Monotonicity and Relative Scope Entailments[1]

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Summary This paper explores the hypothesis that simple monotonicity properties of quantifiers in natural language determine to a large extent the entailment relations between their wide/narrow scope readings. We prove that the *disjunctive normal form* of upward monotone quantifiers using principal ultrafilters correlates with whether an object narrow scope reading entails an object wide scope reading. This result naturally extends the familiar entailment relations between $\exists\forall$ and $\forall\exists$ quantification in first order logic into arbitrary "finitely based" upward monotone determiners (over possibly infinite models), which are precisely defined.

Given a simple sentence of the form *Subject-Verb-Object*, we are interested in the logical relations between the *object narrow scope* (ONS) and the *object wide scope* (OWS) readings of the sentence. In [4], Zimmermann (1993) fully characterizes the class of "scopeless" object quantifiers – those for which the ONS and OWS readings are equivalent for any subject. Zimmermann shows that this class is closely related to the class of (principal) ultrafilters (names). In [3], Westerståhl (1996) fully characterizes the class of "self-commuting" quantifiers, i.e. the quantifiers Q for which ONS and OWS readings are equivalent when Q is substituted for both subject and object. However, as far as we know, the more general problem of characterizing (possibly one-way) entailment relations between ONS and OWS readings has not been given serious attention.

Global determiners (see [2]) are functors that map any non-empty domain E to a binary relation over $\wp(E)$. Any set $Q_E \subseteq \wp(E)$ is called a *generalized quantifier* (GQ) *over* E. Thus, a determiner D_E over E maps any $A \subseteq \wp(E)$ to the generalized quantifier $D_E(A)$ over E.

Let Q_1 and Q_2 be the GQs over E that the subject and object respectively denote in a given model. The ONS and OWS readings of the sentence in this model with respect to a binary relation $R \subseteq E \times E$, are defined using the following polyadic GQs over $E \times E$:

$$(1)\quad \begin{array}{lcll} Q_1\text{-}Q_2(R) & \stackrel{def}{=} & Q_1(\{x \in E : Q_2(\{y \in E : R(y)(x)\})\}) & \text{(ONS reading)} \\ Q_1{\sim}Q_2(R) & \stackrel{def}{=} & Q_2(\{y \in E : Q_1(\{x \in E : R(y)(x)\})\}) & \text{(OWS reading)} \end{array}$$

Let D_1 and D_2 be global determiners that correspond to the subject and object determiners respectively. The polyadic determiners $D_1\text{-}D_2$ and $D_1{\sim}D_2$, which give rise to ONS and OWS readings respectively, are defined as ternary relations between $A, B \subseteq E$ and $R \subseteq E \times E$.

$$(2)\quad \begin{array}{lcl} D_1\text{-}D_2(A)(B)(R) & \stackrel{def}{=} & ((D_1(A))\text{-}(D_2(B))(R) \\ D_1{\sim}D_2(A)(B)(R) & \stackrel{def}{=} & ((D_1(A)){\sim}(D_2(B))(R) \end{array}$$

[1] A full version of this abstract, including proofs of the propositions, appears at `http://www.technion.ac.il/~alon_a` This research was partly supported by grant no. 1999210 ("Extensions and Implementations of Natural Logic") from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel. The third author was also partly supported by the fund for the promotion of research at the Technion, research no. 120-042, and by an NWO grant for visiting the UiL OTS of the Utrecht University in summer 2001.

A quantifier Q_E is called *upward monotone* (MON$\uparrow$) if it is closed under supersets. A global determiner D is called *upward right monotone* (MON$\uparrow$) if for every $A \subseteq E$, the quantifier $D_E(A)$ is upward monotone.

We would like to characterize whether the relation D_1-D_2 is contained in the relation $D_1{\sim}D_2$. When D_1 is *every* (*some*) and D_2 is *some* (*every*), it is well-known that the answer is negative (positive) respectively. For instance, the ONS reading of the sentence *some student saw every teacher* entails, but is not entailed by, its OWS reading. We show that in fact, in sentences with upward monotone subjects and objects, the *existential* determiner is the basis for the class of *subject* determiners that guarantee entailment from the ONS reading to the OWS reading. Symmetrically, for upward monotone subjects and objects, the *universal* determiner is the basis for the class of *object* determiners that guarantee entailment from the ONS reading to the OWS reading.

