GLP Lecture 1: Calibration of Proof-theoretical Strength

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In meta-mathematical language:

\[ \mathcal{F} \vdash \text{Con}(\mathcal{R}) \]

where \( \mathcal{F} \) is some undisputed part of mathematics consisting of finitary methods only, and \( \mathcal{R} \) denotes ‘real’ mathematics
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However, partial realizations of Hilbert’s programme have been obtained.

Most notably, Gentzen’s consistency proof for Peano Arithmetic (1936)
Peano Arithmetic (PA) is the formal arithmetical theory in the language \( \{0, S, +, \cdot, 2^x\} \) axiomatized by the regular axioms for the constant and function symbols together with full induction:

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\varphi(0, \vec{y}) \land \forall x \ [\varphi(x, \vec{y}) \rightarrow \varphi(Sx, \vec{y})] \rightarrow \forall x \varphi(x, \vec{y}).
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Here PR-TI(\( \epsilon_0 \)) is transfinite induction up to \( \epsilon_0 \) for primitive recursive (p.r.) predicates

\[
\forall \alpha \in S [\forall \beta < \alpha A(\beta) \rightarrow A(\alpha)] \rightarrow \forall \alpha A(\alpha)
\]

where \( S \) is some set on which \( < \) defines a (p.r.) well-order of order type \( \epsilon_0 \) and \( A \) is a p.r. predicate.
\[
\mathcal{F} + \text{PR-TI}(\epsilon_0) \vdash \text{Con}(\text{PA})
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With \(\mathcal{F}\) some finitistic part of mathematics (for example Primitive Recursive Arithmetic).
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It is tempting to conceive of PR-TI(\(\epsilon_0\)) as the non-finitistic part encompassed by PA.
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It is tempting to conceive of \(\text{PR-TI}(\varepsilon_0)\) as the non-finitistic part encompassed by \(\text{PA}\).

And in analogy to this, one can define a norm that measures proof strengths for theories \(T\) as follows:

\[|T|_{\text{con}} := \min\{\alpha \mid \text{PRA} + \text{PR-TI}(\alpha) \vdash \text{Con}(T)\}\]
The norm $|T|_{\text{con}}$ is very sensitive to

- The way ordinals are notated
- The way these notations are represented in a theory dealing with natural numbers (PRA)

Ad (a.) Recall that an ordinal is just defined as a transitive set that is well-ordered by $\in$. They live out there but to pick out one particular ordinal one needs a recipe. A uniform recipe makes up an ordinal notation system.

(where $x$ being transitive means $\forall y \in x \forall z (z \in y \rightarrow z \in x)$, that is, each element $y$ of $x$ is also a subset of $x$)

Ad (b.): There are pathological orderings known (Kreisel) such that $\omega$ would be $|T|_{\text{Con}}$ for any $T$. 
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We define $n <_T m$ iff

\[
m < n \text{ and } \forall x < \max\{n, m\} \neg \text{Proof}_T(x, \downarrow 0 = 1) \quad (PRA \text{ proves})\]

The ordering $<_T$ looks like

\[
0 <_T 1 <_T 2 <_T \ldots
given\]

in case $T$ is consistent

\[
x <_{T,x} 0 >_{T,x} 1 >_{T,x} 2 >_{T,x} \ldots
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in case $x_0$ is the smallest proof of $0 = 1$

Now, in PRA:

\[
\text{If } \exists x \text{ Proof}(x, \downarrow 0 = 1) \text{ then for any } z:\n\neg \forall y < T z \neg \text{Proof}(y, \downarrow 0 = 1).
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As there are arbitrary large proofs of anything that has a proof.

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\forall y < T z \neg \text{Proof}(y, \downarrow 0 = 1) \rightarrow \neg \text{Proof}(z, \downarrow 0 = 1).
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By induction along $<_T$ we prove in PRA consistency of $T$.

Note that, as $T$ is consistent, $\text{OT}(\mathbb{N},<_T) = \omega$. 

