# Gödel

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## 1 Kurt Gödel

Kurt Friedrich Gödel, "established, beyond comparison, as the most important logician of our times," as Solomon Feferman wrote in 1986, founded the modern, metamathematical era in mathematical logic. His Incompleteness Theorems, arguably the most significant achievements in logic since, perhaps, those of Aristotle, are among the handful of landmark theorems in twentieth century mathematics, a century which saw an unprecedented number of important theorems. His work touched every field of mathematical logic, if it was not in most cases their original stimulus. In his philosophical work Gödel formulated and defended mathematical Platonism, involving the view that mathematics is a descriptive science, and that the concept of mathematical truth is an objective one; on the basis of that viewpoint he laid the foundation for the program of conceptual analysis within set theory. He adhered to Hilbert's "original rationalistic conception" in mathematics; he was prophetic in anticipating and emphasizing the importance of large cardinals in set theory before their importance became clear.

## 2 Biographical Sketch

Kurt Gödel was born on April 28, 1906 in what was then the Austro-Hungarian city of Brünn, and what is now Brno in the Czech Republic.

Gödel's father Rudolf August was a businessman of a reportedly quite practical turn of mind, according to Gödel's brother Rudolf, and his mother Marianne was a well-educated and cultured woman to whom Gödel remained close thoughout his life, as witnessed by the long and wide-ranging correspondence between them. The family was well off, and Gödel's childhood was an uneventful one, with one important exception; namely, from about the age of four Gödel suffered frequent episodes of poor health, including contracting rheumatic fever at age six, and such health problems as he suffered then as well as others of various kinds were to plague him his entire life.

Health problems notwithstanding, Gödel proved to be an exemplary student at primary school and later the Gymnasium, excelling especially in mathematics, languages and religion. Upon his graduation from the Gymnasium in Brno in 1924 Gödel enrolled in the University of Vienna, attending lectures on physics, his initial field of interest, lectures on philosophy given by Heinrich Gomperz, and lectures on mathematics.

Incidentally, with no small effort on the part of Marianne Schimanovich Gödel's university transcripts have been located – the document listing what courses he took at the University of Vienna, and it turns out that Gödel took an enormous number of courses in physics, right up through general relativity, well after he decided not to go into the area.

Philipp Furtwängler was one of his mathematics professors, and Furtwängler's course on class field theory almost tempted Gödel to pursue his studies in that area. Gödel learned his logic from Rudolph Carnap and from Hans Hahn,[1] eventually graduating under Hahn with a Dr.Phil. in mathematics in 1929, having submitted a dissertation in which he proved the completeness theorem for first order logic. Gödel's university years are notable for marking the beginning of his attendance at meetings of the Vienna Circle, a group around Moritz Schlick now mostly identified with the doctrine of logical positivism.

The 1930's were a prodigious decade for Gödel, though also an extremely turbulent one. After turning in his dissertation in 1929, he published his groundbreaking incompleteness theorems in 1931, on the basis of which he was granted his Habilitation in 1932 and a Privatdozentur at the University of Vienna in 1933. Along with attending meetings of the Vienna Circle, he attended Karl Menger's colloquium from 1929 (where he met Alfred Tarski for the first time). He made three visits to the United States during this decade, the first and second to the Institute for Advanced Study in Princeton, New Jersey in 1933 and 1935, the third in 1938 to both the Institute

[1] Hans Hahn worked in set theory and functional analysis, and is the co-author of the Hahn-Banach Theorem.

and to Notre Dame, the latter at the invitation of Karl Menger. Among his mathematical achievements at the decade's close is the proof of the consistency of both the Axiom of Choice and Cantor's Continuum Hypothesis with the Zermelo-Fraenkel axioms for set theory, obtained in 1935 and 1937, respectively. Gödel also published a number of significant papers on modal and intuitionistic logic and arithmetic during this period. Principal among the latter is Gödel's 1933 "On Intuitionistic Arithmetic and Number Theory," in which Gödel shows that classical first order arithmetic is interpretable in Heyting arithmetic by a simple translation. The 1930's also saw the publication by Gödel on a wide range of other topics in logic and mathematics, ranging from the decision problem for the predicate calculus, to those on the length of proofs, to those on differential and projective geometry. By the end of the decade both Gödel's advisor Hans Hahn and Moritz Schlick (the latter was assassinated by an ex-student) had died, two events which led to a personal crisis for Gödel (as would the death of Einstein in 1955).

