Quantifiers and Functions in Intuitionistic Logic

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Quantifiers are complicated.

Even more so in intuitionistic logic:

In classical logic the following formulas hold.

- $\exists x\varphi(x) \leftrightarrow \neg\forall x\neg\varphi(x)$,
- $\exists x\big(\varphi(x) \to \forall y\varphi(y)\big)$,
- $\forall x\big(\varphi(x) \vee \neg\varphi(x)\big)$.

In intuitionistic logic these formulas do not hold, although the following do.

- $\exists x\varphi(x) \to \neg\forall x\neg\varphi(x)$,
- $\forall x\neg\neg\big(\varphi(x) \vee \neg\varphi(x)\big)$.

The Brouwer–Heyting–Kolmogorov interpretation

A proof of

$\varphi \wedge \psi$	*is given by presenting a proof of* φ *and a proof of* ψ.
$\varphi \vee \psi$	*is given by presenting either a proof of* φ *or of* ψ.
$\varphi \to \psi$	*is a construction which permits us to transform any proof of* φ *into a proof of* ψ.
$\forall x \varphi(x)$	*is a construction that transforms a proof of* $d \in D$ *into a proof of* $\varphi(d)$.
$\exists x \varphi(x)$	*is given by providing a* $d \in D$ *and a proof of* $\varphi(d)$.
$\perp$	*has no proof.*

$\neg\varphi$ *is defined as* $\varphi \to \perp$.

In classical logic the following formulas hold.

- $\exists x \varphi(x) \leftrightarrow \neg\forall x \neg\varphi(x)$,
- $\exists x \big(\varphi(x) \to \forall y \varphi(y)\big)$,
- $\neg\neg\forall x \big(\varphi(x) \vee \neg\varphi(x)\big)$.

In intuitionistic logic these formulas do not hold, because

- $\exists x \varphi(x) \leftrightarrow \neg\forall x \neg\varphi(x)$
  knowing that there is no proof of $\forall x \neg\varphi(x)$ does not provide a d and a proof of $\varphi(d)$
- $\exists x \big(\varphi(x) \to \forall y \varphi(y)\big)$,
  we could be in the situaton that we neither have a proof of $\forall y \varphi(y)$ nor a $d \in D$ such that $\neg\varphi(d)$
- $\neg\neg\forall x \big(\varphi(x) \vee \neg\varphi(x)\big)$.
  even though for every d, $\neg\neg(\varphi(d) \vee \neg\varphi(d))$ holds, there may never be a point at which $\varphi(d)$ has been decided for all $d \in D$

Intuitively: $\forall x \exists y \varphi(x, y)$ *means that there is a function* f *such that* $\forall x \varphi(x, f(x))$. *Namely,* fx *is the* y *such that* $\varphi(x, y)$ *holds.*

(*from now on, we write* fx *for* $f(x)$)

Ex $\forall x \exists y(y$ *is a parent of* $x)$ *and* fx *denotes the mother of* x, *or* fx *denotes the father of* x.

Thm $\forall x \exists y \varphi(x, y)$ *is satisfiable* (*holds in a model*) *if and only if* $\forall x \varphi(x, fx)$ *is satisfiable for any function symbol* f *that does not occur in* $\varphi(x, y)$.

Last requirement necessary: if $\varphi(x, y)$ *is* $fx \neq y$, *then statement not true.*

Ex *Let* $\varphi(x, y) = R(x, y)$. *Models* $M = (D, I(R))$, *where* $I(R) \subseteq D \times D$.

- *Model* $M = (\mathbb{N}, <)$. *Then* $M \models \forall x \exists y R(x, y)$ *and* $M' \models \forall x \varphi(x, fx)$ *for* M' *being* M *but with* f *interpreted as* $f(n) = n + 1$, *or any function monotone in* $<$.
- *Model* $M = (\mathbb{Z}, I(R))$, *where* $I(R)(x, y)$ *iff* $(x + y = 0)$. *Then* $M \models \forall x \exists y R(x, y)$ *and* $M' \models \forall x \varphi(x, fx)$ *for* M' *being* M *but with* f *interpreted as* $f(n) = -n$.

