Provability logics with transfinitely many modalities in the foundations of mathematics

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- Foundations of mathematics: iterated reflection principles.
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In meta-mathematical language:

\[ \mathcal{F} \vdash \text{Con}(\mathcal{R}) \]

where \( \mathcal{F} \) is some undisputed part of mathematics consisting of finitary methods only, and \( \mathcal{R} \) denotes ‘real’ mathematics.
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Most notably, Gentzen’s consistency proof for Peano Arithmetic (1936)
Peano Arithmetic (PA) is the formal arithmetical theory in the language \( \{0, S, +, \cdot, 2^x\} \) axiomatized by the regular axioms for the constant and function symbols together with full induction:

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\varphi(0, \vec{y}) \land \forall x \left[ \varphi(x, \vec{y}) \rightarrow \varphi(Sx, \vec{y}) \right] \rightarrow \forall x \varphi(x, \vec{y}).
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Here PR-TI(\(\epsilon_0\)) is transfinite induction up to \(\epsilon_0\) for primitive recursive (p.r.) predicates

\[
\forall \alpha \in S \left[ \forall \beta < \alpha A(\beta) \rightarrow A(\alpha) \right] \rightarrow \forall \alpha A(\alpha)
\]

where \(S\) is some set on which \(\prec\) defines a (p.r.) well-order of order type \(\epsilon_0\) and \(A\) is a p.r. predicate.
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It is tempting to conceive of \( \text{PR-TI}(\varepsilon_0) \) as the non-finitistic part encompassed by \( \text{PA} \).

And in analogy to this, one can define a norm that measures proof strengths for theories \( T \) as follows:

\[
| T |_{\text{con}} := \min\{\alpha \mid \text{PRA} + \text{PR-TI}(\alpha) \vdash \text{Con}(T)\}
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Ad (b.): There are pathological orderings known (Kreisel) such that $\omega$ would be $|T|_{\text{Con}}$ for any $T$
Kreisel’s pathological ordering for a consistent theory $T$: 

- $n < T m$ iff $n < m$ and $\forall x < \max\{n, m\} \neg \text{Proof}_T(x, \lceil 0 = 1 \rceil)$ 
- $m < n$ and $\exists x < \max\{n, m\} \text{Proof}_T(x, \lceil 0 = 1 \rceil)$ 
- $\forall n < T m \left[ \forall n' < n \neg \text{Proof}_T(n', \lceil 0 = 1 \rceil) \rightarrow \forall m' < m \neg \text{Proof}_T(m', \lceil 0 = 1 \rceil) \right]$ provably in PRA holds for any $m$ by construction of $< T$:
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3. $\forall n < T^m [\forall n' < n \neg \text{Proof}_T(n', \psi(0 = 1)) \rightarrow [\forall m' < m \neg \text{Proof}_T(m', \psi(0 = 1))]]$ provably in PRA holds for any $m$ by construction of $<_T$: if $\exists m' < m \text{Proof}_T(m', \psi(0 = 1))$ then $m + 1 < T^m$ whence by the induction hypothesis $\forall m' < m + 1 \neg \text{Proof}_T(m', \psi(0 = 1))$ which is a contradiction.

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There are some technical details here as well-foundedness is a $\Pi^1_1$ predicate and as such not definable in first-order theories.
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Let \( S \) be a set of true \( \Sigma^1_1 \) sentences, then, under some fairly reasonable conditions

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We can define the proof theoretic measure

$$|T|_{\text{it}} := \min\{\alpha \mid \mathcal{F}_\alpha \vdash \text{Con}(T)\}$$

where $\mathcal{F}$ is a suitably chosen finitistic fragment of arithmetic.
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However, provability logics yield two main advantages:

- All the calculations involved in determining $|T|_{it}$ can be done within these logics.
- The logics suggest a very natural ordinal notation which is completely unambiguous up to the Feferman-Shütte ordinal $\Gamma_0$. 
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➤ We need some notation and terminology to make this precise.
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- For a formula \( \varphi \), we denote the representation by \( \lfloor \varphi \rfloor \).
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Using coding techniques, syntactic objects like formulas and proofs can be represented in number theories:

- For a formula \( \varphi \), we denote the representation by \( \downarrow \varphi \).
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- For elementary represented theories \( T \), one can write down a formula \( \text{Proof}_T(p, \Gamma \varphi) \) that is true only when \( p \) is the code of a proof in \( T \) of a formula \( \varphi \).
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Using coding techniques, syntactic objects like formulas and proofs can be represented in number theories:

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6. We will write $\Box_T \varphi$ for $\exists p \ \text{Proof}_T(p, \Gamma \varphi^\frown)$. 

