chi \in L_{CCP}$.

Next, we show that $\varphi \overset{KP}{\Rightarrow} \psi$ iff $\varphi \overset{HS}{\Rightarrow} \psi$:

For all $\varphi, \psi \in L_3$, by soundness of KP rewriting:

$\varphi \overset{KP}{\Rightarrow} \psi$

iff $KP(\varphi) \overset{KP}{\Rightarrow} KP(\psi)$, or, using our notation: $(\varphi_1 : \varphi_2) \overset{KP}{\Rightarrow} (\psi_1 : \psi_2)$

iff $\varphi_1 \wedge \varphi_2 \overset{bi}{\Rightarrow} \psi_1 \wedge \psi_2$

iff for every HS interpretation $[[\cdot]]^{HS}_W$: $[[\varphi_1 \wedge \varphi_2]]^{HS}_W \subseteq [[\psi_1 \wedge \psi_2]]^{HS}_W$

We conclude:

$\varphi \overset{KP}{\Rightarrow} \psi$ *iff* for every HS interpretation $[\![\cdot]\!]^{HS}$: $[\![\varphi_1 \wedge \varphi_2]\!]^{HS} \subseteq [\![\psi_1 \wedge \psi_2]\!]^{HS}$ *(iv)*

By definition of HS interpretations, for any $W \neq \varnothing$ and HS interpretation $[\![\cdot]\!]^{HS}_W$:

$[\![\varphi_1 \wedge \varphi_2]\!]^{HS}_W \subseteq [\![\psi_1 \wedge \psi_2]\!]^{HS}_W$

iff $[\![\varphi_1]\!]^{HS}_W \cap [\![\varphi_2]\!]^{HS}_W \subseteq [\![\psi_1]\!]^{HS}_W \cap [\![\psi_2]\!]^{HS}_W$

iff for every $X \subseteq W$: if $X \subseteq [\![\varphi_1]\!]^{HS}_W \cap [\![\varphi_2]\!]^{HS}_W$ then $X \subseteq [\![\psi_1]\!]^{HS}_W \cap [\![\psi_2]\!]^{HS}_W$

iff for every $X \subseteq W$: if $X \subseteq [\![\varphi_1]\!]^{HS}_W$ and $X \subseteq [\![\varphi_2]\!]^{HS}_W$
then $X \subseteq [\![\psi_1]\!]^{HS}_W$ and $X \subseteq [\![\psi_2]\!]^{HS}_W$

iff for every $X \subseteq W$: if $X \subseteq [\![\varphi_1]\!]^{HS}_W$ and $X \cap [\![\varphi_2]\!]^{HS}_W = X$
then $X \subseteq [\![\psi_1]\!]^{HS}_W$ and $X \cap [\![\psi_2]\!]^{HS}_W = X$

iff for every $C \in L_2$: if $[\![C]\!]^{HS} \subseteq [\![\varphi_1]\!]^{HS}$ and $[\![C]\!]^{HS} \cap [\![\varphi_2]\!]^{HS} = [\![C]\!]^{HS}$
then $[\![C]\!]^{HS} \subseteq [\![\psi_1]\!]^{HS}$ and $[\![C]\!]^{HS} \cap [\![\psi_2]\!]^{HS} = [\![C]\!]^{HS}$

By definition of HS interpretations:

iff for every $C \in L_2$: if $[\![C[(\varphi_1 : \varphi_2)]]\!]^{HS} = [\![C]\!]^{HS}$ then $[\![C[(\psi_1 : \psi_2)]]\!]^{HS} = [\![C]\!]^{HS}$

By our notation:

iff for every $C \in L_2$: if $[\![C[KP(\varphi)]]\!]^{HS} = [\![C]\!]^{HS}$ then $[\![C[KP(\psi)]]\!]^{HS} = [\![C]\!]^{HS}$

