Gödel’s Contribution to Set Theory

Benedikt Löwe
VvL/OzSL Gödel Centenary Celebration
Utrecht, May 26, 2006
14:00-15:00
Gödel’s Work.

- Completeness Theorem (1929)
- Incompleteness Theorem (1931)
- Consistency of CH (1937)
- Gödel Universes (1947)

Set Theory (1).

Georg Cantor (1845-1918)

Two protagonists of set theory.

- The notion of transfinite counting (ordinal numbers),
- the notion of size comparison between infinite sets (cardinal numbers).
Set Theory (2).

Ordinal Numbers.

0, 1, 2, 3, 4, ... \( \infty = \mathbb{N} \)

\( \infty, \infty + 1, \infty + 2, \infty + 3, \infty + 4, ... \) \( \infty + \infty \)

Cardinality.

Two sets \( X \) and \( Y \) are equinumerous if there is a map \( \pi : X \rightarrow Y \) which is one-to-one and onto.
Two sets $X$ and $Y$ are equinumerous if there is a map $\pi : X \to Y$ which is one-to-one and onto.

- Finite sets are equinumerous if and only if they have the same number of elements,
- the (infinite) set of natural numbers is not equinumerous to any finite set,
- there are infinite sets not equinumerous to the set of natural numbers: $\mathbb{R}$, $\varphi(\mathbb{N})$. 
Two sets $X$ and $Y$ are **equinumerous** if there is a map $\pi : X \rightarrow Y$ which is one-to-one and onto. There are infinite sets not equinumerous to the set of natural numbers: $\mathbb{R}$, $\wp(\mathbb{N})$.

A set $X$ is called **countable** if it is equinumerous to $\mathbb{N}$.

Are the **transfinite ordinals** countable?

Some are. **But not all!**
Two sets $X$ and $Y$ are **equinumerous** if there is a map $\pi : X \to Y$ which is one-to-one and onto. There are uncountable infinite sets: $\mathbb{R}$, $\wp(\mathbb{N})$. Some transfinite ordinals are countable.

**Theorem.** The set $\omega_1$ of countable ordinals is not countable. It is a transfinite ordinal ("the smallest uncountable ordinals").

**Question** ("the Continuum Problem"). What is the relationship between $\omega_1$ and $\mathbb{R}$ (or $\wp(\mathbb{N})$)? Are they equinumerous?
Hilbert’s Problems.

David Hilbert (1862-1943)
1900 International Congress of Mathematicians, Paris:
Hilbert poses 23 problems for the 20th century.

Hilbert #1. *Die Untersuchungen von Cantor machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher niemandem gelungen ist; dieser Satz lautet: ‘Jedes System von unendlich vielen reellen Zahlen ist entweder der Menge der ganzen natürlichen Zahlen oder der Menge sämtlicher reeller Zahlen äquivalent.’*

Hilbert #2. Consistency of Peano arithmetic
1931-1937.

"Jetzt, Mengenlehre"

1933-1934. First Princeton visit.
Spring 1935. Consistency of AC.
1936. Rekawinkel sanatorium.
1937-1940.

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<th>Date</th>
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<tr>
<td>Jun 14, 1937</td>
<td>Breakthrough</td>
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<tr>
<td>Sep 20, 1938</td>
<td>Gödel marries Adele</td>
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<td>1938-1939</td>
<td>Third Princeton visit</td>
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<td>Nov 9, 1938</td>
<td>PNAS Announcement</td>
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<td>1939</td>
<td>Visit to Notre Dame</td>
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<td>1939-1940</td>
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<td>Transsiberian Emigration</td>
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<td>1940-1946</td>
<td>Fourth Princeton visit</td>
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<td>July 1940</td>
<td>Publication of the <em>Annals</em> monograph</td>
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Gödel’s Result.

**Theorem.** There is a model of all axioms of set theory that satisfies the continuum hypothesis CH (“the set of reals and the set of countable ordinals are equinumerous”).

**Corollary.** The axioms of set theory do not refute CH.

**Letter to Menger** (Dec 15, 1937). “Right now, I am trying to also prove the independence of the Continuum Hypothesis.”

- Gödel did not succeed: letters to Church and Rautenberg,
- Cohen proved the independence in 1963.
Inner Models (1).

Axioms of set theory are closure axioms: sets determine by formulas other sets.
Given a model $V$ of set theory, you can consider the closure of the ordinals under all operations corresponding to the axioms. Call this closure $L$.
Since $L$ is closed under all operations demanded by the axioms, all axioms are valid in $L$. **Relativization of formulas!**
Every element of $L$ is given by an ordinal and a formula. Therefore the axiom of choice holds in $L$ (a global wellordering of the universe).

**Therefore:** If $V$ is a model of ZF, then $L$ is a model of ZFC.
Inner Models (2).

Given a model $V \models ZF$, you can consider the closure $L$ of the ordinals under all operations corresponding to the axioms. Then $L \models ZFC$.

The set $L_\alpha$ of all sets constructed in at most $\alpha$ steps by the closure operations is definable. There is a formula $\varphi$ such that $X \models \varphi$ if and only if $X$ is some $L_\alpha$. (*)

Show by induction that $\alpha$ and $L_\alpha$ are equinumerous. (**) Consider an arbitrary $x \subseteq \mathbb{N}$ such that $x \in L$. There is some $\alpha$ such that $x \in L_\alpha$. Clearly, $L_\alpha \models \varphi$.

Let $X \subseteq L_\alpha$ be the Skolem hull of $x$. $X$ is countable and $X \models \varphi$. (*) and (**) show that $X = L_\xi$ for some countable $\xi$. So in $L$, $\varphi(\mathbb{N}) \subseteq L_{\omega_1}$. But by (**), $L_{\omega_1}$ and $\omega_1$ are equinumerous.
Importance.

- Relativization in set theory.
- Construction of models with definable well-ordering of the reals: consequences for topology and analysis (there is non-measurable $\Delta^1_2$ set).
- The inner model technique is amenable for other axiom systems: Silver 1971, Jensen 1980s, Mitchell & Steel 1990s.
Hilbert (1930). “Für uns gibt es kein ignorabimus.”

Gödel (1931). There are sentences undecided on the basis of the standard axioms of set theory.

Gödel (1937) / Cohen (1963). The continuum problem (Hilbert #1) is undecidable on the basis of the standard axioms of set theory.

Scenarios.

- It’s the fault of the continuum problem. (Feferman. “The continuum hypothesis is an inherently vague problem.”)
- It’s the fault of the standard axioms of set theory. (Gödel’s Programme)
Gödel’s Programme

“Search for new insights that solve the continuum problem!”

Large Cardinal Axioms (= “strong axioms of infinity”)

“There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory.”
Trivia

Kurt Gödel Research Center for Mathematical Logic