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- Bounded formulas define the elementary predicates.
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This fact is formalizable in EA whence for $\sigma \in \Sigma_1$

$$\text{EA} \vdash \sigma \rightarrow \Box_{\text{EA}} \sigma$$
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Moreover, weak theories like EA prove all the Tarski Truth Conditions for these predicates, e.g.,

\[ \text{EA} \vdash \text{True}_{\Pi_n}(\psi \land \chi) \iff [\text{True}_{\Pi_n}(\psi) \land \text{True}_{\Pi_n}(\chi)] \]

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The complexity of $\text{True}_{\Pi_n}$ is itself $\Pi_n$
Using partial truth predicates,
\([n]T\varphi : \varphi \text{ is provable in the theory whose axioms are those of } T \text{ together with all true } \Pi_n \text{ sentences.} \)
Using partial truth predicates,
\([n]_T \varphi : \varphi \text{ is provable in the theory whose axioms are those of } T \text{ together with all true } \Pi_n \text{ sentences.}\)

We sometimes write \([0]_T \varphi \) for \(\Box_T \varphi\)
Using partial truth predicates,

\[ [n]_T \varphi : \varphi \text{ is provable in the theory whose axioms are those of } T \text{ together with all true } \Pi_n \text{ sentences.} \]

We sometimes write \([0]_T \varphi\) for \(\square_T \varphi\)

We abbreviate \(\neg[n]_T \neg \varphi\), that is, the \(n\)-consistency of \(\varphi\), by \(\langle n \rangle_T \varphi\)
Using partial truth predicates, $[n]_T \varphi : \varphi$ is provable in the theory whose axioms are those of $T$ together with all true $\Pi_n$ sentences.

We sometimes write $[0]_T \varphi$ for $\Box_T \varphi$.

We abbreviate $\neg[n]_T \neg \varphi$, that is, the $n$-consistency of $\varphi$, by $\langle n \rangle_T \varphi$.

$\langle n \rangle_T \top$ will stand for $T$ is $n$-consistent.
Uniform reflection over $T$ denoted by $\text{RFN}(T)$ is the scheme

$$\forall \vec{x} \left( \square_T \varphi(\vec{x}) \rightarrow \varphi(\vec{x}) \right)$$
Uniform reflection over $T$ denoted by $\text{RFN}(T)$ is the scheme

$$\forall \overrightarrow{x} \left( \square_T \varphi(\overrightarrow{x}) \rightarrow \varphi(\overrightarrow{x}) \right)$$

Restricted reflection over $T$ denoted by $\text{RFN}_{\Sigma_n}(T)$ is the scheme

$$\forall \overrightarrow{x} \left( \square_T \varphi(\overrightarrow{x}) \rightarrow \varphi(\overrightarrow{x}) \right) \text{ with } \varphi \in \Sigma_n$$
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Restricted reflection over $T$ denoted by $\text{RFN}_{\Sigma_n}(T)$ is the scheme

$$\forall \vec{x} \ (\square_T \varphi(\vec{x}) \rightarrow \varphi(\vec{x})) \quad \text{with} \quad \varphi \in \Sigma_n$$

It is an easy theorem that $\text{RFN}_{\Sigma_n}(T)$ is equivalent to Kleene’s rule for $\Sigma_n$ formulas:

$$\frac{\forall \vec{x} \ \square_T \varphi(\vec{x})}{\forall \vec{x} \ \varphi(\vec{x})} \quad \text{with} \quad \varphi \in \Sigma_n.$$
From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory EA.
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Theorem:
\[ \langle n \rangle_T^\top \equiv RFN_{\Sigma_n}(T) \]
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\[ \langle n \rangle_T \top \equiv RFN_{\Sigma_n}(T) \]

**Proof:** Suppose $[n]_T \bot$, then

\[ [0]_T(\pi \rightarrow \bot) \] for some $\Pi_n$ sentence $\pi$ (possibly non-standard)
From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory EA.