In 1940 he and his wife Adele, whom he had married in 1938, emigrated to the United States, where they both were eventually granted American citizenship in April 1948. The change in atmosphere induced by moving from what was then a capital of world culture, though this was of course in the process of being destroyed by the Nazis, to the sleepy academic town of Princeton, New Jersey, at least Adele found difficult to cope with. (And knowing Princeton as I do, I must say I quite sympathize.)

Upon their arrival Gödel took up an appointment as an ordinary member at the Institute for Advanced Study; he would become a permanent member of the Institute in 1946 and would be granted his professorship in 1953. He would remain at the Institute until his retirement in 1976. Gödel never returned to Europe. Various difficulties concerning his position at the University of Vienna, as well as the general situation in Austria just then, a situation which became increasingly dire for Gödel in that he was found fit for military service by the Nazi government in 1939, were undoubtedly decisive in influencing his decision to leave Austria.

So Gödel and his wife were out walking one night when they were actually beaten up by some members of the Hitler youth, or rather Gödel was – Adele tried to fend them off.

The Gödels' immigration to the United States was a long and difficult episode in their life, and is recounted by John Dawson in his biography of Gödel called "Logical Dilemmas,"[2] as well as by Solomon Feferman in "Gödel's Life and Work,"[3] to both of which the reader is referred. Because they had to avoid crossing the Atlantic they actually crossed Russia to Japan, then sailed to San Francisco.

Gödel's early years at the Institute were notable for his close friendship with his daily walking partner Albert Einstein – who said at one point that the only reason he went to the Institute anymore was to walk home with Gödel – , as well as for his turn to philosophy of mathematics, a field on which Gödel began to concentrate almost exclusively from about 1943. The initial period of his subsequent lifelong involvement with philosophy was a fruitful one (in terms of publications): in 1944 he published his first philosophical paper, entitled "On Russell's Mathematical Logic," which Kreisel has called a kind of survey of the history of logic, and in 1947 Gödel published his second, entitled "What is Cantor's Continuum Hypothesis?"[4] in which he suggests that even so that the continuum hypothesis, which we will hear about later from Benedikt, will turn out to be independent of the axioms of ZFC set theory, still it is a meaningful question, unlike for example the question of the truth of the parallel postulate in geometry, which he says has lost its meaning, because of the independence proofs. So sometimes an independence result entails a loss of meaning of the question of the truth of an axiom, and sometimes it doesn't. In fact Gödel gives a very careful analysis of this phenomenon in the paper. In 1949 he published his third, entitled "A Remark on the Relationship between Relativity Theory and Idealistic Philosophy." He argues there that relativity theory actually vindicates Kant's subjective view of time, a view Gödel himself doesn't actually agree with. The paper must have been occasioned by results on rotating universes in relativity he had obtained in 1949, which were first published in 1949 in an article entitled: "An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation."[5]

Among Gödel's other significant philosophical works of the 1940's must be counted his 1941 lecture at Yale entitled "In What Sense is Intuitionistic Logic Constructive?",[6] in which the notion: "computable function of finite type" is introduced. The corresponding paper is entitled "Über eine bisher

[2]see [?]
[3]see [?]
[4]A revision of the latter was published in 1964.
[5]All three of these are published in [?].
[6]published in [?]

noch nicht benützte Erweiterung des finiten Standpunktes," and was published in 1958. The interpretation of classical arithmetic into intuitionistic arithmetic in it became known as the "Dialectica Interpretation," after the journal in which the article was published.[7] The analysis of the notion of finite, or finitary intuition given there has led some to think of this as Gödel's finest work in philosophy.

Finally the 1940's saw the beginning of Gödel's intensive study of Leibniz, which, Gödel reports, occupied the period from 1943 to 1946.[8]

The 1950's saw a deepening of Gödel's involvement with philosophy: In 1951 Gödel delivered a philosophical lecture at Brown University, later known as the Gibbs Lecture, entitled "Some basic Theorems on the Foundations of Mathematics and Their Philosophical Implications." In this lecture he presents his famous "disjunctive theorem," a philosophical theorem according to Gödel, about the question whether the mind is a finite machine. This is in line with the idea, taken from Leibniz but also present always in some form in the background of Gödel's thinking, of developing philosophy in an exact manner. From 1953 to 1959 Gödel worked on a submission to the Schilpp volume on Rudolph Carnap entitled "Is Mathematics a Syntax of Language?" The answer Gödel offers in the paper to the question is no – mathematics is irreducibly contentual. The second incompleteness theorem is used in the argument, the suggestion being that even if one could reduce mathematics to a formal game of symbols, i.e. a meaningless system of rules governing a syntax of finite strings of symbols, one would always have to import concepts exterior to the system, in order to recognize its consistency.