We use the fact (cf. [1]) that any upward monotone quantifier Q_E can be represented as a union of intersections of principal ultrafilters.

Fact 1 *Let Q_E be an upward monotone GQ over E. Then $Q_E = \cup_{N \in M} \cap_{x \in N} I_x$, for some subset M of $\wp(E)$, where I_y is the principal ultrafilter $\{A \subseteq E : y \in A\}$ generated by $y \in E$.*

We call M the *signature* of a *disjunctive normal form* (DNF) of an upward monotone quantifier. We define a hierarchy of the upward monotone quantifiers by requiring a DNF for a quantifier $Q_E \in$ MON$\uparrow$, with a signature M that satisfies certain conditions. The classes in the hierarchy, with the respective conditions on M that defines them, are listed below.

- TRIV$_0$: $M = \emptyset$: Q_E is empty
- TRIV$_1$: $\emptyset \in M$: Q_E is equal to $\wp(E)$
- PUF: $M = \{\{a\}\}$ for some $a \in E$:
  Q_E is the *principal ultrafilter* I_a *generated by* a;
- PUF$_\cap$: $M = \{A\}$ for some $A \subseteq E$:
  Q_E is an intersection of PUFs: the *principal filter* F_A *generated by* A
- PUF$_\cup$: M is a (possibly empty) collection of singletons in $\wp(E)$:
  Q_E is a *union of PUFs.*

Obviously, the following relations hold between these classes of GQs: PUF $\subset$ PUF$_\cap$ $\subset$ MON$\uparrow$; PUF $\subset$ PUF$_\cup$ $\subset$ MON$\uparrow$; TRIV$_0$ $\subset$ PUF$_\cup$; TRIV$_1$ $\subset$ PUF$_\cap$.

Further, observe the following simple facts.

Fact 2 *A quantifier Q is in* PUF$_\cup$ *iff* $Q = \cup_{\{x\} \in Q} I_x$.

Fact 3 *A quantifier Q is in* PUF$_\cap$ *iff* $Q = \cap_{x \in \cap Q} I_x$ $(= F_{\cap Q})$.

Consider now the following simple relation between the above hierarchy and scope entailments.

Lemma 4 *Let $Q_1, Q_2 \subseteq \wp(E)$ be upward monotone GQs over E. If $Q_1 \in$* PUF$_\cup$ *or $Q_2 \in$* PUF$_\cap$ *then Q_1-$Q_2 \subseteq Q_1{\sim}Q_2$.*

We use our classification of MON$\uparrow$ local quantifiers in order to classify MON$\uparrow$ global determiners as follows. For any global determiner D:

D is $\mathrm{PUF}^1_{\cup}$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_{\cup} \cup \mathrm{TRIV}_1$.

D is $\mathrm{PUF}^0_{\cap}$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_{\cap} \cup \mathrm{TRIV}_0$.

D is TRIV_0 (TRIV_1) iff for all $A \subseteq E$: $D_E(A)$ is in TRIV_0 (TRIV_1).
When D is TRIV_0 or TRIV_1 we say that D is *trivial*.

D is $\mathrm{TRIV}^{\exists}_0$ ($\mathrm{TRIV}^{\exists}_1$) iff there exist $A \subseteq E$ s.t. $D_E(A)$ is in TRIV_0 (TRIV_1).

Thus, a determiner is called $\mathrm{PUF}^1_{\cup}$ ($\mathrm{PUF}^0_{\cap}$) when it generates only $\mathrm{PUF}_{\cup}$ ($\mathrm{PUF}_{\cap}$) and trivial quantifiers. Note that a determiner is classified as $\mathrm{PUF}^1_{\cup}$ ($\mathrm{PUF}^0_{\cap}$) or TRIV_0 (TRIV_1) according to its behavior on *all* domains and arguments. By contrast, for classifying a determiner as $\mathrm{TRIV}^{\exists}_0$ ($\mathrm{TRIV}^{\exists}_1$), it is sufficient to find *one* domain and one argument for which it is TRIV_0 (TRIV_1). The usefulness of both "universal" and "existential" classifications of determiners will be clarified as we go along.