**Theorem:**

$\langle n \rangle_T \top \equiv RFN_{\Sigma_n}(T)$

**Proof:** Suppose $[n]_T \bot$, then

$[0]_T(\pi \rightarrow \bot)$ for some $\Pi_n$ sentence $\pi$ (possibly non-standard)

thus, $[0]_T \neg \pi$
From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory EA.

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**Proof:** Suppose $[n]_T \bot$, then
\[ [0]_T (\pi \rightarrow \bot) \] for some $\Pi_n$ sentence $\pi$ (possibly non-standard)

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whence $[0]_T \text{True}_{\Sigma_n}(\neg \pi)$. 

From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory EA.

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We obtain $\text{True}_{\Sigma_n}(\neg \pi)$ using $RFN_{\Sigma_n}(T)$
▶ From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory EA.

▶ Theorem: 
\[ \langle n \rangle_T \top \equiv RFN_{\Sigma_n}(T) \]

▶ Proof: Suppose $[n]_T \bot$, then 
\[ [0]_T (\pi \rightarrow \bot) \] for some $\Pi_n$ sentence $\pi$ (possibly non-standard)

▶ thus, $[0]_T \neg \pi$

▶ whence $[0]_T \text{True}_{\Sigma_n}(\neg \pi)$.

▶ We obtain $\text{True}_{\Sigma_n}(\neg \pi)$ using $RFN_{\Sigma_n}(T)$

▶ contradicting $\text{True}_{\Pi_n}(\pi)$
From now on, $T$ will be a consistent theory in the language of arithmetic that contains the theory $EA$.

Theorem: 
\[
\langle n \rangle_T \top \equiv RFN_{\Sigma_n}(T)
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Proof: Suppose $[n]_T \bot$, then 
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[0]_T (\pi \rightarrow \bot) \text{ for some } \Pi_n \text{ sentence } \pi \text{ (possibly non-standard)}
\]

thus, $[0]_T \neg \pi$

whence $[0]_T \text{True}_{\Sigma_n}(\neg \pi)$.

We obtain $\text{True}_{\Sigma_n}(\neg \pi)$ using $RFN_{\Sigma_n}(T)$

contradicting $\text{True}_{\Pi_n}(\pi)$

whence $\neg [n]_T \bot$, i.e., $\langle n \rangle_T \top$
Theorem:
\[
\langle n \rangle_T \top \equiv RFN_{\Sigma_n}(T)
\]
Theorem:
\[ \langle n \rangle^T \top \equiv RFN_{\Sigma_n}(T) \]

For the other direction, we need a very easy lemma:
\[ EA \vdash \sigma \rightarrow [n]^T \sigma \text{ for } \sigma \in \Sigma_{n+1} \]
Theorem:
\[ \langle n \rangle_T^{\top} \equiv RFN_{\Sigma_n}(T) \]

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\[ \text{EA} \vdash \sigma \rightarrow [n]_T \sigma \text{ for } \sigma \in \Sigma_{n+1} \]

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Theorem:
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For the other direction, we need a very easy lemma:
EA ⊢ σ → [n]_T σ for σ ∈ \( \Sigma_{n+1} \)

Suppose \[0\]_T φ with φ ∈ \( \Sigma_n \)

suppose, for a contradiction, that \neg φ
Theorem:
\[ \langle n \rangle_T^T \equiv RFN_{\Sigma_n}(T) \]

For the other direction, we need a very easy lemma:
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Suppose \([0]_T \varphi\) with \(\varphi \in \Sigma_n\)