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Beklemishev has provided pathological representations for arbitrary large $\beta < \omega_1^{\text{CK}}$ such that PRA together with transfinite induction along $\beta$ does not prove $\text{Con}(PA)$. 

Gentzen: $\text{F} + \text{PR-TI}(\epsilon_0) \vdash \text{Con}(PA)$

Consequently, $\text{PA} \nvdash \text{PR-TI}(\alpha)$ for any $\alpha < \epsilon_0$

This leads to another measure for prove-strength of a theory $T$: the supremum of the order types of those recursive well-orders that are provably (in $T$) well founded

$|T| := \sup \{ \alpha | \alpha \text{ is the ordertype of a, provably in } T, \text{recursive well-order} \}$

There are some technical details here as well-foundedness is a $\Pi_1$ predicate and as such not definable in first-order theories.
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\[ \text{Let } S \text{ be a set of true } \Sigma_1^1 \text{ sentences, then, under some fairly reasonable conditions} \]

\[ |T|_{\text{sup}} = |T + S|_{\text{sup}} \]
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- $T_0 := T$

\[ T_\alpha := T_{\alpha+1} + \text{Con}(T_\alpha) \]

\[ T_\lambda := \bigcup_{\beta < \lambda} T_\beta \text{ for limit } \lambda \]

We can define the proof theoretic measure $|T|_\text{it} := \min \{ \alpha : F_\alpha \vdash \text{Con}(T) \}$ where $F$ is a suitably chosen finitistic fragment of arithmetic.
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It is to be expected that $|T|_{it}$ is more fine-grained than the other notions as it is defined in terms of a central notion: \textit{consistency}
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We can expect that $|T|_{it}$ is again very sensible to pathological orderings and representations thereof.
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However, provability logics yield two main advantages:

- All the calculations involved in determining $|T|_{it}$ can be done within these logics.
- The logics suggest a very natural ordinal notation which is completely unambiguous up to the Feferman-Shütte ordinal $\Gamma_0$. 
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In particular, the fragments $\Pi^0_n$ can be fully characterized in terms of consistency statements.

We need some notation and terminology to make this precise.
Using coding techniques, syntactic objects like formulas and proofs can be represented in number theories:
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Using coding techniques, syntactic objects like formulas and proofs can be represented in number theories:

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Using coding techniques, syntactic objects like formulas and proofs can be represented in number theories:

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- $\text{Proof}_T(p, \lbrack \varphi \rbrack)$ is a decidable formula.
- We will write $\square_T \varphi$ for $\exists p \text{ Proof}_T(p, \lbrack \varphi \rbrack)$. 

Joost J. Joosten

Modal Logic Course, 2011
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Weak theories like EA prove all true $\Pi_0$ statements $\psi$, that is,

$$\mathbb{N} \models \psi \implies \text{EA} \vdash \psi$$
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This fact is formalizable in EA whence for $\sigma \in \Sigma_1$

$$\text{EA} \vdash \sigma \rightarrow \square_{\text{EA}} \sigma$$
From Tarski’s Theorem on the undefinability of truth, we know that there is no arithmetical formula True(\(x\)) such that

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From Tarski’s Theorem on the undefinability of truth, we know that there is no arithmetical formula $\text{True}(x)$ such that:

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However, there are *partial truth predicates*:

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Moreover, weak theories like EA prove all the Tarski Truth Conditions for these predicates, e.g.,
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\text{EA} \vdash \text{True}_{\Pi_n}(\neg \psi \land \chi) \iff [\text{True}_{\Pi_n}(\neg \psi) \land \text{True}_{\Pi_n}(\neg \chi)]
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The complexity of $\text{True}_{\Pi_n}$ is itself $\Pi_n$
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\(\langle n \rangle_T \top\) will stand for \(T\) is \(n\)-consistent
Uniform reflection over $T$ denoted by RFN($T$) is the scheme

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Uniform reflection over $T$ denoted by $\text{RFN}(T)$ is the scheme