Gödel published neither of these two important manuscripts in his lifetime, although both would appear on two lists which were found in the Gödel Nachlass, entitled "Was ich publizieren könnte."[9]

The decade's close saw the inclusion of an important new element in Gödel's philosophical scheme: namely he came to emphasize the role of the subject. Platonism is often thought of as a doctrine asserting the existence of abstract objects, with less emphasis placed on the role of the subject, or even on the relationship of the subject to this realm of objects. In converting to Husserlian phenomenology as a means to systematize the Leibnizian views that had been in place since the 1930s, Gödel incorporated the subjective

[7] see [**?**]

[8] Though according to Menger, Leibniz was an important interest of Gödel's already in the 1930s. See [**?**].

[9] In English: What I could publish.

element into his realist or platonist framework – and perhaps this is indeed the original meaning of Platonism.

Gödel's final years are notable for his circulation of two manuscripts: "Some considerations leading to the probable conclusion that the true power of the continuum is $\aleph_2$," his attempt to derive the value of the continuum from the so-called scale axioms of Hausdorff, and his "ontologischer Beweis," which he entrusted to Dana Scott in 1970 (though it appears to have been written earlier).[10], an exercise in exact theology and modal logic in the spirit of Leibniz and others. Taken together, the two manuscripts are the fitting "last words" of someone who, in a fifty year involvement with mathematics and philosophy, pursued, or more precisely, *sought the grounds* for pursuing those two subjects under the single heading: "strenge Wissenschaft" – an attitude, or turn of mind, or wish, if you will, Gödel held from his start in 1929, when at the age of twenty-three he opened his doctoral thesis with some philosophical remarks (which we consider below).

Gödel died in Princeton on January 14, 1978 at the age of 71. His death certificate records the cause of death, tragically, as "starvation and inanition, due to personality disorder." His wife Adele survived him by three years.

Before leaving Gödel's biography to take up some of the ideas surrounding the completeness theorem, I would like to ask the question, what kind of person was Gödel? There are of course many anecdotes told about him by people who knew him – and many others have passed judgement on his character, most conspicuously his two biographers to date (John Dawson and Rebecca Goldstein).

But one very revealing remark in my judgement is from a man named Dominic Angelucci, an extremely colorful Princeton character who happens to be the security guard, to this day, at the Institute for Advanced Study in Princeton. In the 1970s Dominic was given the job of picking Gödel up from his house every morning and driving him to the Institute, and then driving him home at the end of the day. So he saw quite a bit of Gödel. I asked him one time, what kind of person was Gödel? and Dominic answered: some of the people around here, they are very smart but they have their noses stuck up in the air. Gödel was not like that. He was a very nice man.

[10]See [**?**].

## 3 The Completeness Theorem and the emergence of the Incompleteness Theorems

I would now like to talk about the completeness theorem, in particular how the analysis of finite provability in it led inevitably (I shall argue) to the discovery of incompleteness.

First, the completeness theorem is as follows:

A “logical expression” in Gödel’s terminology is a well-formed first order formula without identity. An expression is “refutable” if its negation is provable, “valid” if it is true in every interpretation and “satisfiable” if it is true in some interpretation. The Completeness Theorem is stated as follows:

**Theorem 1.** *Every valid logical expression is provable. Equivalently, every logical expression is either satisfiable or refutable,*

A few observations before moving on to incompleteness: first, note that the completeness theorem tells us the following: if a first order sentence holds in all models, it must be because of a uniform reason (proof) and not for an accidental reason which happens in each model or in each case for a different reason. This observation is due to Andreas Blass, see ‘Some Semantical Aspects of Linear Logic,’ [**?**]. But I think it has a very Gödelian ring to it.