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Suppose \[0\]_T \varphi \text{ with } \varphi \in \Sigma_n

suppose, for a contradiction, that \( \neg \varphi \)

as \( \neg \varphi \in \Sigma_{n+1} \)

we have \([n]_T \neg \varphi\), whence

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EA \vdash \sigma \rightarrow [n]_T \sigma \text{ for } \sigma \in \Sigma_{n+1}

Suppose [0]_T \phi \text{ with } \phi \in \Sigma_n

suppose, for a contradiction, that \( \neg \phi \)

as \( \neg \phi \in \Sigma_{n+1} \)

we have [n]_T \neg \phi, whence

[n]_T \bot

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All of the steps can be done within EA!
Let $I\Sigma_n$ be as PA but now the induction axioms restricted to $\Sigma_n$ formulas.
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Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$
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Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

It is not hard to see that

$$\forall x \Box_{EA}((\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)]) \rightarrow \varphi(\dot{x}))$$
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It is not hard to see that
$$\forall x \square_{EA}(\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)]) \rightarrow \varphi(\dot{x})$$

Note, the complexity of this formula
$$\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)]) \rightarrow \varphi(\dot{x})$$ 'is' $\Sigma_{n+1}$
Let $I\Sigma_n$ be as PA but now the induction axioms restricted to $\Sigma_n$ formulas

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\[ (\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)]) \rightarrow \varphi(\dot{x}) \text{ 'is' } \Sigma_{n+1} \]

By Kleene’s rule:
\[ \varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)] \rightarrow \forall x \varphi(x) \]
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Note, the complexity of this formula

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By Kleene’s rule: $\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)] \rightarrow \forall x \varphi(x)$

Note, this direction is fully formalizable in EA
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

For the other direction, suppose $\square_T \sigma$ with $\sigma \in \Sigma_{n+1}$
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

For the other direction, suppose $\Box_T \sigma$ with $\sigma \in \Sigma_{n+1}$

$\exists p \text{ Proof}_T(p, \Box \sigma)$
Theorem: $I \Sigma_n \equiv \text{RFN}_{\Sigma_{n+1}}(\text{EA})$

For the other direction, suppose $\Box_T \sigma$ with $\sigma \in \Sigma_{n+1}$

$\exists p \text{ Proof}_T(p, \neg \sigma)$

Now, employ cut-elimination to obtain a cut-free proof of $\sigma$
Theorem: \( I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA) \)

For the other direction, suppose \( \Box_T \sigma \) with \( \sigma \in \Sigma_{n+1} \)

\( \exists p \) Proof \( T(p, \framesigma) \)

Now, employ cut-elimination to obtain a cut-free proof of \( \sigma \)

Now, proof by induction on \( p \) that
\[
\text{Cut-Free-Proof}_T(p, \chi) \rightarrow \text{True}_{\Sigma_{n+1}}(\framesigma)
\]
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

For the other direction, suppose $\Box_T \sigma$ with $\sigma \in \Sigma_{n+1}$

$\exists p \text{ Proof}_T(p, \neg\neg \sigma)$

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This requires $\Sigma_{n+1}$ induction
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

For the other direction, suppose $\square_T \sigma$ with $\sigma \in \Sigma_{n+1}$

$\exists p \text{ Proof}_T(p, \lnot \sigma)$

Now, employ cut-elimination to obtain a cut-free proof of $\sigma$

Now, proof by induction on $p$ that $\text{Cut-Free-Proof}_T(p, \chi) \rightarrow \text{True}_{\Sigma_{n+1}}(\lnot \chi)$

This requires $\Sigma_{n+1}$ induction

With techniques from proof-theory, this can actually be brought back to $\Sigma_n$ induction
Theorem: \( \exists \sum_n \equiv \text{RFN}_{\sum_{n+1}}(\Sigma) \)

For the other direction, suppose \( \square_T \sigma \) with \( \sigma \in \sum_{n+1} \)

\[ \exists \rho \text{ Proof}_T(\rho, \neg \sigma) \]

Now, employ cut-elimination to obtain a cut-free proof of \( \sigma \)

Now, proof by induction on \( \rho \) that

\[ \text{Cut-Free-Proof}_T(\rho, \chi) \rightarrow \text{True}_{\sum_{n+1}}(\neg \chi) \]