Thm $\forall x \exists y \varphi(x, y)$ *is satisfiable (holds in a model) if and only if* $\forall x \varphi(x, fx)$ *is satisfiable for any function symbol* f *that does not occur in* $\varphi(x, y)$.

In a model for $\forall x \exists y \varphi(x, y)$, f *chooses, for every* x, *a witness* fx *such that* $\varphi(x, fx)$.

Cor $\exists x \forall y \varphi(x, y)$ *holds (in all models) if and only if* $\exists x \varphi(x, fx)$ *holds for a function symbol* f *not in* φ.

Prf *By contraposition.* $\exists x \forall y \varphi(x, y)$ *does not hold iff* $\forall x \exists y \neg\varphi(x, y)$ *is satisfiable, iff* $\forall x \neg\varphi(x, fx)$ *is satisfiable for any* f *not in* φ, *iff there is no* f *not in* φ *such that* $\exists x \varphi(x, fx)$ *holds.* $\dashv$

In a counter model to $\exists x \forall y \varphi(x, y)$, f *chooses, for every* x, *a counter witness* fx *such that* $\neg\varphi(x, fx)$.

Dfn $\vdash_{CQC}$ *denotes derivability in classical predicate logic CQC.*

Thm
$\vdash_{CQC} \exists x \forall y \varphi(x, y)$ *iff* $\vdash_{CQC} \exists x \varphi(x, fx)$ *for a function symbol* f *not in* φ.

Thm
$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \varphi(x_1, y_1, x_2, y_2)$ *satisf. iff* $\forall x_1 \forall x_2 \varphi(x_1, fx_1, x_2, gx_1x_2)$ *satisf.*

(gx_1x_2 *short for* $g(x_1, x_2)$)

Cor
$\vdash_{CQC} \exists x_1 \forall y_1 \exists x_2 \forall y_2 \varphi(x_1, y_1, x_2, y_2)$ *iff* $\vdash_{CQC} \exists x_1 \exists x_2 \varphi(x_1, fx_1, x_2, gx_1x_2)$ *for some function symbols* f, g *not in* φ.

Thm For any formula φ *and any theory* T:

$$T \vdash_{CQC} \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \varphi(x_1, y_1, \dots, x_n, y_n)$$
$$\Longleftrightarrow$$
$$T \vdash_{CQC} \exists x_1 \dots \exists x_n \varphi(x_1, f_1x_1, \dots, x_n, f_nx_1 \dots x_n)$$

for some function symbols $f_1, f_2, \dots, f_n$ *not in* φ *and* T.

For some of Skolem's articles, see Richard Zach's Logic Blog.

Question *Does there exist the same connection between functions and quantifiers in intuitionistic logic?*

Answer *No, but . . . see rest of the talk.*

In a constructive reading:

- *a proof of* $\forall x \varphi(x)$ *consists of a construction that from a proof that* d *belongs to the domain produces a proof of* $\varphi(d)$.
- *a proof of* $\exists x \varphi(x)$ *consists of a construction of an element* d *in the domain and a proof of* $\varphi(d)$.

Thus in a constructive reading, a proof of $\forall x \exists y \varphi(x, y)$ *consists of a construction that for every* d *in the domain produces an element* e *in the domain and a proof of* $\varphi(d, e)$.

Heyting Arithmetic, the constructive theory of the natural numbers, is consistent with Church Thesis, which states that if $\forall x \exists y \varphi(x, y)$*, then there exists a total computable function* h *such that* $\forall x \varphi(x, hx)$.

Question *Does Skolemization hold in IQC?*

For any formula φ *and any theory* T*:*

$$T \vdash_{IQC} \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \varphi(x_1, y_1, \ldots, x_n, y_n)$$
$$\Longleftrightarrow$$
$$T \vdash_{IQC} \exists x_1 \ldots \exists x_n \varphi(x_1, f_1 x_1, \ldots, x_n, f_n x_1 \ldots x_n)$$

for some function symbols $f_1, f_2, \ldots, f_n$ *not in* φ *and* T*?*

Answer *No. Counterexample:*

$$\nvdash_{IQC} \exists x \forall y \big(\varphi(x) \to \varphi(y)\big) \qquad \vdash_{IQC} \exists x \big(\varphi(x) \to \varphi(fx)\big).$$

Fact *In intuitionistic predicate logic IQC not every formula has a prenex normal form.*

Dfn *An occurrence of a quantifier $\forall x$ ($\exists x$) in a formula is strong if it occurs positively (negatively) in the formula, and weak otherwise.*

Ex $\exists x$ *and* $\forall y$ *occur strong in* $\exists x\varphi(x) \to \forall y\psi(y)$ *and weak in* $\exists x\varphi \wedge \neg\forall y\psi(y)$.