By definition of HS calculus:

iff for every $C \in L_2$: if $[\![HS(C[\varphi])]\!]^{HS} = [\![C]\!]^{HS}$ then $[\![HS(C[\psi])]\!]^{HS} = [\![C]\!]^{HS}$

By soundness of HS calculus:

iff for every $C \in L_2$: if $[\![C[\varphi]]\!]^{HS} = [\![C]\!]^{HS}$ then $[\![C[\psi]]\!]^{HS} = [\![C]\!]^{HS}$

By definition of HS interpretations:

iff for every $\chi \in L_{CCP}$: if $[\![\chi[\varphi]]\!]^{HS} = [\![\chi]\!]^{HS}$ then $[\![\chi[\psi]]\!]^{HS} = [\![\chi]\!]^{HS}$

We conclude that for any HS interpretation $[\![\cdot]\!]^{HS}$:

$[\![\varphi_1 \wedge \varphi_2]\!]^{HS} \subseteq [\![\psi_1 \wedge \psi_2]\!]^{HS}$ holds

iff for every $\chi \in L_{CCP}$: if $[\![\chi[\varphi]]\!]^{HS} = [\![\chi]\!]^{HS}$ then $[\![\chi[\psi]]\!]^{HS} = [\![\chi]\!]^{HS}$

From this biequivalence and (iv) we conclude:

$\varphi \overset{KP}{\Rightarrow} \psi$

iff every HS interpretation $[\![\cdot]\!]^{HS}$ and for every $\chi \in L_{CCP}$:
if $[\![\chi[\varphi]]\!]^{HS} = [\![\chi]\!]^{HS}$ then $[\![\chi[\psi]]\!]^{HS} = [\![\chi]\!]^{HS}$

iff $\varphi \overset{HS}{\Rightarrow} \psi$ □

G Proof of theorem 2

Theorem 2 *For $\kappa \in L_3$, let C be a weakest formula in L_2 that KP-admits κ, and let β be a strongest KP-presupposition of κ in L_2. Then $C \equiv \beta \equiv \alpha_\kappa^{KP}$.*

Proof. By definition of KP-admittance:

$C \in L_2$ *KP-admits* κ

iff $\alpha^{KP}_{(\top : C) \wedge \kappa} \equiv \top$

iff $KP((\top : C) \wedge \kappa) \overset{KP}{\equiv} (\top : \beta)$ for some $\beta \in L_2$

By definition of conjunction in KP calculus:

iff $(\alpha_\kappa^{KP} \vee \neg C : C \wedge \beta_\kappa^{KP}) \overset{KP}{\equiv} (\top : \beta)$ for some $\beta \in L_2$

iff $[\![(\alpha_\kappa^{KP} \vee \neg C : C \wedge \beta_\kappa^{KP})]\!]^{KP} \neq *$ for every KP interpretation

iff $[\![\alpha_\kappa^{KP} \vee \neg C]\!]^{bi} = 1$ for every bivalent interpretation

iff $C \overset{bi}{\Rightarrow} \alpha_\kappa^{KP}$

We conclude:

$C \in L_2$ KP-admits κ *iff* $C \overset{bi}{\Rightarrow} \alpha_\kappa^{KP}$.

Therefore:

$C \in L_2$ is a weakest formula in L_2 that KP-admits κ *iff* $C \overset{bi}{\equiv} \alpha_\kappa^{KP}$ *(i)*

By definition of KP-presupposition:

κ KP-presupposes $\beta \in L_2$

iff $\kappa \overset{KP}{\Rightarrow} (\top : \beta)$ and $\neg\kappa \overset{KP}{\Rightarrow} (\top : \beta)$

By soundness of KP calculus:

iff $(\alpha_\kappa^{KP} : \beta_\kappa^{KP}) \overset{KP}{\Rightarrow} (\top : \beta)$ and $(\alpha_\kappa^{KP} : \neg\beta_\kappa^{KP}) \overset{KP}{\Rightarrow} (\top : \beta)$

By definition of KP-entailment and transplication:

iff $\alpha_\kappa^{KP} \wedge \beta_\kappa^{KP} \overset{bi}{\Rightarrow} \beta$ and $\alpha_\kappa^{KP} \wedge \neg\beta_\kappa^{KP} \overset{bi}{\Rightarrow} \beta$

iff $\alpha_\kappa^{KP} \overset{bi}{\Rightarrow} \beta$.