This requires \( \sum_{n+1} \) induction

With techniques from proof-theory, this can actually be

brought back to \( \sum_n \) induction

Note that the proof can only be formalized in a setting where

cut-elimination can be proved
Theorem: $I\Sigma_n \equiv RFN_{\Sigma_{n+1}}(EA)$

For the other direction, suppose $\Box_T \sigma$ with $\sigma \in \Sigma_{n+1}$

$\exists p \ Proof_T(p, \neg \neg \sigma)$

Now, employ cut-elimination to obtain a cut-free proof of $\sigma$

Now, proof by induction on $p$ that
$Cut-Free-Proof_T(p, \chi) \rightarrow True_{\Sigma_{n+1}}(\neg \neg \chi)$

This requires $\Sigma_{n+1}$ induction

With techniques from proof-theory, this can actually be brought back to $\Sigma_n$ induction

Note that the proof can only be formalized in a setting where cut-elimination can be proved

that is, the sup-exp function must be provably total
Summarizing: $I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA)$
Summarizing: $I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA)$

Using similar techniques one can prove an analogous for the induction rules:
▶ Summarizing: $IΣ_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFNΣ_{n+1}(EA)$

▶ Using similar techniques one can prove an analogous for the induction rules:

▶ $IΣ^R_n$ is the closure of EA under the rule $\frac{\varphi(0) \land \forall x (\varphi(x) \to \varphi(x+1))}{\forall x \varphi(x)}$
Summarizing: \( I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA) \)

Using similar techniques one can prove an analogous for the induction rules:

\( I\Sigma^R_n \) is the closure of EA under the rule \( \frac{\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1))}{\forall x \varphi(x)} \)

**Theorem**

\( I\Sigma^R_n \equiv \Pi_{n+1} - RR^n(EA) \)
Summarizing: $IΣ_n ≜ \langle n+1 \rangle_{EA} \top ≜ RFN_{Σ_{n+1}}(EA)$

Using similar techniques one can prove an analogous for the induction rules:

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$$\frac{\varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(x+1))}{\forall x \varphi(x)}$$

**Theorem**

$$IΣ^R_n ≜ Π_{n+1}−RR^n(EA)$$

Here $Π_{n+1}−RR^n(EA)$ is the rule

$$\frac{\pi}{\langle n \rangle \pi} \quad \text{with} \quad \pi \in Π_{n+1}$$
▶ Summarizing: \( I\Sigma_n \equiv \langle n+1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA) \)

▶ Using similar techniques one can prove an analogous for the induction rules:

▶ \( I\Sigma_n^R \) is the closure of EA under the rule

\[
\frac{\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1))}{\forall x \varphi(x)}
\]

▶ Theorem

\( I\Sigma_n^R \equiv \Pi_{n+1} - RR^n(EA) \)

▶ Here \( \Pi_{n+1} - RR^n(EA) \) is the rule

\[
\frac{\pi}{\langle n \rangle \pi}
\]

with \( \pi \in \Pi_{n+1} \)

▶ It is not hard to see that \( RFN_{\Sigma_{n+1}}(EA) \vdash \pi \rightarrow \langle n \rangle \pi \) for \( \pi \in \Pi_{n+1} \) whence
Summarizing: \( I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA) \)

Using similar techniques one can prove an analogous for the induction rules:

\( I\Sigma^R_n \) is the closure of EA under the rule

\[
\varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(x+1)) \frac{\forall x \varphi(x)}{\langle n \rangle \pi}
\]

Theorem

\( I\Sigma^R_n \equiv \Pi_{n+1} - RR^n(EA) \)

Here \( \Pi_{n+1} - RR^n(EA) \) is the rule

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It is not hard to see that \( RFN_{\Sigma_{n+1}}(EA) \vdash \pi \rightarrow \langle n \rangle \pi \) for \( \pi \in \Pi_{n+1} \) whence

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Theorem

\[ I\Sigma^R_n \equiv \Pi_{n+1}^- \text{RR}^n(EA) \]
Theorem

\[ I\Sigma_n^R \equiv \Pi_{n+1} - RR^n(EA) \]