In $\exists x\forall y\exists z\varphi(x, y, z)$, $\forall y$ *is a strong occurrence and the two existential quantifiers occur weakly.*

Dfn An occurrence of $\forall x$ ($\exists x$) in a formula is *strong* if it occurs positively (negatively) in the formula, and *weak* otherwise.

φ^s is the *skolemization* of φ if it does not contain strong quantifiers and there are formulas

$$\varphi = \varphi_1, \dots, \varphi_n = \varphi^s$$

such that φ_{i+1} is the result of replacing the leftmost strong quantifier

$$Qx\psi(x, \bar{y}) \text{ in } \varphi_i \text{ by } \psi(f_i(\bar{y}), \bar{y}),$$

where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi(x, \bar{y})$ occurs, and f_i does not occur in any φ_j with $j \leq i$.

Ex $\exists x(\exists y\varphi(x, y) \to \forall z\psi(x, z))^s = \exists x(\varphi(x, fx) \to \psi(x, gx))$.

In case φ is in prenex normal form, this definition of Skolemization coincides with the earlier one.

Fact For any formula φ and any theory T: $T \vdash_{CQC} \varphi \Leftrightarrow T \vdash_{CQC} \varphi^s$.

Question Does $T \vdash_{IQC} \varphi \Leftrightarrow T \vdash_{IQC} \varphi^s$ hold?

Dfn For a theory T, Skolemization is *sound* if

$$T \vdash \varphi \Rightarrow T \vdash \varphi^s$$

and *complete* if

$$T \vdash \varphi \Leftarrow T \vdash \varphi^s.$$

A theory *admits* Skolemization if Skolemization is both sound and complete.

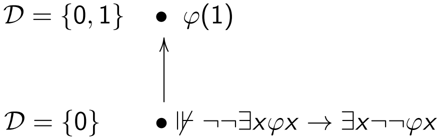
Many nonclassical theories (including IQC) do not admit Skolemization: it is sound but not complete for such theories.

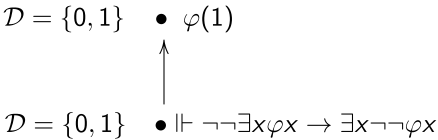
For infix formulas in general not $\Leftarrow$. Examples are

$$\begin{array}{rl} DLEM & \neg\neg\forall x(\varphi x \vee \neg\varphi x) \\ CD & \forall x(\varphi x \vee \psi) \rightarrow \forall x \varphi x \vee \psi \\ EDNS & \neg\neg\exists x \varphi x \rightarrow \exists x \neg\neg\varphi x. \end{array}$$

From now on, φx abbreviates $\varphi(x)$.

Ex $\vdash_{IQC} \neg\neg\varphi c \to \exists x \neg\neg\varphi x$ *but* $\nvdash_{IQC} \neg\neg\exists x\varphi x \to \exists x\neg\neg\varphi x$.





Note If all elements in the domains occur in the domain at the root, then there is no counter model to EDNS.

Extend IQC with an existence predicate E: Et is interpreted as t exists.

Dfn (Scott 1977) The logic IQCE has quantifier rules:

$$\frac{\varphi t \wedge Et}{\exists x \varphi x} \qquad \frac{\exists x \varphi x \qquad \begin{matrix}[\varphi y, Ey] \\ \vdots \\ \psi\end{matrix}}{\psi} \qquad\qquad \frac{\begin{matrix}Ey \\ \vdots \\ \varphi y\end{matrix}}{\forall x \varphi x} \qquad \frac{\forall x \varphi x \wedge Et}{\varphi t} \;.$$

IQCE has a well–behaved sequent calculus.

Note IQCE is conservative over IQC.