We conclude:

κ KP-presupposes $\beta \in L_2$ *iff* $\alpha_\kappa^{KP} \overset{bi}{\Rightarrow} \beta$

Therefore:

$\beta \in L_2$ is a strongest KP-presupposition of κ iff $\beta \overset{bi}{\equiv} \alpha_\kappa^{KP}$ *(ii)*

Theorem 2 follows from (i) and (ii). □

H Proof of fact 6

Fact 6 $\varphi \vee \psi \overset{K}{\equiv} \neg((\neg\varphi) \wedge \neg\psi) \qquad \varphi \rightarrow \psi \overset{K}{\equiv} (\neg\varphi) \vee \psi$

Proof. For any $\kappa \in L_3$ and $C \in L_2$, we denote:

$$K(C[\kappa]) = (K^1(C[\kappa]) \colon K^2(C[\kappa]))$$

Now we note, by definition of K-calculus:

$$\begin{aligned} & K^1(C[\neg((\neg\varphi) \wedge \neg\psi)]) \\ &\quad = K^1(C[(\neg\varphi) \wedge \neg\psi]) \\ &\quad = K^1(C[\neg\varphi]) \wedge K^1((C \wedge K^1(C[\neg\varphi]) \wedge K^2(C[\neg\varphi]))[\neg\psi]) \\ &\quad = K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \neg K^2(C[\varphi]))[\neg\psi]) \\ &\quad = K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \neg K^2(C[\varphi]))[\psi]) \\ &\quad = K^1(C[\varphi \vee \psi]) \end{aligned}$$

For all $\kappa \in L_3$ and $C \in L_2$ we have:

- By definition:
  $K^2(C[\neg\varphi]) = \neg K^2(C[\varphi])$, if $\kappa = \neg\varphi$
- By simple induction on the structure of κ:
  $K^2(C[\varphi \,\mathsf{op}\, \psi]) = K^2(C[\varphi]) \,\mathsf{op}\, K^2(C[\psi])$, if $\kappa = \varphi \,\mathsf{op}\, \psi$

Thus, in particular:

$$\begin{aligned} & K^2(C[\neg((\neg\varphi) \wedge \neg\psi)]) \\ &\quad = \neg((\neg K^2(C[\varphi])) \wedge \neg K^2(C[\psi])) \\ &\quad \overset{bi}{\equiv} K^2(C[\varphi]) \vee K^2(C[\psi]) \\ &\quad = K^2(C[\varphi \vee \psi]) \end{aligned}$$

We conclude:

$$\begin{aligned} & K(\neg((\neg\varphi) \wedge \neg\psi)) \\ &\quad = K(\top[\neg((\neg\varphi) \wedge \neg\psi)]) \\ &\quad \equiv K(\top[\varphi \vee \psi]) \\ &\quad = K(\varphi \vee \psi) \end{aligned}$$

The proof of the equivalence $\varphi \rightarrow \psi \overset{K}{\equiv} (\neg\varphi) \vee \psi$ is using similar considerations.