Second observation: the theorem gives a complete analysis of the notion of a finite first order proof. It has, then, a very important corollary, taking note of the fact that Gödel generalized the result to countably many formulas: and this is the existence of non-standard models, for theories such as Peano Arithmetic and also for set theory. So ZFC for example has countable models, a somewhat paradoxical fact which was already proved by Skolem in 1922, after Lowenheim. And Peano Arithmetic has continuum many non-isomorphic countable models. Thus categoricity – the property of a set of axioms of uniquely characterizing a structure (so a theory is categorical if it has only one model up to isomorphism) – fails for alot of the theories we are interested in. Theories that were thought to be categorical. This particular fact took a while to penetrate the logical scene of the 1930s, but once it did it was very important.

As to undecidability and how this emerges, I now turn to an observation in the introduction to Gödel’s thesis:

# 4 An Observation in the Introduction to Gödel's Thesis

As background, Gödel is reacting to a precept of formalism, that consistency is a sufficient ground for existence. Another way of putting this is, one needs only to establish the *consistency* of an axiom with one's preferred axiom system, say, Zermelo-Frankel set theory, to infer the existence of those objects to which the axioms refer. Gödel is clearly weighing in on the Frege-Hilbert dispute about this very issue, where Hilbert argued for the idea that consistency entails existence, and Frege argued against. Gödel argues against this assertion as follows:

> L.E. Brouwer, in particular, has emphatically stressed that from the consistency of an axiom system we cannot conclude without further ado that a model can be constructed. But one might perhaps think that the existence of the notions introduced through an axiom system is to be defined outright by the consistency of the axioms and that, therefore, a proof has to be rejected out of hand. This definition (if only we impose the self-evident requirement that the notion of existence thus introduced obeys the same operation rules as does the elementary one), however, manifestly presupposes the axiom that every mathematical problem is solvable. Or, more precisely, it presupposes that we cannot prove the unsolvability of any problem. For, if the unsolvability of some problem (in the domain of real numbers, say) were proved, then, from the definition above, there would follow the existence of two non-isomorphic realizations of the axiom system for the real numbers, while on the other hand we can prove the isomorphism of any two realizations. We cannot at all exclude out of hand, however, a proof of the unsolvability of a problem if we observe that what is at issue here is only unsolvability by certain *precisely stated formal* means of inference. . . .

For emphasis we restate the argument: suppose Hilbert's principle — consistency implies existence — is true. Then assuming that there is a proposition $\phi$ about the reals, such that both $\phi$ and $\neg\phi$ are consistent with Hilbert's second order axiomatization, it then follows that there must be two non-isomorphic models $M$ and $N$ of those axioms, by the principle. But Hilbert's second order axiomatization of the reals is categorical,[11] meaning: there is only one model up to isomorphism. We thus obtain a contradiction. So Hilbert's principle must be false. Note that the existence of $M$ and $N$ would follow also from Gödel's own Completeness Theorem, if a first order axiomatization of the reals were used. But then we would not have categoricity and no contradiction would follow.

It is important to note that Gödel assumes the existence of a proposition $\phi$ such that both $\phi$ and $\neg\phi$ are consistent with the axioms. Should Gödel not abandon this assumption rather than Hilbert's Principle, that is, doesn't the fact that Hilbert's axiomatization is categorical force him to conclude that there will be no unsolvable problems about the reals?

What is at issue here is the first order second order distinction and the fact that it is really only with the completeness theorem that the distinction crystallizes, because of categoricity. (The failure of categoricity for set theory was very important to Skolem.) So the completeness theorem is a kind if hierarchy result.

Of course this kind of remark is very easy to make in retrospect, but I believe Gödel saw things this way, though he doesn't express this explicitly here. He seems also to have noticed in these remarks that once you have proved the completeness theorem, it is now, and only now, that it is meaningful to talk about unsolvability: Why? The completeness theorem shows that first order logic is maximal, in the sense of being a proof system strong enough to prove everything that holds in all models. Therefore it becomes meaningful to ask can we NOW decide every proposition. If the proof system is so weak (as in the case of full second order logic) that it cannot prove what holds in the model(s), why would the question of incompleteness be interesting?

We know now that no effective second order axiomatization of the reals admits a completeness theorem, unless we resort to the non-standard models of Henkin, in which case, however, we lose categoricity.

On the philosophical side of things, one may ask, why did Gödel feel pressed to argue against the idea that consistency implies existence? In other words, what is this argument doing in his thesis in the first place?