Ex $\nvdash_{IQCE} \neg\neg\exists x \varphi x \to \exists x \neg\neg\varphi x$ *and* $\nvdash_{IQCE} \neg\neg(Ec \wedge \varphi c) \to \exists x \neg\neg\varphi x$.

$\nvdash_{IQCE} \forall x(\varphi x \vee \psi) \to \forall x \varphi x \vee \psi$ *and* $\nvdash_{IQCE} \forall x(\varphi x \vee \psi) \to (Ec \to \varphi c) \vee \psi$.

Dfn *The eskolemization of φ is a formula φ^e without strong quantifiers such that there are formulas*

$$\varphi = \varphi_1, \ldots, \varphi_n = \varphi^e$$

such that φ_{i+1} is the result of replacing the leftmost strong quantifier $Qx\psi(x, \bar{y})$ in φ_i by

$$\begin{cases} E(f\bar{y}) \to \psi(f\bar{y}, \bar{y}) & \text{if } Q = \forall \\ E(f\bar{y}) \wedge \psi(f\bar{y}, \bar{y}) & \text{if } Q = \exists, \end{cases}$$

where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi(x, \bar{y})$ occurs, and f_i does not occur in any φ_j with $j \leq i$.

If only existential qfs are replaced, the result is denoted by φ^E.

Thm *If φ is in prenex normal form, then $\vdash_{IQCE} \varphi \Leftrightarrow \vdash_{IQCE} \varphi^e$.*

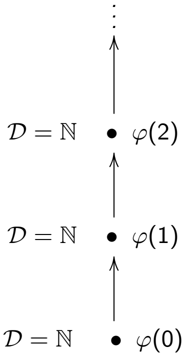
Thm (Baaz&Iemhoff 2011)
For theories T not containing the existence predicate:

$$T \vdash_{IQC} \varphi \;\Leftrightarrow\; T \vdash_{IQCE} \varphi \;\Leftrightarrow\; T \vdash_{IQCE} \varphi^E.$$

Note *For universal quantifiers eskolemization is not complete:*

$$\nvdash_{IQCE} \forall x \neg\neg\varphi x \to \neg\neg\forall x \varphi x \qquad \vdash_{IQCE} \forall x \neg\neg\varphi x \to \neg\neg(Ec \to \varphi c).$$

Ex *Model in which* $\forall x \neg\neg\varphi x \to \neg\neg\forall x \varphi$ *is not forced:*
(Note that for no $d \in \mathbb{N}$, $\varphi(d) \leftrightarrow \forall x \varphi(x)$ *is forced in the model.)*



- *For which (intermediate) logics is eskolemization complete?*
- *For which Skolem functions is skolemization sound and complete?*
- *Are there useful alternative skolemization methods?*

Dfn A model has the *witness property* if for all nodes k refuting a formula $\forall x \varphi x$ there is a $l \succcurlyeq k$ and $d \in D_l$ such that $l \nVdash \varphi d$ and $l \Vdash \varphi d \leftrightarrow \forall x \varphi x$.

Note Every conversely well-founded model has the witness property.

A theory has the *witness property* if it is sound and complete w.r.t. a class of well-founded models that all have the witness property.

Thm (Baaz&Iemhoff 2016) For all theories T with the witness property:

$$T \vdash \varphi \Leftrightarrow T \vdash \varphi^e.$$

Cor Eskolemization is sound and complete for all theories complete w.r.t. a class of well-founded and conversely well-founded models. This holds in particular for theories with the finite model property.

The previous results have the following thoeerems, proved by Craig Smoryński in the 1970s, as a corollary.

Thm (Smoryński)
The constructive theory of decidable equality is decidable.

Thm (Smoryński)
The constructive theory of decidable monadic predicates is decidable.

- *For which (intermediate) logics is eskolemization complete?*
  At least those with the witness property.
- *For which Skolem functions is skolemization sound and complete?*
- *Are there useful alternative skolemization methods?*

For which Skolem functions is skolemization sound and complete?

Aim *Extend IQCE in a minimal way to a theory, say IQCO, that admits a translation, say $(\cdot)^o$, close to Skolemization.*

Dfn $\mathcal{L}$ can be any first-order language, $\mathcal{L}_s$ consists of (Skolem) function symbols and $\mathcal{L}_o$ consists of the constant ι, unary predicates E and W and binary predicate $\preccurlyeq$ and binary function $\langle\cdot,\cdot\rangle$.