□

I Proof of theorem 3

Theorem 3 *For any $\kappa \in L_3$, let C be a weakest formula in L_2 that K-admits κ, and let α be a strongest K-presupposition of κ in L_2. Then we have:*

$$\alpha \equiv \alpha_\kappa^K,\ \alpha_\kappa^K \Rightarrow C, \text{ and } C \equiv \alpha_\kappa^{KP}.$$

Proof. First, we show that $\alpha \equiv \alpha_\kappa^K$:

By definition of K-presupposition we have:

$\alpha \in L_2$ is a K-presupposition of κ

iff $\kappa \overset{K}{\Rightarrow} (\top{:}\alpha)$ and $\neg\kappa \overset{K}{\Rightarrow} (\top{:}\alpha)$

iff $K(\kappa) \overset{T}{\Rightarrow} K((\top{:}\alpha))$ and $K(\neg\kappa) \overset{T}{\Rightarrow} K((\top{:}\alpha))$

(see def. 17 and def. 11, where T=class of interpretations of $L_2 \times L_2$)

iff $(\alpha_\kappa^K{:}\beta_\kappa^K) \overset{T}{\Rightarrow} (\top{:}\alpha)$ and $(\alpha_\kappa^K{:}\neg\beta_\kappa^K) \overset{T}{\Rightarrow} (\top{:}\alpha)$

By definition of transplication and T-entailment:

iff $\alpha_\kappa^K \wedge \beta_\kappa^K \overset{bi}{\Rightarrow} \top \wedge \alpha$ and $\alpha_\kappa^K \wedge \neg\beta_\kappa^K \overset{bi}{\Rightarrow} \top \wedge \alpha$

iff $\alpha_\kappa^K \overset{bi}{\Rightarrow} \alpha$

Thus, if $\alpha \in L_2$ is a strongest presupposition of κ then $\alpha \overset{bi}{\equiv} \alpha_\kappa^K$.

Next, we show that $C \equiv \alpha_\kappa^{KP}$:

By definition of K-admittance:

C K-admits κ

iff $\alpha^K_{(\top\,:\,C)\,\wedge\,\kappa} \equiv \top$

iff there is $\gamma \in L_2$ s.t. $K((\top{:}C) \wedge \kappa) \equiv (\top{:}\gamma)$

By definition of K-calculus:

iff there is $\gamma \in L_2$ s.t. $WK((\top{:}C) \wedge K(C[\kappa])) \equiv (\top{:}\gamma)$

iff there is $\delta \in L_2$ s.t. $K(C[\kappa]) \equiv (\top{:}\delta)$ *(i)*

We will now show:

for any $\delta \in L_2$: $K(C[\kappa]) \equiv (\top : \delta)$ *iff* $K(C[\mathit{KP}(\kappa)]) \equiv (\top : \delta)$ *(ii)*

We prove (ii) inductively on the structure of κ:

$\underline{\kappa = (\kappa_1 : \kappa_2)}$:

By definition of KP calculus:

$$K(C[\kappa]) = K(C[(\kappa_1 : \kappa_2)]) \overset{(def)}{=} K(C[\mathit{KP}((\kappa_1 : \kappa_2))]) = K(C[\mathit{KP}(\kappa)])$$

$\underline{\kappa = \neg\varphi}$:

By definition of K-calculus and induction hypothesis (ii) for φ:

$K(C[\neg\varphi]) \equiv (\top : \delta)$

iff $K(C[\varphi]) \equiv (\top : \neg\delta)$ *(def. K-calculus)*

iff $K(C[(\mathit{KP}(\varphi))]) \equiv (\top : \neg\delta)$ *(by (ii), inductively)*

iff $C \Rightarrow \alpha_\varphi^{KP}$ and $\beta_\varphi^{KP} \equiv \neg\delta$ *(def. K-calculus)*

By definition $\alpha_{\neg\varphi}^{KP} = \alpha_\varphi^{KP}$ and $\beta_{\neg\varphi}^{KP} = \neg\beta_\varphi^{KP}$, hence:

iff $C \Rightarrow \alpha_{\neg\varphi}^{KP}$ and $\beta_{\neg\varphi}^{KP} \equiv \delta$

Thus, by definition of K-calculus:

iff $K(C[(\alpha_{\neg\varphi}^{KP} : \beta_{\neg\varphi}^{KP})]) \equiv (\top : \delta)$

iff $K(C[\mathit{KP}(\neg\varphi)]) \equiv (\top : \delta)$

$\underline{\kappa = \varphi \wedge \psi}$:

By definition of K-calculus and WK-calculus, for some $\delta_1, \delta_2 \in L_2$:

$K(C[\varphi \wedge \psi]) \equiv (\top : \delta)$

iff $K(C[\varphi]) \equiv (\top : \delta_1)$ and $K((C \wedge \delta_1)[\psi]) \equiv (\top : \delta_2)$ and $\delta \equiv \delta_1 \wedge \delta_2$

By the induction hypothesis (ii) on φ and ψ:

iff $K(C[\mathit{KP}(\varphi)]) \equiv (\top : \delta_1)$ and $K((C \wedge \delta_1)[\mathit{KP}(\psi)]) \equiv (\top : \delta_2)$ and $\delta \equiv \delta_1 \wedge \delta_2$

By definition of K-calculus:

iff $(C \Rightarrow \alpha_\varphi^{KP}$ and $\beta_\varphi^{KP} \equiv \delta_1)$ and $(C \wedge \delta_1 \Rightarrow \alpha_\psi^{KP}$ and $\beta_\psi^{KP} \equiv \delta_2)$ and $\delta \equiv \delta_1 \wedge \delta_2$

iff $C \Rightarrow \alpha_{\varphi}^{KP}$ and $C \wedge \beta_{\varphi}^{KP} \Rightarrow \alpha_{\psi}^{KP}$ and $\delta \equiv \beta_{\varphi}^{KP} \wedge \beta_{\psi}^{KP}$

iff $C \Rightarrow \alpha_{\varphi}^{KP} \wedge (\neg\beta_{\varphi}^{KP} \vee \alpha_{\psi}^{KP})$ and $\delta \equiv \beta_{\varphi}^{KP} \wedge \beta_{\psi}^{KP}$

By def. of KP calculus $\alpha_{\varphi\wedge\psi}^{KP} = \alpha_{\varphi}^{KP} \wedge (\neg\beta_{\varphi}^{KP} \vee \alpha_{\psi}^{KP})$ and $\beta_{\varphi\wedge\psi}^{KP} = \beta_{\varphi}^{KP} \wedge \beta_{\psi}^{KP}$, thus:

iff $C \Rightarrow \alpha_{\varphi\wedge\psi}^{KP}$ and $\delta \equiv \beta_{\varphi\wedge\psi}^{KP}$

Thus, by definition of K-calculus:

iff $K(C[(\alpha_{\varphi\wedge\psi}^{KP} : \beta_{\varphi\wedge\psi}^{KP})]) \equiv (\top : \delta)$

iff $K(C[KP(\varphi \wedge \psi)]) \equiv (\top : \delta)$

This concludes our proof of (ii).

From (i) we conclude:

C K-admits κ

iff there is $\delta \in L_2$ s.t. $K(C[\kappa]) \equiv (\top : \delta)$

And by (ii):

iff there is $\delta \in L_2$ s.t. $K(C[(\alpha_{\kappa}^{KP} : \beta_{\kappa}^{KP})]) \equiv (\top : \delta)$

Thus, by definition of K-calculus:

C K-admits κ *iff* $C \Rightarrow \alpha_{\kappa}^{KP}$.

We conclude:

C is a weakest formula in L_2 that K-admits κ *iff* $C \equiv \alpha_{\kappa}^{KP}$.

<u>Lastly, we show that $\alpha_{\kappa}^{K} \Rightarrow \alpha_{\kappa}^{KP}$:</u>

By lemma 1 below we have:

$K(\alpha_{\kappa}^{K}[\kappa]) \equiv (\top : \beta_{\kappa}^{K})$

Thus, α_{κ}^{K} K-admits κ. And we have shown above that α_{κ}^{KP} is equivalent to any weakest proposition that K-admits κ. Thus, we conclude:

$\alpha_{\kappa}^{K} \Rightarrow \alpha_{\kappa}^{KP}$. □

Lemma 1 *For any* $\kappa \in L_3$*:* $K(\alpha^K_\kappa[\kappa]) \equiv (\top : \beta^K_\kappa)$.