Our main claim is that this typology of determiners allows us to determine in which cases of MON$\uparrow$ determiners D_1 and D_2, the ONS reading D_1-D_2 entails (or is entailed by) the OWS reading $D_1 \sim D_2$. Before proving that, there is one qualification concerning this result that we should explain. We will assume that both D_1 and D_2 are *finitely based*, in a sense that is defined below. This restriction is needed because MON$\uparrow$ determiners such as *infinitely many* behave with respect to relative scope entailments differently than MON$\uparrow$ determiners such as *at least three*. Consider the following examples.

(3) a. Infinitely many students saw John or Mary.
    b. At least three students saw John or Mary.

(4) a. Infinitely many students saw at least one of the two students.
    b. At least three students saw at least one of the two students.

In (3a), the ONS reading entails the OWS reading: if there are infinitely many students that have the property *saw John or saw Mary*, then either John or Mary has the property *was seen by infinitely many students*. But this is obviously not the case in (3b). A similar contrast is observed between (4a) and (4b), under a Russellian treatment of the definite article. For instance:

(5) $\mathbf{at\_least\_one\_of\_the\_n}'(A)(B) = 1 \;\Leftrightarrow\; |A| = n \wedge A \cap B \neq \emptyset$

(6) $\mathbf{each\_of\_the\_n}'(A)(B) = 1 \;\Leftrightarrow\; |A| = n \wedge A \subseteq B$

We observe that MON$\uparrow$ determiners such as *infinitely many* show scope entailments that are different than those of similar "finite" determiners. Such "infinite" determiners, which are common in the mathematical jargon, are much less common – and have a much less defined meaning – in everyday speech. This is in contrast to more ordinary determiners such *at least three* or *every*, which English speakers use by and large with the same meaning as logicians do. The formal distinction between determiners that is held responsible for this difference is defined as follows.

Definition 1 (FB **quantifiers**) *Let E be a non-empty domain. A sequence $A_i|_{i=1}^{\infty}$ of subsets of E is called* properly monotone *if $A_i \subset A_{i+1}$ for every $i \geq 1$, or $A_i \supset A_{i+1}$ for every $i \geq 1$.*
Two properly monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ are called mutually monotone *if $A_i \subset B_j$ for all $i, j \geq 1$, or $A_i \supset B_j$ for all $i, j \geq 1$.*
A quantifier Q_E over E is called finitely based (FB) *iff for any two mutually monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ s.t. Q_E is constant on both sequences, Q_E sends both sequences to the same value.*

By "constancy" of a quantifier Q_E on a set $\mathcal{X} \subseteq \wp(E)$, we of course mean: $\mathcal{X} \subseteq Q_E$ or $Q_E \cap \mathcal{X} = \emptyset$. In the first case say we say that Q_E sends $\mathcal{X}$ to 1. In the second case say we say that Q_E sends $\mathcal{X}$ to 0.
The definition of FB determiners is derived from the definition of FB quantifiers.

Definition 2 (FB **determiners**) *A global determiner D is* FB *iff for any domain E, $D_E(E)$ is an* FB *quantifier.*

Note that this definition pays attention only to the behavior of D_E on the whole E domain, and does not take into account proper subsets of E. Thus, a determiner such as *all* is provably FB, even though on the domain of natural numbers, the quantifier *all odd natural numbers* is *not* FB. This is in accordance with the intuition that the determiner *all* does not inherently pertain to infinite sets. By contrast, the determiner *all but finitely many* provably maps any infinite domain to a non-FB quantifier, hence it is not FB itself.

