On Many-valued modal logics: decidability and RE

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Barcelona

February 14, 2019

Contents

Modal logics

- Modal logics expand CPL with non “truth-functional” operators
- K models naturally notions like "possibly/necessarily", "sometimes/always"; many other modal operators/logics are considered in the literature (deontic logics, doxastic logics)
- Many applications developed (eg. program verification, AI modellings)

(partially) why? offer a much higher expressive power than CPL and (generally) much lower complexity than FOCL (most well-known and used modal logics are decidable).

- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability

Many-valued logics

- Many-valued logics valuate the formulas out of $\{0, 1\}(\top, \bot)$, but over richer algebraic structures.
- Huge family of logics (different classes of algebras for evaluation). Allow modeling vague/uncertain/incomplete knowledge and probabilistic notions
- Applications in industry/AI etc. + (classical) mathematical interest for its relation with Universal Algebra and more specific algebraic fields (eg. lattice-ordered abelian groups with unit).
- Many well-known infinitely-valued cases still decidable (Ł, Gödel, Product, H-BL...)

Many-valued modal logics

- Natural idea: expansion of MV logics with modal-like operators.
- syntactically/algebraically not so clear what does this mean in the basic case
- usual approach: valuation of Kripke models: both the accessibility relation ($\rightarrow$ fuzzy relation) and the formulas at each world
- systematic study is under development:
  - Some modal MV logics have been axiomatised, but most have not.*
  - In many (most) cases, there is no (usual) finite model property or (known) R.E/finite axiomatization $\Rightarrow$ decidability??

* if we knew more on the axiomatizations $\Longrightarrow$ algebraic semantics, decidability/complexity, duality, etc.

What do we know?

- (local) modal Gödel logics are decidable, even if they **do not enjoy the FMP** (Caicedo et. al., 2013)
- Theorems of modal Łukasiewicz logic (for models with $\{0, 1\}$-valued accessibility relation) are decidable via some Hajek's results.(2005)

- Gödel modal logics with valued accessibility have beenn axiomatized (Caicedo et. al, 2011), but not if we restrict to $\{0, 1\}$-accesibility.
- Łukasiewicz modal logics have not been axiomatized -only its corresponding no-compact/infinitary (Teheux, 2013).
- Product modal logics either -only their infinitary correspondent, and adding rational constants (Vidal, 2017).
- can we say something else??

The non-modal part

Definition

A FL_{ew} algebra $\mathbf{A}$ is $\langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \odot, 1\rangle$ is a commutative monoid
- $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$ (residuation law)
- $0 \leq x \leq 1 \ \forall x \in A$.

$\Gamma \models_{\mathcal{C}} \varphi$ iff for any homomorphism $h\colon Fm \rightarrow \mathbf{A}$, for any $\mathbf{A} \in \mathcal{C}$, if $h(\Gamma) \subseteq \{1\}$ then $h(\varphi) = 1$.

well known cases

* $[\mathbf{0}, \mathbf{1}]_{Ł}$ (where $x \odot y = \max\{0, x + y - 1\}$), $[\mathbf{0}, \mathbf{1}]_{\Pi}$ (where $\odot = \cdot$), $[\mathbf{0}, \mathbf{1}]_{\mathbf{G}}$ (where $\odot = \wedge$).
* 1-generated subalgebras of $[0, 1]_{\Pi}$: $\{0, 1\} \cup \{a^k \colon k \in \omega\}$ for some $a \in (0, 1)$.

MV-Kripke models

Language: $\odot, \to, \wedge, 0$ plus two unary (modal) symbols $(\Box, \Diamond)$.

Definition

A (crisp) **A**-*Kripke model* $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ where:

- $R \subseteq W \times W$ (Rus stands for $\langle u, s\rangle \in R$)
- $e : W \times Var \to A$ uniquely extended by:
  - $e(u, \varphi \star \psi) = e(u, \varphi) \star e(u, \psi)$, for $\star \in \{\odot, \wedge, \to\}$,
  - $e(u, \Box\varphi) = inf\{e(s, \varphi) : Rus\}$
  - $e(u, \Diamond\varphi) = sup\{e(s, \varphi) : Rus\}$

safe whenever $e(u, \Box\varphi), e(u, \Diamond\varphi)$ are defined in every world.

- $\mathcal{M}_{\mathcal{C}}$ = class of safe Kripke models over algebras in $\mathcal{C}$, and $\omega\mathcal{M}_{\mathcal{C}}$ the finite ones.
- $\mathcal{M}_{Ł}$, $\mathcal{M}_{\Pi}$, $\mathcal{M}_{\Pi_1}$ = classes of Kripke models valued respectively over $[0,1]_{Ł}$, $[0,1]_{\Pi}$ and 1-gen. subalgebras of $[0,1]_{\Pi}$.