Proof. For any $\kappa \in L_3$ and $C \in L_2$, we denote:

$K(C[\kappa]) = (K^1(C[\kappa]) : K^2(C[\kappa]))$

Thus, by definition of K-calculus:

$\alpha^K_\kappa = K^1(\top[\kappa])$ and $\beta^K_\kappa = K^2(\top[\kappa])$.

Our proof relies on the following facts, for any $\kappa \in L_3$ and $C, C' \in L_2$:

$K^2(C[\kappa]) \equiv \beta^K_\kappa$ *(i)*

$K^1((C \wedge K^1(C[\kappa]))[\kappa]) \equiv \top$ *(ii)*

The proof of (i) is simply inductive on the structure of κ, and is spared here.

From claims (i) and (ii), the proof of Lemma 1 will follow, since:

$$
\begin{aligned}
&K(\alpha^K_\kappa[\kappa]) \\
&\stackrel{(def)}{=} (K^1(\alpha^K_\kappa[\kappa]) : K^2(\alpha^K_\kappa[\kappa])) \\
&\stackrel{(def)}{=} (K^1((K^1(\top[\kappa]))[\kappa]) : K^2(\alpha^K_\kappa[\kappa])) \\
&\stackrel{(i)}{\equiv} (K^1((K^1(\top[\kappa]))[\kappa]) : \beta^K_\kappa) \\
&\equiv (K^1((\top \wedge K^1(\top[\kappa]))[\kappa]) : \beta^K_\kappa) \\
&\stackrel{(ii)}{\equiv} (\top : \beta^K_\kappa)
\end{aligned}
$$

The proof of claim (ii) uses Lemma 2 below, and is by induction on the structure of κ:

$\underline{\kappa = (\kappa_1 : \kappa_2)}$:

If $C \Rightarrow \kappa_1$ then by definition $K^1(C[\kappa]) = K^1(C[(\kappa_1 : \kappa_2)]) = \top$.

Thus: $K^1((C \wedge K^1(C[\kappa]))[\kappa]) = K^1(C[\kappa]) = \top$.

If $C \not\Rightarrow \kappa_1$ then by definition $K^1(C[\kappa]) = K^1(C[(\kappa_1 : \kappa_2)]) = \kappa_1$.

And since $C \wedge \kappa_1 \Rightarrow \kappa_1$:

$K^1((C \wedge K^1(C[\kappa]))[\kappa]) = K^1((C \wedge \kappa_1)[(\kappa_1 : \kappa_2)]) = \top$.

<u>$\kappa = \neg\varphi$:</u>

By definition of K-calculus $K^1(C[\neg\varphi]) = K^1(C[\varphi])$, thus:

$K^1((C \wedge K^1(C[\kappa]))[\kappa]) \ = \ K^1((C \wedge K^1(C[\neg\varphi]))[\neg\varphi])$

$\quad = K^1((C \wedge K^1(C[\varphi]))[\neg\varphi])$

Again, by definition of K-calculus:

$\quad = K^1((C \wedge K^1(C[\varphi]))[\varphi])$

By the induction hypothesis:

$\quad = \top$

<u>$\kappa = \varphi \wedge \psi$:</u>

By definition of K-calculus:

$\quad K^1(C[\varphi \wedge \psi])$

$\quad = K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge K^2(C[\varphi]))[\psi])$

By fact (i):

$\quad = K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta_\varphi^K)[\psi])$

We conclude:

$\quad K^1((C \wedge K^1(C[\kappa]))[\kappa])$

$\quad = K^1((C \wedge K^1(C[\varphi \wedge \psi]))[\varphi \wedge \psi])$

$\quad = K^1((C \wedge K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta_\varphi^K)[\psi]))[\varphi \wedge \psi])$