$k\!:\!A(\bar{a})$ is short for $A(\langle k, a_1\rangle, \ldots, \langle k, a_n\rangle)$ and should be thought of as $k \Vdash A(\bar{a})$. $k\!:\!E\bar{a}$ is short for $k\!:\!Ea_1, \ldots, k\!:\!Ea_n$.

IQCO is an extenion of IQCE, with axioms stating that $\preccurlyeq$ is a preorder with root ι, that $\langle\cdot,\cdot\rangle$ is upwards persistent,

$$(k\!:\!P\bar{x}) \wedge (k\!:\!E\bar{x}) \wedge k \preccurlyeq l \rightarrow l\!:\!P\bar{x},$$

and that terms in $\mathcal{L}$ exist,

$$k\!:\!E\bar{x} \rightarrow k\!:\!Ef(\bar{x}) \quad (\text{for all } f \in \mathcal{L}).$$

In particular, for closed terms t in $\mathcal{L}$, $k\!:\!Et$ is an axiom.

Dfn CQCO is the classical version of IQCO.

IQCO is a formula based version of Sara Negri's labelled calculus but for IQCE instead of IQC. CQCO and IQCO have cut–free sequent calculi.

Dfn The orderization of φ is $\varphi^o = \varphi^{ks} = (\varphi^k)^s$, where $(\cdot)^k$ is defined as

$$\begin{array}{lcl} P^k & = & k:P \ (P \text{ atomic}) \\ \multicolumn{3}{l}{(\cdot)^k \text{ commutes with } \wedge \text{ and } \vee} \\ (\varphi \to \psi)^k & = & \forall l \succcurlyeq k(\varphi^l \to \psi^l) \\ (\exists x \varphi x)^k & = & \exists x(k:Ex \wedge \varphi^k x) \\ (\forall x \varphi x)^k & = & \forall l \succcurlyeq k \forall x(l:Ex \to \varphi^l x). \end{array}$$

Ex For $f, g \in \mathcal{L}_s$,

$$\big(\exists x \forall y P(x,y)\big)^o = \big(\exists x \forall y P(x,y)\big)^{ks} =$$
$$\Big(\exists x(k:Ex \wedge \forall l \succcurlyeq k \forall y \big(l:Ey \to l:P(x,y)\big)\Big)^s =$$
$$\exists x\big(k:Ex \wedge \big(fx \succcurlyeq k \wedge fx:Egx \to fx:P(x,gx)\big)\big).$$

Ex Because of the existence of a counter model to $Px \vee \neg Px$:

$$\nvdash_{IQCO} (Px \vee \neg Px)^o.$$

Thm (Baaz&Iemhoff 2008)
Any theory T in $\mathcal{L}$ admits orderization, i.e. for all φ in $\mathcal{L}$:

$$T \vdash_{IQC} \varphi \Leftrightarrow T^k \vdash_{IQCO} \varphi^k \Leftrightarrow T^k \vdash_{CQCO} \varphi^k \Leftrightarrow T^k \vdash_{CQCO} \varphi^o \Leftrightarrow T^k \vdash_{IQCO} \varphi^o.$$

Note *For T^k close to T, the above is a genuine Skolemization theorem.*

Dfn *If T contains equality, T_k is the extension of T with, for all $f \in \mathcal{L}$, the axioms*

$$(\bar{x} = \bar{y} \to f\bar{x} = f\bar{y})^k .$$

Lem *T_k is $\mathcal{L}$-conservative over T.*

Thm *For any theory T in $\mathcal{L}$ such that the antecedents of the axioms only contain predicates on free variables and the succedents are atomic formulas that derive T^k, T_k admits orderization:*

$$T \vdash_{IQC} \varphi \Leftrightarrow T_k \vdash_{IQCO} \varphi^k \Leftrightarrow T_k \vdash_{CQCO} \varphi^k \Leftrightarrow T_k \vdash_{CQCO} \varphi^o \Leftrightarrow T_k \vdash_{IQCO} \varphi^o.$$

Applications: the intuitionistic theory of apartness, of groups, ...

Note In IQCO the Skolem functions are relations that are not necessarily functional.