Let us consider an example for a pair of FB/non-FB determiners that belong in the same class of the above hierarchy. Consider first the determiner *infinitely many*. Let $\mathbf{N}$ by the set of natural numbers, with $\mathbf{N}_O \subset \mathbf{N}$ the set of odd natural numbers. Consider two sequences $(\mathbf{N}_O \cap [1..2i])|_{i=1}^{\infty}$ – the increasing sequence of sets of odd numbers; and $(\mathbf{N}_O \cup [1..2i])|_{i=1}^{\infty}$ – the unions of the odd numbers with elements in the increasing sequence of sets of even numbers. These two sequences are mutually monotone, but the denotation of *infinitely many natural numbers* on the domain $E = \mathbf{N}$ is constantly false on the first sequence but constantly true on the second sequence. Consequently, the determiner *infinitely many* is *not* FB. It is impossible to find two such sequences for the determiner *at least three*: trivially, for any domain E, the quantifier $\mathbf{at\_least\_3}'_E(E)$ cannot be false over an infinite properly monotone sequence. Consequently, the determiner *at least three* is FB. Note however that, for each of the determiners *infinitely many* and *at least three*, there are quantifiers that the determiner forms that belong in the class $\mathrm{MON}\uparrow \setminus (\mathrm{PUF}_{\cup} \cup \mathrm{PUF}_{\cap})$. Hence, both determiners are in the class $\mathrm{MON}\uparrow \setminus (\mathrm{PUF}^1_{\cup} \cup \mathrm{PUF}^0_{\cap})$. Some more examples for FB and non-FB determiners are given below. We note without proof that the class of FB determiners is closed under complements and finite intersections and unions.

FB Determiners: at least/at most/exactly 3; all; all but at least/most 3.
Non-FB Determiners: (in)finitely many; all but (in)finitely many.

We observe the following fact about upward monotone FB quantifiers.

Lemma 5 *Let Q be an FB upward monotone quantifier over a denumerable domain E. If $C_1 \supset C_2 \supset \ldots$ is a properly decreasing infinite sequence of sets in Q, then there is a finite set $A \subseteq C_1$ in Q.*

For the statement of our main claim, recall the following definitions, which are standard in GQ theory ([2]). For any global determiner D:

D satisfies *extension* (EXT) iff for all $A, B \subseteq E \subseteq E'$:$D_E(A)(B)$=$D_{E'}(A)(B)$.

D is *isomorphism invariant* (ISOM) iff for all bijections $\pi : E \to E'$, for all $A, B \subseteq E$: $D_{E'}(\{\pi(x) : x \in A\})(\{\pi(y) : y \in B\}) = D_E(A)(B)$.

D is *conservative* (CONS) iff for all $A, B \subseteq E$: $D_E(A)(B) = D_E(A)(A \cap B)$.

As in other works on GQ theory, we restrict our attention to determiners in natural language that are EXT, ISOM and CONS.

It is now possible to move on to our main claim.

Theorem 6 *Let D_1 and D_2 be two global* MON$\uparrow$ *determiners that satisfy* FB, EXT *and* CONS. *Then D_1-$D_2 \subseteq D_1{\sim}D_2$ for any domain E iff both following conditions hold: (1) D_1 is* $\mathrm{PUF}^1_\cup$ *or D_2 is* $\mathrm{PUF}^0_\cap$*; and (2) D_1 is not* $\mathrm{TRIV}^\exists_1$ *or D_2 is not* $\mathrm{TRIV}^\exists_0$.

The proof of the "if" direction is quite direct. To prove the "only if" direction, we make use of the following two lemmas, which rely on the FB property.

Lemma 7 *Let D be an* FB *determiner in* MON$\uparrow$ $\setminus\mathrm{PUF}^1_\cup$ *that satisfies* EXT *and* CONS. *Then there are $A \subseteq E$, for which there is $B \in D_E(A)$ s.t. $|B| \geq 2$ and for every $X \subset B$: $X \notin D_E(A)$.*

Lemma 8 *Let D be an* FB *determiner in* MON$\uparrow$ $\setminus\mathrm{PUF}^0_\cap$ *that satisfies* EXT *and* CONS. *Then there is a domain E and $A \subseteq E$, for which there are $B_1, B_2 \in D_E(A)$ s.t. $B_1 \cap B_2 \notin D_E(A)$.*

Theorem 6 characterizes all the logical cases of upward right-monotone subject and object determiners that make the ONS reading entail (or be entailed by) the OWS reading. Simple cases like that are when the subject determiner is $\mathrm{PUF}^1_\cup \setminus \mathrm{TRIV}^\exists_1$ or when the object determiner is $\mathrm{PUF}^0_\cap \setminus \mathrm{TRIV}^\exists_0$. That is: when the subject always denotes a $\mathrm{PUF}_\cup$ quantifier or the object always denotes a $\mathrm{PUF}_\cap$ quantifier. This is the case in the following sentences.