RFN_{\Sigma_{n+1}}(EA) turns out to be \( \Pi_{n+1} \) conservative over
EA + \( \Pi_{n+1} - RR^n(EA) \)
Theorem

$I\Sigma_n^R \equiv \Pi_{n+1}^- \text{RR}^n(EA)$

$\text{RFN}_{\Sigma_{n+1}}(EA)$ turns out to be $\Pi_{n+1}$ conservative over $EA + \Pi_{n+1}^- \text{RR}^n(EA)$

We write

$EA + \text{RFN}_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1}^- \text{RR}^n(EA)$
Theorem

\[ I \Sigma_n^R \equiv \Pi_{n+1} - \text{RR}^n(EA) \]

- RFN_{\Sigma_n^{n+1}}(EA) turns out to be \( \Pi_{n+1} \) conservative over \( EA + \Pi_{n+1} - \text{RR}^n(EA) \)
- We write
  \[ EA + \text{RFN}_{\Sigma_n^{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - \text{RR}^n(EA) \]
- This is formalizable in \( EA^+ \), and can be generalized to theories other than \( EA \)
Theorem

\[ I \Sigma^R_n \equiv \Pi_{n+1} - RR^n(EA) \]

- RFN_{\Sigma_{n+1}}(EA) turns out to be \( \Pi_{n+1} \) conservative over \( EA + \Pi_{n+1} - RR^n(EA) \)

- We write

\[ EA + RFN_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - RR^n(EA) \]

- This is formalizable in \( EA^+ \), and can be generalized to theories other than EA

- Here \( EA^+ \) is the theory EA together with the axiom stating that super-exponentiation is a total function
\[ \text{EA + RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA + } \Pi_{n+1} - \text{RR}_n(\text{EA}) \]
\[ EA + \text{RFN}_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - \text{RR}^n(EA) \]

From this follows

\[ \langle n + 1 \rangle^\top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \} \]
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Preliminaries and definitions
Equivalences
The Reduction Property

- $\text{EA} + \text{RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA} + \Pi_{n+1} - \text{RR}^n(\text{EA})$
- From this follows
  \[
  \langle n + 1 \rangle \top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \}
  \]
- The **Reduction Property** follows from the generalized theorem:
  \[
  \langle n + 1 \rangle \varphi \equiv_n \{ \langle n \rangle Q^n_k(\varphi) \mid k < \omega \}
  \]
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The Reduction Property

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Preliminaries and definitions

Equivalences

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$$\langle n+1 \rangle \varphi \equiv_n \{ \langle n \rangle Q^n_k(\varphi) | k < \omega \}$$

- $Q^n_0(\varphi) := \langle n \rangle \varphi$
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Preliminaries and definitions
Equivalences
The Reduction Property

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- $Q^n_{k+1}(\varphi) := Q^n_k(\varphi) \land \langle n \rangle Q^n_k(\varphi)$
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Preliminaries and definitions
Equivalences
The Reduction Property

▶ EA + RFN_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - RR^n(EA)
▶ From this follows
\[ \langle n+1 \rangle \top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \} \]
▶ The **Reduction Property** follows from the generalized theorem:
\[ \langle n+1 \rangle \varphi \equiv_n \{ \langle n \rangle Q^n_k(\varphi) \mid k < \omega \} \]
▶ \[ Q^n_0(\varphi) := \langle n \rangle \varphi \]
\[ Q^n_{k+1}(\varphi) := Q^n_k(\varphi) \land \langle n \rangle Q^n_k(\varphi) \]
▶ As the Reduction Property is provable in $EA^+$ we can provably substitute the one for the other in $RFN_{\Pi_{m+1}}$
Proof-strength of theories

Reflection, Consistency and Arithmetic

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The Reduction Property

\[ \text{EA} + \text{RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA} + \Pi_{n+1} - \text{RR}^n(\text{EA}) \]

From this follows

\[ \langle n + 1 \rangle \top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \} \]

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\[ \langle n + 1 \rangle \varphi \equiv_n \{ \langle n \rangle Q^n_k(\varphi) \mid k < \omega \} \]

\[ Q^n_0(\varphi) := \langle n \rangle \varphi \]

\[ Q^n_{k+1}(\varphi) := Q^n_k(\varphi) \land \langle n \rangle Q^n_k(\varphi) \]

As the Reduction Property is provable in \( \text{EA}^+ \) we can provably substitute the one for the other in \( \text{RFN}_{\Pi_{m+1}} \).

