The Universal Model for the negation-free fragment of IPC

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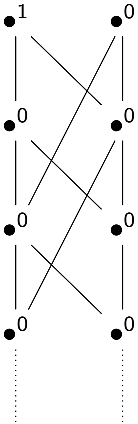
The n-universal model for IPC, $\mathcal{U}(n) = (U(n), R, V)$ is the "least" model of IPC that witnesses the failure of every unprovable formula of IPC.

- The first layer $\mathcal{U}(n)^1$ consists of 2^n nodes with the 2^n different n-colors under the discrete ordering.
- Under each element w in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.
- Under any finite anti-chain X with at least one element in $\mathcal{U}(n)^k \setminus \mathcal{U}(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.

The whole model $\mathcal{U}(n)$ is the union of its layers.

The Rieger-Nishimura ladder:



Lemma

For any finite rooted Kripke n-model $\mathfrak{M}$, there exists a unique $w \in \mathcal{U}(n)$ and a p-morphism of $\mathfrak{M}$ onto $\mathcal{U}(n)_w$.

Theorem

For any n-formula φ, $\mathcal{U}(n) \models \varphi$ iff $\vdash_{IPC} \varphi$.

Proposition

For every $w \in \mathcal{U}(n)$ we have that

- $V(\varphi_w) = R(w)$, *where* $R(w) = \{w' \in \mathcal{U}(n) | wRw'\}$;
- $V(\psi_w) = \mathcal{U}(n) \backslash R^{-1}(w)$, *where* $R^{-1}(w) = \{w' \in \mathcal{U}(n) | w'Rw\}$.

For any node w in an n-model $\mathfrak{M}$, if $w \prec \{w_1, \ldots, w_m\}$, then we let
$\mathrm{prop}(w) := \{p_i | w \models p_i, 1 \leq i \leq n\}$,
$\mathrm{notprop}(w) := \{q_i | w \not\models q_i, 1 \leq i \leq n\}$,
$\mathrm{newprop}(w) := \{r_j | w \not\models r_j$ and $w_i \models r_j$ for each $1 \leq i \leq m$, for $1 \leq j \leq n\}$.

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge \mathrm{prop}(w) \wedge \bigwedge \{\neg p_k | p_k \in \mathrm{notprop}(w), 1 \leq k \leq n\},$$

and

$$\psi_w := \neg \varphi_w.$$

If $d(w) > 1$, and $\{w_1, \ldots, w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w := \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$

and

$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

Lemma

For any $w \in \mathcal{U}(n)$, let φ_w be the de Jongh formula of w, then we have that $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$.

Lemma

$Upper(\mathcal{H}(n))$ is isomorphic to $\mathcal{U}(n)$.

Definition (Top-Model Property)

φ has the *top-model property* (TMP), if for all $\mathfrak{M}$, w, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^+, w \models \varphi$, where $\mathfrak{M}^+$ is obtained by adding a top point t such that all proposition letters are true in t.

Proposition

1. *If $\varphi \in [\vee, \wedge, \rightarrow]$ then it has the TMP, and so has $\bot$.*
2. *For any formula φ, there exists a formula $\varphi^* \in [\vee, \wedge, \rightarrow]$ or $\varphi^* = \bot$ such that for any top-model $(\mathfrak{M}^+, w)$, $(\mathfrak{M}^+, w) \models \varphi \leftrightarrow \varphi^*$.*

The n-universal model for the negation-free fragment of IPC, $\mathcal{U}^{\star}(n) = (U^{\star}(n), R^{\star}, V^{\star})$, is a generated submodel of the universal model for IPC. It is (generated by):

$$\{u \in U(n) : \neg uRw_0\}$$

where w_0 is the maximal element of $\mathcal{U}(n)$ that satisfies all propositional atoms.

- The first layer $\mathcal{U}^\star(n)^1$ consists of $2^n - 1$ nodes with all the different n-colors – *excluding the color* $1 \ldots 1$ – under the discrete ordering.
- Under each element w in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}^\star(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.
- Under any finite anti-chain X with at least one element in $\mathcal{U}^\star(n)^k \setminus \mathcal{U}^\star(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}^\star(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.

The whole model $\mathcal{U}^\star(n)$ is the union of its layers.

Definitions
Examples

The 1-universal model is a singular point:

For $n \geq 2$ it is infinite.

Definitions

Positive morphisms

Definition

A *positive morphism* is a partial function $f : (W, R, V) \to (W', R', V')$ such that:

1. $\mathrm{dom}(f) \supseteq \{w \in W : \exists p \in \mathrm{Prop}(w \notin V(p))\}$.
2. If $w, v \in \mathrm{dom}(f)$ and wRv then $f(w)R'f(v)$.
3. If $w \in \mathrm{dom}(f)$ and $f(w)R'v$ then there exists some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu (**back**).
4. If $w \in \mathrm{dom}(f)$ and vRw, then $v \in \mathrm{dom}(f)$ (**downwards closed**).
5. For every $p \in \mathrm{Prop}$ we have $w \in V(p) \iff f(w) \in V'(p)$.