Modal logics over residuated lattices

Let $\mathcal{A}$ be a class of R.Ls and $\mathcal{K}$ be a class of **A**-Kripke models for $\mathbf{A} \in \mathcal{A}$.

Definition

- **(Global modal logic)**: $\Gamma \vdash^{g}_{\mathcal{K}} \varphi$ iff for all $\mathfrak{M} \in \mathcal{K}$,

$$[\forall u \in W \;\; e(u, [\Gamma]) \subseteq \{1\}] \text{ implies } [\forall u \in W \;\; e(u, \varphi) = 1]$$

- **(Local modal logic)**: $\Gamma \vdash^{l}_{\mathcal{K}} \varphi$ iff for all $\mathfrak{M} \in \mathcal{K}$ and for all $u \in W$,

$$e(u, [\Gamma]) \subseteq \{1\} \text{ implies } e(u, \varphi) = 1$$

Undecidable deductions

Theorem

The following are undecidable:

1. $\vdash^{g}_{\mathcal{M}_{Ł}}$, $\vdash^{g}_{\mathcal{M}_{\Pi}}$ and $\vdash^{g}_{\mathcal{M}_{\Pi_1}}$, and their restrictions to finite models.
2. $\vdash^{l}_{4\mathcal{M}_{Ł}}$, $\vdash^{l}_{4\mathcal{M}_{\Pi}}$ and $\vdash^{l}_{4\mathcal{M}_{\Pi_1}}$, and their restrictions to finite models ($4\mathcal{M}$ denotes the transitive models in $\mathcal{M}$).

More generally: for $\mathcal{A}$ class of R.L such that

- $\forall n \in \omega$ there is $\mathbf{A} \in \mathcal{A}$ and $a \in A$ such that $a^{n+1} < a^n$.
- $\forall \mathbf{A} \in \mathcal{A}, \forall a, b \in A$, if $b \leq a^n$ for all n, then $b \odot a = b$.

Lemma ($\star$)

1. $\vdash^{g}_{\mathcal{M}_{\mathcal{A}}}$ and $\vdash^{g}_{\omega\mathcal{M}_{\mathcal{A}}}$ are undecidable;
2. $\vdash^{l}_{4\mathcal{M}_{\mathcal{A}}}$ and $\vdash^{l}_{\omega 4\mathcal{M}_{\mathcal{A}}}$ are undecidable;

Some general ideas of the proof...

- Post correspondence problem: given $\langle v_1, w_1\rangle, \dots, \langle v_n, w_n\rangle$ of pairs of numbers in some base $s > 1$, it is **undecidable** whether there exist $i_1, \dots, i_k$ with $i_j \in \{1, \dots, n\}$ such that $v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}$.
- OBS: Monoidal operation over non contractive elements can uniquely express concatenations of numbers as the above ones.

Given an instance P of the PCP we can define Γ_P, φ_P such that

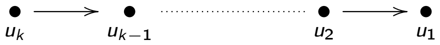
$$P \text{ is SAT} \iff \Gamma_P \nVdash^{g}_{\mathcal{M}_\mathcal{A}} \varphi_P \iff \Gamma_P \nVdash^{g}_{\omega\mathcal{M}_\mathcal{A}} \varphi_P$$

Similar for the local transitive case.

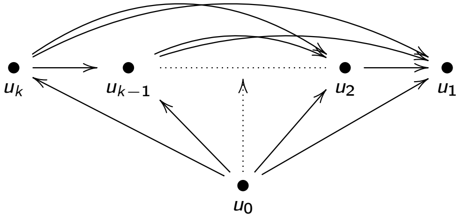
- The $\Rightarrow$ direction exploits non-contractivity of some algebra in the class.

...Some general ideas of the proof

- The $\Leftarrow$ direction uses weakly saturation and non-contractivity to prove that if $\Gamma_P \not\Vdash_{\mathcal{K}} \varphi_P$ then it happens in a model with structure



  with an evaluation that is then easily turnable into a solution of P.
- For the local case, the approach is very similar, getting a model of the form:



In general

- If $\models_{\mathcal{C}}$ is decidable, then $\not\vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is recursively enumerable.
- For the cases in the previous lemma, $\vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is undecidable!

Lemma

If $\mathcal{C}$ of R.L is as in Lemma $(\star)$ and $\models_{\mathcal{C}}$ is decidable, then $\vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is not R.E, and so, not axiomatizable.

Corollary

$\vdash^{g}_{\omega\mathcal{M}_{\text{Ł}}}$, $\vdash^{g}_{\omega\mathcal{M}_{\Pi}}$ and $\vdash^{g}_{\omega\mathcal{M}_{\Pi_1}}$ are not R.E, and so, is not axiomatizable.