As provably \( \text{RFN}_{\Pi_{m+1}} \equiv \text{RFN}_{\Sigma_m} \) we obtain the following **Important Corollary**

\[ \text{EA}^+ \vdash \langle m \rangle \langle n + 1 \rangle \varphi \leftrightarrow \forall k \langle m \rangle Q^n_k(\varphi) \]
We are now almost ready for our consistency proof of PA and a scent of $\epsilon_0$ is already in the air.
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The logic GLP.
Language of modal logic GLP: unary modalities [0], [1], [2],... axiomatized by

\[
\begin{align*}
\langle n \rangle \phi & \rightarrow \phi \\
\langle n \rangle \phi & \rightarrow \langle m \rangle \phi & \text{for } m \geq n \\
\langle n \rangle \phi & \rightarrow \langle m \rangle \phi & \text{for } m > n
\end{align*}
\]

Closed under the rules of Modus Ponens and of necessitation for each modality: \( \phi \langle n \rangle \phi \rightarrow \langle m \rangle \phi \)
Language of modal logic GLP: unary modalities $[0], [1], [2], \ldots$ axiomatized by

$\left[ n \right] (\varphi \rightarrow \psi) \rightarrow (\left[ n \right] \varphi \rightarrow \left[ n \right] \psi)$ for all $n$

- $[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)$ for all $n$
- $[n]\varphi \rightarrow [n][n]\varphi$ for all $n$
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\[ \langle n \rangle \varphi \rightarrow [m]\langle n \rangle \varphi \] for \( m > n \)
Language of modal logic GLP: unary modalities

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1. \([n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)\) for all \(n\)
2. \([n]\varphi \rightarrow [n][n]\varphi\) for all \(n\)
3. \([n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi\) for all \(n\)
4. \([n]\varphi \rightarrow [m]\varphi\) for \(m \geq n\)
5. \(\langle n\rangle\varphi \rightarrow [m]\langle n\rangle\varphi\) for \(m > n\)

Closed under the rules of Modus Ponens and of necessitation
for each modality: \(\frac{\varphi}{[n]\varphi}\)
GLP is a decidable logic
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The closed fragment suffices to do all our calculations
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In particular the fragment $S$ consisting of all formulas like $\langle n_0 \rangle \ldots \langle n_m \rangle \top$
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We define an order $<_0$ on $S$ as follows

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Using techniques from modal logic – in particular modal semantics – one can show

$<_0$ is a well-order on $S$ of order type $\epsilon_0$ which is modulo provably equivalence in GLP linear.
Using GLP, one can show that $S$ is closed under the operation $\langle n + 1 \rangle \alpha \mapsto Q^n_k(\alpha)$.
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Moreover, for every non-empty $\langle n + 1 \rangle \alpha \in S$ we have

$$Q^n_k(\alpha) <_0 \langle n + 1 \rangle \alpha$$
We show that \( \text{EA}^+ + \text{TI}^R(\Pi_1, <_0) \vdash \text{Con}(\text{PA}) \)
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where $TI^R(\Pi_1, <_0)$ stands for

$$\forall \beta <_0 \alpha \varphi(\beta) \rightarrow \varphi(\alpha)$$

with $\varphi(x) \in \Pi_1$
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$$\forall \beta <_0 \alpha \varphi(\beta) \rightarrow \varphi(\alpha) \quad \forall \alpha \in S \varphi(\alpha)$$

with $\varphi(x) \in \Pi_1$

Proof: $\text{EA}^+ \vdash \forall n \diamond \langle n \rangle^T \iff \text{Con}(\text{PA})$ as we have seen that $I\Sigma_n \equiv \langle n + 1 \rangle^T$ is provable in $\text{EA}^+$
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Proof: $\text{EA}^+ \vdash \forall n \Diamond \langle n \rangle \top \iff \text{Con}(\text{PA})$ as we have seen that $I\Sigma_n \equiv \langle n + 1 \rangle \top$ is provable in $\text{EA}^+$