If the models are descriptive we furthermore require for every $Q \in \mathcal{Q}$ that $W \setminus R^{-1}(f^{-1}[W' \setminus Q]) \in \mathcal{P}$.

These maps restrict strong partial Esakia morphisms.

Lemma

Let $f : (W, R, V) \to (W', R', V')$ *be a positive morphism. Then for every* $\varphi \in [\vee, \wedge, \to]$ *and* $w \in \mathrm{dom}(f)$ *we have that*

$$(W, R, V), w \models \varphi \quad \textit{if and only if} \quad (W', R', V'), f(w) \models \varphi.$$

Proof.

If $(W', R', V'), f(w) \models \varphi \to \psi$ then if $(W, R, V), v \models \varphi$ with wRv, then either $v \in \mathrm{dom}(f)$ and we use the induction hypothesis, or $v \notin \mathrm{dom}(f)$, i.e. it satisfies all propositional atoms and hence $(W, R, V), v \models \psi$. □

Lemma

There exists a positive morphism $F : \mathcal{U}(n) \to \mathcal{U}^\star(n)$, that is onto and for every $w \in \mathrm{dom}(F)$ we have that $F \restriction \mathcal{U}(n)_w$ is onto $\mathcal{U}^\star(n)_{F(w)}$.

Proof.

We construct F by induction on the levels of $\mathcal{U}(n)$. If $w \prec \{w_1, \ldots, w_k\}$, take $A \subseteq F[\{w_1, \ldots, w_k\}]$ the set that contains the $R^\star$-minimal elements of $F[\{w_1, \ldots, w_k\}]$. If A is empty then let $F(w)$ to be the element of $\mathcal{U}^\star(n)$ with depth 1, with the same color as w. If $A = \{u\}$ and u has the same color as w then let $F(w) = u$. Otherwise by the construction of $\mathcal{U}^\star(n)$ there a unique $v \prec A$ (by the induction hypothesis about F) with the same color as w and we let $F(w) = v$. □

Theorem

For any finite rooted intuitionistic n-model $\mathfrak{M} = (M, R, V)$ such that for some $x \in M$ and $p \in \mathrm{Prop}$ with $x \notin V(p)$, there exists unique $w \in U^\star(n)$ and positive morphism of $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_w$.

Proof.

We know there is a unique such p-morphism to the universal model. We take the composition with F. It is still unique since otherwise if g_1, g_2 were different positive morphism, since $\mathrm{dom}(g_1) = \mathrm{dom}(g_2) = \{x \in M : \exists p \in \mathrm{Prop}(x \notin V(p))\}$, we would have two different p-morphisms from $\mathrm{dom}(g_1)$ to $\mathcal{U}(n)$, a contradiction. □

Theorem

For every n-formula $\varphi \in [\vee, \wedge, \rightarrow]$, $\mathcal{U}^\star(n) \models \varphi$ *if and only if* $\vdash_{IPC} \varphi$.

Proof.

Follows from previous Lemma. □

We have that $(\mathcal{U}^\star(n))^+$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$, whose domain consist of the elements of $U(n)$ whose only successor of depth 1 satisfies all propositional atoms. Let's call this generated submodel $\mathcal{M}$.

Definition

If $d(w) = 1$ then define

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{notprop}(w))$$

and

$$\psi^\star_w = \varphi^\star_w \to \bigwedge_{i \in n} p_i.$$

Definition

If $d(w) > 1$ then let $w \prec \{w_1, \ldots, w_r\}$ and define

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i \leq r} \psi^\star_{w_i} \to \bigvee_{i \leq r} \varphi^\star_{w_i})$$

and

$$\psi^\star_w = \varphi^\star_w \to \bigvee_{i \leq r} \varphi^\star_{w_i}.$$

Proposition

For every $w \in \mathcal{U}^\star(n)$ we have that

- $V^\star(\varphi_w) = R^\star(w)$
- $V^\star(\psi_w) = \mathcal{U}^\star(n) \setminus (R^\star)^{-1}(w)$

Proof.

We can show that for every world w in $\mathcal{M}$, φ_w is top-model equivalent to $\varphi_w^\star$. And since $\varphi_w^\star$ is negation free it is satisfied in a world of $(\mathcal{U}^\star(n))^+$ if and only if it is satisfied in the same world in $\mathcal{U}^\star(n)$. □

We denote the n-Henkin model for the $[\vee, \wedge, \to]$ fragment of IPC with $\mathcal{H}^\star(n)$. We write

$$\mathrm{Cn}^\star_n(\varphi) = \{\psi \in [\vee, \wedge \to] : \psi \text{ is an } n\text{-formula and } \vdash_{IPC} \varphi \to \psi\}$$

and we write

$$\mathrm{Th}^\star_n(\mathfrak{M}, w) = \{\varphi \in [\vee, \wedge \to] : \varphi \text{ is an } n\text{-formula and } \mathfrak{M}, w \models \varphi\}.$$

Proposition

For any point $w \in \mathcal{U}^\star(n)$, $\mathrm{Th}^\star_n(\mathcal{U}^\star(n), w) = \mathrm{Cn}^\star_n(\varphi^\star_w)$.