[11]This fact was claimed by Hilbert about his axiomatization in 'Über den Zahlbegriff,' and there are indications of a proof in unpublished lecture notes from the late 1890's. By the late 1920's it had become 'folklore.'

I would speculate that Gödel may have been worried that what the completeness theorem precisely does is to verify the principle that consistency implies existence: if a theory is consistent, Gödel shows that it has a model. (One should mention Skolem here, though his notion of consistency was somewhat semantic.)

This paragraph was famously not included in the article based on his thesis. Some speculate that his advisor Hans Hahn advised him to leave it out, but in fact we don't know the reason.

In this connection it is perhaps worth pointing out that in some sense this had happened before, in connection with the so-called "ignorabimus debate" in the 1870s. That is to say, once a careful analysis enables one to see how restricted one's methods are, the question of what is going to be undecided by those methods suggests itself very naturally.

The ignorabimus debate, briefly, was initiated in 1872 by an experimental psychologist called Emil du Bois-Reymond, in a lecture entitled: "On the Limits of the Knowledge of Nature."

In the lecture du Bois-Reymond argues, indeed he states that this is provable, in a way exactly "similar to that of a mathematician who has proved the unsolvability of a problem," that science, based as it is on Newtonian mechanics, will have to adopt the doctrine "ignorabimus", a doctrine Hilbert famously renounces in 1900 and then again in his retirement adress in 1928: in other words there will be unsolvable questions. du Bois-Reymond's examples were the nature of matter and force, the origin of consciousness and thirdly the genesis of motion.

The idea, put simply, is this: if the methods are given by Newtonian mechanics, then the limitations of those methods are such that they do not suffice to answer every question natural science may pose, such as the origin of consciousness.

Similarly if the notion of finite proof is that given by the completeness theorem, it is clear, or at least the idea naturally suggests itself, that there will be unsolvable mathematical questions; questions, as Gödel put it, about the reals. I have also pointed out that it is important that one's methods are in some sense maximal, otherwise one would simply extend them.

In the interest of comprehensiveness, here is a second philosophical argument from Gödel's thesis:

## 5 Second Argument

Here Gödel observes that one might advance the following objection to his Completeness Theorem: doesn't the use of the law of excluded middle in its proof 'invalidate the entire completeness proof"?[12] The Completeness Theorem asserts

> 'a kind of decidability,' namely every quantificational formula is either provable or a counterexample to it can be given, whereas 'the principle of the excluded middle seems to express nothing other than the decidability of every problem.'

Thus the proof may be circular: one uses the decidability of every question in order to prove just that assertion. But, Gödel remarks, what he has shown is the provability of a valid formula from 'completely *specified, concretely enumerated* inference rules,"[13] not merely from all rules imaginable; whereas the law of excluded middle is used informally (in the metalanguage as one would say nowadays), meaning that the notion of decidability or solvability asserted by the law is left unspecified. As Gödel puts it:

> ... what is affirmed (by the law of excluded middle) is the solvability not at all through specified means but only through all means that are *in any way imaginable* ... [14]

The step forward provided by Gödel is therefore that if we assume solvability by all means imaginable, then we have, in the case of a sentence of first order predicate calculus, then we have a kinf of reduction, namely we have solvability by very specific means laid out carefully beforehand.

Another way of putting this is: if a first order sentence holds in all models, it must be because of a uniform reason (proof) and not for an accidental reason which happens in each model or in each case for a different reason. This observation is due to Andreas Blass, see 'Some Semantical Aspects of Linear Logic,' [**?**]. But I think it has a very Gödelian ring to it.

[12]See p.63 of [**?**].

[13]italics Gödel's

[14]Gödel remarks in a footnote to this passage that the notion of provability by any means imaginable is perhaps 'too sweeping.' Nevertheless, this does not affect the basic distinction that Gödel wishes to make, between the formal and informal notions of provability.

In retrospect, a large part of Gödel's later thinking is contained in these two arguments, in ovo so to speak.

The first argument also raises the purely historical question of what was known to Gödel about the various axiomatizations due to Hilbert. Another interesting aspect of the incompleteness theorem involves what Gödel knew of Skolem's work prior to writing his thesis, part of which contains a construction very close to Skolem's.

This has been cleared up recently by a recent paper, which describes...