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It is an easy theorem that $\text{RFN}_{\Sigma_n}(T)$ is equivalent to Kleene’s rule for $\Sigma_n$ formulas:

$$\frac{\forall \vec{x} \circ_T \varphi(\vec{x})}{\forall \vec{x} \varphi(\vec{x})} \quad \text{with} \quad \varphi \in \Sigma_n.$$
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All of the steps can be done within EA!
Let $I \Sigma_n$ be as PA but now the induction axioms restricted to $\Sigma_n$ formulas.
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It is not hard to see that

$$\forall x \Box_{EA}((\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x+1)]) \rightarrow \varphi(\dot{x}))$$
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Note, the complexity of this formula

$(\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x+1)]) \rightarrow \varphi(\dot{x})$ ‘is’ $\Sigma_{n+1}$
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$$(\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)]) \rightarrow \varphi(\dot{x}) \; \text{‘is’} \; \Sigma_{n+1}$$

By Kleene’s rule: $\varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)] \rightarrow \forall x \varphi(x)$
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Note, this direction is fully formalizable in EA
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Proof-strength of theories
Reflection, Consistency and Arithmetic

Preliminaries and definitions
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This requires $\Sigma_{n+1}$ induction
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With techniques from proof-theory, this can actually be brought back to $\Sigma_n$ induction
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Note that the proof can only be formalized in a setting where cut-elimination can be proved
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that is, the sup-exp function must be provably total
Summarizing: $I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA)$
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Using similar techniques one can prove an analogous for the induction rules:
Summarizing: \( I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA) \)

Using similar techniques one can prove an analogous for the induction rules:

\( I\Sigma_n^R \) is the closure of \( EA \) under the rule

\[
\frac{\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1))}{\forall x \varphi(x)}
\]
Summarizing: $I\Sigma_n \equiv \langle n+1 \rangle_{\text{EA}} \top \equiv \text{RFN}_{\Sigma_{n+1}}(\text{EA})$

Using similar techniques one can prove an analogous for the induction rules:

$I\Sigma^R_n$ is the closure of EA under the rule

$\frac{\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1))}{\forall x \varphi(x)}$

Theorem

$I\Sigma^R_n \equiv \Pi_{n+1} - \text{RR}^n(\text{EA})$
Summarizing: $I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN\Sigma_{n+1}(EA)$

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$I\Sigma^R_n \equiv \Pi_{n+1} - RR^n(EA)$

Here $\Pi_{n+1} - RR^n(EA)$ is the rule

\[
\frac{\pi}{\langle n \rangle_{EA} \pi} \quad \text{with} \quad \pi \in \Pi_{n+1}
\]
Summarizing: $I\Sigma_n \equiv \langle n + 1 \rangle_{EA} \top \equiv RFN_{\Sigma_{n+1}}(EA)$

Using similar techniques one can prove an analogous for the induction rules:

$I\Sigma_n^R$ is the closure of EA under the rule:

\[
\begin{align*}
\varphi(0) & \land \forall x (\varphi(x) \rightarrow \varphi(x+1)) \\
\forall x \varphi(x) &
\end{align*}
\]

**Theorem**

$I\Sigma_n^R \equiv \Pi_{n+1} - RR^n(EA)$

Here $\Pi_{n+1} - RR^n(EA)$ is the rule:

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It is not hard to see that $RFN_{\Sigma_{n+1}}(EA) \vdash \pi \rightarrow \langle n \rangle \pi$ for $\pi \in \Pi_{n+1}$ whence
Summarizing: $I\Sigma_n \equiv \langle n+1 \rangle_{\text{EA}} \top \equiv \text{RFN}_{\Sigma_{n+1}}(\text{EA})$

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$I\Sigma^R_n$ is the closure of $\text{EA}$ under the rule

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**Theorem**

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Here $\Pi_{n+1} - \text{RR}^n(\text{EA})$ is the rule

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\[ I\Sigma^R_n \equiv \Pi_{n+1} - RR^n(EA) \]
Theorem

\[ I \Sigma_n^R \equiv \Pi_{n+1} - \text{RR}^n(\text{EA}) \]