Proof.

If $\nvdash_{IPC} \varphi^\star_w \to \sigma$, then this is witnessed in some world v of $\mathcal{U}^\star(n)$. We have that $v \in R^\star(w)$, hence $\sigma \notin \mathrm{Th}^\star_n(\mathcal{U}^\star(n), w)$. □

Proposition

For any $w \in \mathcal{U}^\star(n)$ we have $\mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi^\star_w)} \cong (\mathcal{U}^\star(n)_w)^+$.

Proof.

We have that $g : (\mathcal{U}^\star(n)_w)^+ \to \mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi^\star_w)}$, such that $g(v) = \mathrm{Cn}^\star_n(\varphi^\star_v)$ and the topmost element is mapped to the set of all negation-free formulas, is the isomorphism. If $\Gamma \supseteq \mathrm{Cn}^\star(\varphi^\star_w)$, then $\Gamma = \mathrm{Cn}^\star(\varphi^\star_v)$ for $wR^\star v$, or it contains all propositional atoms: If there is some v such that $\varphi_v \in \Gamma$ but for all immediate successors of v, v_i $\varphi^\star_{v_i} \notin \Gamma$ $(i \in n+1)$ for $\sigma \in \Gamma$ we have $\sigma \wedge \varphi^\star_v \nvdash_{IPC} \varphi^\star_{v_0} \vee \cdots \vee \varphi^\star_{v_n}$. Then this is witnessed in $\mathcal{U}^\star(n)$, exactly at v, hence $\sigma \in \mathrm{Cn}^\star(\varphi^\star_v)$. □

Corollary

It is the case that $Upper(\mathcal{H}^\star(n)) \cong (\mathcal{U}^\star(n))^+$.

Corollary

Let $\mathfrak{M} = (M, R, V)$ *be any n-model and let* $x \in M$ *be such that* $\mathfrak{M}, x \models \varphi_w^\star$, *for some* $w \in U^\star(n)$. *Then either there are a unique* $v \in U^\star(n)$ *such that* $wR^\star v$, *and a positive morphism* f *from* $\mathfrak{M}_x$ *onto* $\mathcal{U}^\star(n)_v$ *or* $\mathfrak{M}_x$ *satisfies all negation-free formulas.*

Proof.

Define $f(y) = v$, where $\mathrm{Th}_n^\star(\mathfrak{M}, y) = \mathrm{Cn}_n^\star(\varphi_v^\star)$. □

Theorem

For every descriptive frame $\mathfrak{G}$ and $w \in U^\star(n)$ we have that $\mathfrak{G} \not\models \psi^\star_w$ if and only if there is an n-valuation V on $\mathfrak{G}$ such that $\mathcal{U}^\star(n)_w$ is the image, through a positive morphism, of a generated submodel of $(\mathfrak{G}, V)$.

Proof.

If $w \prec \{w_1, \ldots, w_n\}$, then take the submodel generated by the elements that satisfy $\varphi^\star_w$ but none of the $\varphi^\star_{w_i}$. The previous corollary gives the positive morphism. $\square$

Lemma

If $\mathfrak{F}$ is a descritpive frame with a topmost element, and $f : (\mathfrak{G}, V) \to (\mathfrak{F}, V')$ is a descriptive positive morphism between models, then f can be extended to a descriptive frame p-morphism.

Proof.

If f is a total then it is a frame p-morphism. If f is not total then, extend f to f' such that every $y \in \mathrm{dom}(\mathfrak{G}) \setminus \mathrm{dom}(f)$, $f'(y) = x_0$, where x_0 is the topmost element of $\mathfrak{F}$. To show that it is descriptive we need that $f'^{-1}[Q]$ is admissible, where Q is admissible in $\mathfrak{F}$. But, by the construction of f we have that $f'^{-1}[Q] = f^{-1}[Q] \cup (\mathrm{dom}(\mathfrak{G}) \setminus \mathrm{dom}(f))$, which is admissible by assumption. □

Theorem (Jankov)

For every logic $\mathcal{L} \not\subseteq \mathrm{KC}$ which is complete with respect to a class of Kripke frames there exists some negation-free formula σ such that $\mathcal{L} \vdash \sigma$ while $\mathrm{IPC} \nvdash \sigma$.

Proof.

Let χ be the formula that $\mathcal{L}$ proves. Let $\mathfrak{F}$ a finite KC frame, a counterexample to χ. We give a valuation V to $\mathfrak{F}$ such that at every world a propositional atom is not true. Given a $\mathcal{L}$-frame, $\mathfrak{G}$, if for any valuation it satisfies the same negation free formulas as $(\mathfrak{F}, V)$ then by the previous theorem there is a descriptive positive morphism onto $\mathfrak{F}$. This can be extended to a descriptive p-morphism by the above lemma, a contradiction. □

References

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