We denote:

$\quad \theta = C \wedge K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta_\varphi^K)[\psi])$ *(iii)*

Thus:

$\quad K^1((C \wedge K^1(C[\kappa]))[\kappa])$

$\quad = K^1(\theta[\varphi \wedge \psi])$

By definition of K-calculus:

$\quad = K^1(\theta[\varphi]) \wedge K^1((\theta \wedge K^1(\theta[\varphi]) \wedge K^2(\theta[\varphi]))[\psi])$

By (i):

$\quad = K^1(\theta[\varphi]) \wedge K^1((\theta \wedge K^1(\theta[\varphi]) \wedge \beta_\varphi^K)[\psi])$

Thus, we conclude:

$K^1((C \wedge K^1(C[\kappa]))[\kappa])$

$\quad = K^1(\theta[\varphi]) \wedge K^1((\theta \wedge K^1(\theta[\varphi]) \wedge \beta^{K}_{\varphi})[\psi]) \quad$ *(iv)*

Now we have by our notation (iii):

$K^1(\theta[\varphi])$

$\quad = K^1((C \wedge K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta^{K}_{\varphi})[\psi]))[\varphi])$

By Lemma 2 below:

$K^1((C \wedge K^1(C[\varphi]))[\varphi])$

$\quad \Rightarrow K^1((C \wedge K^1(C[\varphi]) \underline{\wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta^{K}_{\varphi})[\psi])})[\varphi]) = K^1(\theta[\varphi])$

By induction $K^1((C \wedge K^1(C[\varphi]))[\varphi]) = \top$, thus:

$K^1(\theta[\varphi]) \equiv \top \quad$ *(v)*

From (iv) and (v) we conclude:

$K^1((C \wedge K^1(C[\kappa]))[\kappa])$

$\quad \equiv K^1((\theta \wedge \beta^{K}_{\varphi})[\psi])$

By the notation in (iii):

$\quad = K^1((C \wedge K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta^{K}_{\varphi})[\psi]) \wedge \beta^{K}_{\varphi})[\psi])$

By denoting $C_0 = C \wedge K^1(C[\varphi]) \wedge \beta^{K}_{\varphi}$ we get:

$\quad = K^1((C_0 \wedge K^1(C_0[\psi]))[\psi])$

$\quad \equiv \top$ by induction. □

Lemma 2 *For any $\kappa \in L_3$, for any $C \in L_2$ and $C' \in L_2$ s.t. $C \Rightarrow C'$: if $K^1(C'[\kappa]) \equiv \top$ then $K^1(C[\kappa]) \equiv \top$.*

Proof. Inductively on the structure of κ:

$\underline{\kappa = (\kappa_1 : \kappa_2)}$:

Assuming $K^1(C'[\kappa]) = K^1(C'[(\kappa_1 : \kappa_2)]) \equiv \top$, we conclude that $C' \Rightarrow \kappa_1$ by definition of K-calculus.

And since $C \Rightarrow C'$, we have $C \Rightarrow \kappa_1$, hence by definition of K-calculus:

$K^1(C[\kappa]) = K^1(C[(\kappa_1 : \kappa_2)]) \equiv \top$.

$\kappa = \neg\varphi$:

Assuming $K^1(C'[\kappa]) = K^1(C'[\neg\varphi]) \equiv \top$, we conclude $K^1(C'[\varphi]) \equiv \top$ by definition of K-calculus.