RFN_{\Sigma_{n+1}}(\text{EA}) turns out to be \( \Pi_{n+1} \) conservative over \( \text{EA} + \Pi_{n+1} - \text{RR}^n(\text{EA}) \)
Theorem

\[ I^R \equiv \Pi_{n+1} - \text{RR}^n(EA) \]

RFN_{\Sigma_{n+1}}(EA) turns out to be \( \Pi_{n+1} \) conservative over \( EA + \Pi_{n+1} - \text{RR}^n(EA) \)

We write

\[ EA + \text{RFN}_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - \text{RR}^n(EA) \]
Theorem

\[ I\Sigma^R_n \equiv \Pi_{n+1} - \text{RR}^n(EA) \]

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This is formalizable in \( EA^+ \), and can be generalized to theories other than \( EA \)
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This is formalizable in \( EA^+ \), and can be generalized to theories other than \( EA \)

Here \( EA^+ \) is the theory \( EA \) together with the axiom stating that super-exponentiation is a total function
\[ \mathbf{EA} + \mathbf{RFN}_{\Sigma_{n+1}(EA)} \equiv_n \mathbf{EA} + \Pi_{n+1} - \mathbf{RR}^n(\mathbf{EA}) \]
\[ EA + RFN_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - RR^n(EA) \]

From this follows

\[ \langle n + 1 \rangle^\top \equiv_n \{ \langle n \rangle^k^\top \mid k < \omega \} \]
EA + RFN_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1} - RR^n(EA)

From this follows

\langle n + 1 \rangle^\top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \}

Bluffing (fallacious/incomplete argument):
- \(\text{EA} + \text{RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA} + \Pi_{n+1} \neg \text{RR}^n(\text{EA})\)

- From this follows

\[\langle n + 1 \rangle \top \equiv_n \{\langle n \rangle^k \top \mid k < \omega\}\]

- Bluffing (fallacious/incomplete argument):
  - \(\langle 1 \rangle \top \equiv_0 \langle 0 \rangle^\omega \top\)
\[ \text{EA} + \text{RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA} + \Pi_{n+1} - \text{RR}^n(\text{EA}) \]

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Bluffing (fallacious/incomplete argument):

\[ \langle 1 \rangle^\top \equiv_0 \langle 0 \rangle^\omega \top \]

\[ \text{I}^\Sigma_1 \equiv \langle 2 \rangle^\top \equiv_1 \langle 1 \rangle^\omega \top \equiv_0 (\langle 0 \rangle^\omega)^\omega \top \equiv \langle 0 \rangle^\omega^\omega \top \]
Underlying logical framework: Proof-theoretic systems

**Proof-strength of theories**

Reflection, Consistency and Arithmetic

- Preliminaries and definitions
- Equivalences
- The Reduction Property

- **EA + RFN**$_{\Sigma_{n+1}}$(EA) $\equiv_n$ EA + $\Pi_{n+1}$ − RR$^n$(EA)

- From this follows:

\[
\langle n + 1 \rangle \top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \}
\]

- **Bluffing (fallacious/incomplete argument):**
  - $\langle 1 \rangle \top \equiv_0 \langle 0 \rangle^\omega \top$
  - $I\Sigma_1 \equiv \langle 2 \rangle \top \equiv_1 \langle 1 \rangle^\omega \top \equiv_0 (\langle 0 \rangle^\omega)^\omega \top \equiv \langle 0 \rangle^{\omega^\omega} \top$
  - $I\Sigma_2 \equiv \langle 3 \rangle \top \equiv_2 \langle 2 \rangle^{\omega} \top \equiv_1 \langle 0 \rangle^{\omega^\omega} \top \equiv_0 \langle 0 \rangle^{\omega^\omega^\omega} \top$

- Where $\epsilon_0 = \sup \{ \omega, \omega^\omega, \omega^{\omega^\omega}, ... \}$

  This can be conceived as the proof theoretic ordinal of PA
\[ \text{EA + RFN}_{\Sigma_{n+1}}(\text{EA}) \equiv_n \text{EA + } \Pi_{n+1} - \text{RR}^n(\text{EA}) \]