(7) a. Some student saw every/most/at least two teachers.

b. Every/most/at least two student(s) saw every teacher.

However, to characterize completely the cases of MON$\uparrow$ logical determiners for which the ONS reading entails the OWS reading, we have also considered some more complex cases of global determiners. An example for a member in $\mathrm{PUF}^1_\cup \cap \mathrm{TRIV}^\exists_1$ is the determiner *some or every*. Examples for members in $\mathrm{PUF}^0_\cap \cap \mathrm{TRIV}^\exists_0$ are the determiner *some and every* and the determiner *each of the five* (cf. definition (6)). These determiners show entailments from the ONS reading to the OWS reading in sentences such as the following.

(8) Some or (perhaps even) every student saw some or (perhaps even) every teacher.

(9) a. At least two teachers saw some and (in fact) every student.
b. At least two teachers saw each of the five students.

The complete characterization of scope entailments with MON$\uparrow$ determiners explains why there is no entailment from the ONS reading to the OWS reading in simple cases such as the following.

(10) Every/most/at least two student(s) saw some/most/at least two teacher(s).

Also in more complex cases such as the following, there is no entailment from the ONS reading to the OWS reading, as theorem 6 expects.

(11) Some or (perhaps even) every student saw some teacher.

(12) Every student saw some and (in fact) every teacher.

In both cases, when there are no students and no teachers, the ONS reading is true but the OWS reading is false.

Another result concerns the following fact that is mentioned in [3] about *local* quantifiers. Westerståhl calls two quantifiers Q_1 and Q_2 over E *independent* when Q_1-$Q_2 = Q_1{\sim}Q_2$. Then he makes the following claim.

Proposition 9 (Westerståhl) *Let Q_1 and Q_2 be two quantifiers over E that are* MON$\uparrow$*, non-trivial and* ISOM*. Then Q_1 and Q_2 are independent iff* $Q_1 = Q_2 = \mathbf{every}'_E(E) = \wp(E)$ *or* $Q_1 = Q_2 = \mathbf{some}'_E(E) = \wp(E) \setminus \emptyset$.

When we consider global determiners, we call D_1 and D_2 *independent* if D_1-$D_2 = D_1{\sim}D_2$ for any domain E. Theorem 6 entails the following fact about independent determiners. Note the ISOM requirement (as in Westerståhl's proposition), in addition to the requirements in theorem 6.

Corollary 10 *Let D_1 and D_2 be two global* MON$\uparrow$ *determiners that satisfy* FB, EXT, ISOM *and* CONS.
Then D_1 and D_2 are independent iff both of the following conditions hold:

1. *At least one of the following holds: (a) D_1 and D_2 are both* $\mathrm{PUF}^1_\cup$*; or (b) D_1 and D_2 are both* $\mathrm{PUF}^0_\cap$*; or (c) D_1 is trivial; or (d) D_2 is trivial.*
2. *At least one of the following holds: (a) Neither D_1 nor D_2 are* $\mathrm{TRIV}^\exists_0$*; or (b) Neither D_1 nor D_2 are* $\mathrm{TRIV}^\exists_1$.

Examples for identical D_1 and D_2 that are independent are the following cases: $D_1 = D_2 =$ *some, every, some-or-every, some-and-every*. However, independent determiners do not have to be identical. For instance: *each of the two* and *each of the five* are independent determiners, since according to the Russellian definition in (6), they are both in $\mathrm{PUF}^0_\cap \setminus \mathrm{TRIV}^\exists_1$.

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