Partial: $E\bar{x} \Rightarrow Ef(\bar{x})$ *is an axiom only for* $f \in \mathcal{L}$.

Not functional: if equality is present in T, *then*

$$\bar{x} = \bar{y} \Rightarrow f\bar{x} = f\bar{y}$$

only holds for the functions in $\mathcal{L}$ *but not necessarily for the Skolem functions.*

- *For which (intermediate) logics is eskolemization complete?*
  At least those with the witness property.
- *For which Skolem functions is skolemization sound and complete?*
  For relations that are not necessarily functional.
- *Are there useful alternative skolemization methods?*

The answer depends on the meaning of "alternative skolemization method".

Dfn An *alternative Skolemization method* is a computable total translation $(\cdot)^a$ from formulas to formulas such that for all formulas φ, φ^a does not contain strong quantifiers. A theory T *admits* the alternative Skolemization method if

$$T \vdash \varphi \Leftrightarrow T \vdash \varphi^a. \tag{1}$$

The method is *strict* if for every Kripke model K of T and all formulas φ:

$$K \nVdash \varphi^a \;\Rightarrow\; K \nVdash \varphi. \tag{2}$$

Ex Replacing quantifiers $\exists x \psi(x, \bar{y})$ and $\forall x \psi(x, \bar{y})$ by

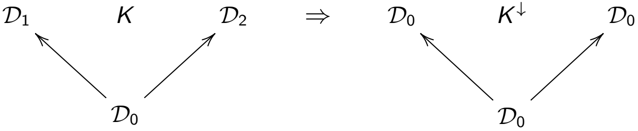
$$\bigvee_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) \text{ and } \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}),$$

respectively, is an alternative Skolemization method, admitted by any intermediate logic with the finite model property (Baaz&Iemhoff 2016).

Dfn *Given a class of Kripke models $\mathcal{K}$, $\mathcal{K}_{cd}$ denotes the set of those models in $\mathcal{K}$ that have constant domains.*

Dfn *For a Kripke model K:*

- *$K^{\downarrow}$ denotes the Kripke model that is the result of replacing every domain in K by the domain at the root of K and defining, for elements $\bar{d}$ in D: $K^{\downarrow}, k \Vdash P(\bar{d})$ iff $K, k \Vdash P(\bar{d})$.*
- *$K^{\uparrow}$ denotes the Kripke model that is the result of replacing every domain in K by the union of all domains in K and defining, for elements $\bar{d}$ in that union: $K^{\downarrow}, k \Vdash P(\bar{d})$ if $K, k \Vdash P(\bar{d})$ and $\bar{d}$ are elements in D_k.*



Dfn *The strong quantifier free fragment (sqff) of a theory consists of those theorems of the theory that do not contain strong quantifiers, and likewise for weak quantifiers.*

Thm *Let T be a theory that is sound and complete with respect to a class of Kripke models $\mathcal{K}$ closed under $\uparrow$ and $\downarrow$, then the sqff of T is sound and complete with respect to $\mathcal{K}_{cd}$, and so is the wqff.*

Thm (I. 2017) *Except for CQC, there is no intermediate logic that is sound and complete with respect to a class of frames and that admits a strict, alternative Skolemization method.*

Cor *The intermediate logics IQC,*

- *QDn (the logic of frames of branching at most n),*
- *QKC (the logic of frames with one maximal node),*
- *QLC (the logic of linear frames),*

and all tabular logics, do not admit any strict, alternative Skolemization method.

- *For which (intermediate) logics is eskolemization complete?*

  At least for those with the witness property.

- *For which Skolem functions is skolemization sound and complete?*

  For relations that are not necessarily functional.

- *Are there useful strict alternative skolemization methods?*

  Not for any intermediate logic that is sound and complete with respect to a class of frames.

- *Skolemization in Gödel predicate logics (Baaz, Metcalfe, Cintula).*
- *Complexity of Skolemization (Baaz&Leitsch).*
- *Deskolemization (Baaz&Hetzl&Weller).*
- *Complexity of Skolemization (Avigad).*

- *Are there useful alternative nonstrict Skolemization methods?*
- *Can the result on orderization be improved?*
- *What are the philosophical implications of the results thus far?*

Finis