And by induction, we have $K^1(C[\varphi]) \equiv \top$, hence by def. of K-calculus:

$$K^1(C[\kappa]) = K^1(C[\neg\varphi]) \equiv \top.$$

$\kappa = \varphi \wedge \psi$:

We assume:

$$K^1(C'[\kappa]) = K^1(C'[\varphi \wedge \psi]) \equiv \top \qquad \textit{(i')}$$

From (i') we conclude by definition of K-calculus:

$$K^1(C'[\kappa]) =$$
$$K^1(C'[\varphi]) \wedge K^1((C' \wedge K^1(C'[\varphi]) \wedge K^2(C'[\varphi]))[\psi]) \equiv \top \qquad \textit{(ii)}$$

Thus, $K^1(C'[\varphi]) \equiv \top$, and we conclude:

– by induction:

$$K^1(C[\varphi]) \equiv \top \qquad \textit{(iii)}$$

– by substitution in (ii):

$$K^1(C'[\kappa]) = \top \wedge K^1((C' \wedge \top \wedge K^2(C'[\varphi]))[\psi]) \equiv \top$$

Using fact (i) from Lemma 1:

$$K^1((C' \wedge \beta_\varphi^K)[\psi]) \equiv \top$$

And by induction, since $C \wedge \beta_\varphi^K \Rightarrow C' \wedge \beta_\varphi^K$:

$$K^1((C \wedge \beta_\varphi^K)[\psi]) \equiv \top \qquad \textit{(iv)}$$

By definition of K-calculus and fact (i) from Lemma 1 we conclude:

$$\begin{aligned} K^1(C[\kappa]) &= K^1(C[\varphi \wedge \psi]) \\ &\overset{(def)}{=} K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge K^2(C[\varphi]))[\psi]) \\ &\overset{(i)}{\equiv} K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge \beta_\varphi^K)[\psi]) \\ &\overset{(iii)}{\equiv} \top \wedge K^1((C \wedge \top \wedge \beta_\varphi^K)[\psi]) \\ &\overset{(iv)}{\equiv} \top \end{aligned}$$

□

A note on Lemma 2 and context-monotonicity: Lemma 2 shows that any expression $C[\kappa]$ with a tautological K-presupposition monotonically retains this property under strengthening of C. However, the K-calculus does not show general downward monotonicity of $C[\kappa]$'s presupposition under strengthening of C. Thus, there exist $C, C' \in L_2$ and $\kappa \in L_3$ s.t. $C \Rightarrow C'$ but $C'[\kappa] \not\Rightarrow C[\kappa]$.

For instance, consider the following example:

$\varphi = (\top{:}a) \to (b{:}c)$

$\psi = (b \wedge (a \to c){:}d)$

where $\kappa = \varphi \wedge \psi$, $C = a \to b$ and $C' = \top$, s.t. a, b, c and d are arbitrary.

Now we have:

$$K^1(C[\kappa]) = K^1(C[\varphi \wedge \psi])$$
$$= K^1(C[\varphi]) \wedge K^1((C \wedge K^1(C[\varphi]) \wedge K^2(C[\varphi]))[\psi])$$
$$= K^1((a \to b)[\varphi]) \wedge K^1(((a \to b) \wedge K^1((a \to b)[\varphi]) \wedge K^2((a \to b)[\varphi]))[\psi])$$
$$= \top \wedge K^1(((a \to b) \wedge \top \wedge (a \to c))[\psi])$$
$$= K^1((a \to (b \wedge c))[\psi])$$

Since $(a \to (b \wedge c)) \not\Rightarrow (b \wedge (a \to c))$:

$$= b \wedge (a \to c)$$

However:

$$K^1(C'[\kappa]) = K^1(\top[\varphi \wedge \psi])$$
$$= K^1(\top[\varphi]) \wedge K^1((\top \wedge K^1(\top[\varphi]) \wedge K^2(\top[\varphi]))[\psi])$$
$$= b \wedge K^1((b \wedge (a \to c))[\psi])$$

Since $(b \wedge (a \to c)) \Rightarrow (b \wedge (a \to c))$:

$$= b \wedge \top$$
$$= b$$

Thus, $C = a \to b \Rightarrow C' = \top$ but $K^1(C'[\kappa]) = b \not\Rightarrow K^1(C[\kappa]) = b \wedge (a \to c)$.

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