However, it is not the case that $\vdash^{g}_{\omega\mathcal{M}_{\text{Ł}}} = \vdash^{g}_{\mathcal{M}_{\text{Ł}}}$, nor for the product case ...

so what about $\vdash^{g}_{\mathcal{M}_{\text{Ł}}}$ and $\vdash^{g}_{\mathcal{M}_{\Pi}}$? (the modal Łukasiewicz/product logics?)

The Łukasiewicz case

A model $\mathfrak{M}$ is witnessed iff for all $v \in W$, φ, there are $w_{\Box\varphi}$, $w_{\Diamond\varphi}$

$$e(v, \Box\varphi) = e(w_{\Box\varphi}, \varphi) \quad \text{and} \quad e(v, \Diamond\varphi) = e(w_{\Diamond\varphi}, \varphi)$$

$wit\mathcal{M}_{Ł}$ be the class of witnessed models over $[0, 1]_{Ł}$.

From (Hájek, 2005) + the standard translation from ML into FOL:

Lemma

$\Gamma \vdash^{g}_{\mathcal{M}_{Ł}} \varphi$ if and only if $\Gamma \vdash^{g}_{wit\mathcal{M}_{Ł}} \varphi$

We have completeness wrt. finite-width models... but the depth might still be infinite

The Łukasiewicz case

Lemma

$\Gamma \vdash^{g}_{\omega\mathcal{M}_Ł} \varphi$ iff $\Gamma, \Upsilon(p,q) \vdash^{g}_{\mathcal{M}_Ł} \varphi \vee \Psi(p,q)$ for any $p, q \notin \mathcal{V}ars(\Gamma, \varphi)$ and

- $\Upsilon(p,q) := \{\Box 0 \vee (p \leftrightarrow \Box p), \Box 0 \vee (\Box p \leftrightarrow \Diamond p), (q \leftrightarrow p \odot \Box q)\}$
- $\Psi(p,q) := p \vee \neg p \vee q \vee \neg q$.

Given a finite set of formulas Γ, φ, whether $\Gamma \equiv \Gamma_0 \cup \Upsilon(p,q)$ and $\varphi \equiv \varphi_0 \vee \Psi(p,q)$ is a decidable process.

Theorem

The finitary companion of the modal Lukasiewicz logic is not RE, and so, is not axiomatizable.

The Product case

! not known anything like the completeness of $\vdash^{g}_{\mathcal{M}_{\Pi}}$ wrt witnessed models (only a partial result, not generalizable, for theorems).
! $\vdash^{g}_{\mathcal{M}_{\Pi_1}}$ has directly quasi-witnessed models -and it is not axiomatizable.

We can split the reduction in two parts

Lemma

$\Gamma \vdash_{\omega\mathcal{M}_{\Pi_1}} \varphi$ iff $\Gamma, \Upsilon(p,q), QW(\Gamma,\varphi) \vdash_{\mathcal{M}_{\Pi_1}} \varphi \vee \Psi(p,q)$ for $p, q, \Upsilon(p,q), \Psi(p,q)$ as in the Ł case and

- $QW(\Gamma,\varphi) \coloneqq \{\neg\Box\chi \rightarrow \Diamond\neg\chi\}_{\Box\chi \in SFm(\Gamma,\varphi)}$.

Corollary

The finitary companion of $\vdash_{\mathcal{M}_{\Pi_1}}$ is not RE, and so, not axiomatizable.

The Product case

To reduce $\vdash_{\mathcal{M}_{\Pi_1}}$ to $\vdash_{\mathcal{M}_\Pi}$ we can use the cancelativity ($\forall a \in [0,1], \neg x \in \{0,1\}$).

Lemma

Given Γ, φ, there is a set of variables $\mathcal{V}'$ defined from $\mathcal{V}ar(\Gamma, \varphi)$ and two sets of formulas $\Sigma(\Gamma, \varphi, \mathcal{V}')$, $\Theta(\varphi, \mathcal{V}')$ such that $\Gamma \vdash_{\mathcal{M}_{\Pi_1}} \varphi$ iff $\Sigma(\Gamma, \varphi, \mathcal{V}') \vdash_{\mathcal{M}_\Pi} \Theta(\varphi, \mathcal{V}')$.

In both steps it is decidable whether some Γ, φ coincide with the corresponding transformed premises/consequence of some Γ_0, φ_0.

Theorem

The finitary companion of $\vdash_{\mathcal{M}_\Pi}$ is not RE, and so, not axiomatizable.

Some final observations

- (Scarpellini, 62) proved that the set of theorems of the FO Ł logic was not RE.
- (Mostowski, 61) gave some examples of non-axiomatizable many-valued logics.
- We close -in a negative way- the search for axiomatic systems of said logics.
- Gödel case? (global)

Moltes gràcies!