Thus, we are finished once we show

$$\forall \beta <_0 \alpha \Diamond \beta \rightarrow \Diamond \alpha$$
\[ \forall \beta < 0 \, \alpha \, \Diamond \beta \rightarrow \Diamond \alpha \]  

(†):
$\forall \beta < 0 \alpha \diamond \beta \rightarrow \diamond \alpha$ (†):

Lemma: EA$^+$ proves $\langle 1 \rangle_{EA^+}$ which is just RFN$_{\Sigma_1}(EA)$
\[ \forall \beta <_0 \alpha \diamond \beta \rightarrow \diamond \alpha \quad \text{(*)):} \]

- Lemma: EA\(^+\) proves \(\langle 1 \rangle_{EA} \top\) which is just RFN\(\Sigma_1\)(EA)
- actually, there is a deep connection between diagonalizing provably total recursive functions and 1-consistency
\[ \forall \beta < 0 \alpha \diamond \beta \rightarrow \diamond \alpha \quad (\dagger) : \]

- Lemma: \( \text{EA}^+ \) proves \( \langle 1 \rangle_{\text{EA}} \top \) which is just \( \text{RFN}_{\Sigma_1}(\text{EA}) \)
- actually, there is a deep connection between diagonalizing provably total recursive functions and 1-consistency
- Thus, for \( \alpha = \top \), we see \( (\dagger) \) as \( \text{EA}^+ \vdash \diamond_{\text{EA}} \top \)
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Lemma: EA\(^+\) proves \(\langle 1 \rangle_{EA}^\top\) which is just RFN\(\Sigma_1\)(EA)

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- If \(\alpha = \Diamond \beta\), by the IH, \(\text{EA}^+ \vdash \Diamond \beta\) whence by RFN\(\Sigma_1\)(EA) we get \(\text{EA}^+ \vdash \Diamond \Diamond \beta\)
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If \( \alpha = \langle n + 1 \rangle \beta \), then by the IH \( EA^+ \vdash \forall k \diamond Q^n_k(\beta) \) as we saw that for each \( k \), \( Q^n_k(\beta) <_0 \langle n + 1 \rangle \beta \)
\begin{itemize}
  \item $\forall \beta <_0 \alpha \diamond \beta \rightarrow \diamond \alpha \quad (\dagger)$:
  \item Lemma: $\text{EA}^+ \text{ proves } \langle 1 \rangle_{\text{EA}} \top$ which is just $\text{RFN}_{\Sigma_1}(\text{EA})$
  \item actually, there is a deep connection between diagonalizing provably total recursive functions and 1-consistency
  \item Thus, for $\alpha = \top$, we see $(\dagger)$ as $\text{EA}^+ \vdash \diamond \text{EA} \top$
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  \item If $\alpha = \langle n + 1 \rangle \beta$, then by the IH $\text{EA}^+ \vdash \forall k \diamond Q^n_k(\beta)$ as we saw that for each $k$, $Q^n_k(\beta) <_0 \langle n + 1 \rangle \beta$
  \item By our important corollary to the Reduction Property, we see
    \[ \diamond \langle n + 1 \rangle \beta \leftrightarrow \forall k \diamond Q^n_k(\beta) \]
  \end{itemize}

and we are done
Introduce modalities $[\alpha]$ for each ordinal $\alpha$ satisfying the GLP axioms
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GLP with modalities for all ordinals is still decidable!
The intuitive reading of $[\alpha]_T \varphi$ is
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$\varphi$ is provable from $T$ together with all true hyperarithmetical sentences of level $\alpha$
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- Proof-theory/ model theory part is being worked on in Moscow/Sevilla
Proof-strength of theories
Reflection, Consistency and Arithmetic
Provability logics
A consistency proof for PA
Beyond PA: Predicative mathematics

Moltes gràcies i bon nadal