From this follows

\[ \langle n + 1 \rangle \top \equiv_n \{ \langle n \rangle^k \top \mid k < \omega \} \]

Bluffing (fallacious/incomplete argument):

1. \[ \langle 1 \rangle \top \equiv_0 \langle 0 \rangle^\omega \top \]
2. \[ I \Sigma_1 \equiv \langle 2 \rangle \top \equiv_1 \langle 1 \rangle^\omega \top \equiv_0 (\langle 0 \rangle^\omega)^\omega \top \equiv \langle 0 \rangle^{\omega^\omega} \top \]
3. \[ I \Sigma_2 \equiv \langle 3 \rangle \top \equiv_2 \langle 2 \rangle^\omega \top \equiv_1 \langle 0 \rangle^{\omega^\omega} \top \equiv_0 \langle 0 \rangle^{\omega^\omega} \top \]
4. \[ \text{PA} \equiv \langle \omega \rangle \top \equiv_0 \langle 0 \rangle^{\varepsilon_0} \top \]
Proof-strength of theories
Reflection, Consistency and Arithmetic

Preliminaries and definitions
Equivalences
The Reduction Property

- \( EA + RFN_{\Sigma_{n+1}}(EA) \equiv_n EA + \Pi_{n+1}-RR^n(EA) \)
- From this follows

\[
\langle n+1 \rangle^T \equiv_n \{ \langle n \rangle^k^T \mid k < \omega \}
\]
- Bluffing (fallacious/incomplete argument):
  - \( \langle 1 \rangle^T \equiv_0 \langle 0 \rangle^\omega^T \)
  - \( I\Sigma_1 \equiv \langle 2 \rangle^T \equiv_1 \langle 1 \rangle^\omega^T \equiv_0 (\langle 0 \rangle^\omega)^\omega^T \equiv \langle 0 \rangle^{\omega^\omega}^T \)
  - \( I\Sigma_2 \equiv \langle 3 \rangle^T \equiv_2 \langle 2 \rangle^\omega^T \equiv_1 \langle 0 \rangle^{\omega^\omega}^T \equiv_0 \langle 0 \rangle^{\omega^{\omega^\omega}}^T \)
  - \( PA \equiv \langle \omega \rangle^T \equiv_0 \langle 0 \rangle^{\epsilon_0} T \)
- Where \( \epsilon_0 = \sup\{ \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots \} \)
Proof-strength of theories
Reflection, Consistency and Arithmetic

1. **Preliminaries and definitions**

2. **Equivalences**

3. **The Reduction Property**

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- **EA + RFN**$_{n+1}$**(EA) \equiv_n EA + \Pi_{n+1} - RR^n(EA)**

- From this follows

\[ \langle n + 1 \rangle \vdash \equiv_n \{ \langle n \rangle^k \vdash | k < \omega \} \]

- **Bluffing (fallacious/incomplete argument):**
  - \[ \langle 1 \rangle \vdash \equiv_0 \langle 0 \rangle^\omega \vdash \]
  - \[ \text{I} \Sigma_1 \equiv \langle 2 \rangle \vdash \equiv_1 \langle 1 \rangle^\omega \vdash \equiv_0 (\langle 0 \rangle^\omega)^\omega \vdash \equiv \langle 0 \rangle^\omega^\omega \vdash \]
  - \[ \text{I} \Sigma_2 \equiv \langle 3 \rangle \vdash \equiv_2 \langle 2 \rangle^\omega \vdash \equiv_1 \langle 0 \rangle^\omega^\omega \vdash \equiv_0 \langle 0 \rangle^\omega^\omega^\omega \vdash \]

- **PA** \[ \equiv \langle \omega \rangle \vdash \equiv_0 \langle 0 \rangle^{\epsilon_0} \vdash \]

- **Where** \[ \epsilon_0 = \sup\{ \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots \} \]

- This can be conceived as the proof theoretic ordinal of **PA**