Inference versus Consequence[*]

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The following passage, hereinafter "the passage", could have been taken from a modern textbook.[1] It is prototypical of current logical orthodoxy:

> The inference
> (*) $A_1, ..., A_k$. Therefore: C
> is valid if and only if
> whenever all the premises $A_1, ..., A_k$ are true, the conclusion C is true also.
> When (*) is valid, we also say that C is a logical consequence of $A_1, ..., A_k$.
> We write $A_1, ..., A_k \models C$.

It is my contention that the passage does not properly capture the nature of inference, since it does not distinguish between valid inference and logical consequence. The view that the validity of inference is reducible to logical consequence has been made famous in our century by Tarski, and also by Wittgenstein in the *Tractatus* and by Quine, who both reduced valid inference to the logical truth of a suitable implication.[2] All three were anticipated by Bolzano.[3]

Bolzano considered *Urteile* (judgements) of the form

A is true

where A is a *Satz an sich* (proposition in the modern sense).[4] Such a judgement is correct (*richtig*) when the proposition A, that serves as the judgemental content, really is true.[5] A correct judgement is an *Erkenntnis*, that is, a piece of knowledge.[6] Similarly, for Bolzano, the general form I of inference

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[1] Could have been so taken and almost was; cf. Neil Tennant, *Natural Logic*, Edinburgh U. P., 1978, p. 2. In order to avoid misunderstanding let me note that I hold Tennant's book in high regard.

[2] Alfred Tarski, 'Über den Begriff der logischen Folgerung', *Actes du Congrès International de Philosophie Scientifique,* Vol. VII, Paris, 1937, pp. 1-11; Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, Routledge and Kegan Paul, London, 1922, §§ 5.11, 5.132; W. V. Quine, *Mathematical Logic* (rev. ed.), Harvard U.P., 1951, p. 7.

[3] Bernard Bolzano, *Wissenschaftslehre*, Vols I - IV, J. von Seidel, Sulzbach, 1837. Cited as **WL**.

[4] A proposition in the old sense is a *judgement*, usually of the [subject/copula/predicate] form **S is P** and its linguistic correlate is a complete declarative sentence, for instance, *Snow is white*. A proposition in the modern sense is not itself a judgement, but serves as the content of a judgement of the modern form **A is true**. Its linguistic correlate is a that-clause, for instance, *that snow is white*. The term 'proposition' without further qualification will be taken in the modern sense of a *Satz an sich* that was introduced by Bolzano (WL § 19).

[5] WL § 34.

[6] WL § 36.

$$\frac{J_1, \ldots, J_k}{J,}$$

where $J_1, \ldots, J_k$ are judgements, becomes I':

$$\frac{A_1 \text{ is true}, \ldots, A_k \text{ is true}}{C \text{ is true},}$$

where $A_1, \ldots, A_k$, and C are propositions. The inference I' is valid when C is a logical consequence of $A_1, \ldots, A_k$.[7] This is the notion of logical consequence that is explained in the passage: whenever all the antecedent propositions are true, the consequent proposition C is true also.[8]

One should note, however, that propositions and judgements are conflated in the passage. The relata in logical consequence are propositions, whereas an inference effects a passage from known judgements to a novel judgement that becomes known in virtue of the inference in question. Frege wrote:

> Ein Schluss …ist eine Urteilsfällung, die auf grund schon früher gefällter Urteile nach logischen Gesetzen vollzogen wird. Jede der Prämissen ist ein bestimmter als wahr anerkannter Gedanke, und im Schlussurteil wird gelichfalls ein bestimmter Gedanke als wahr anerkannt.[9]

An *Erkenntnis* — what is known — is a judgement and may be of the form that a proposition is true.[10] Such a piece of knowledge gets known, or is obtained, in an act of judgement. Similarly, in an inference-act, the conclusion-judgement gets known on the basis of previously known premiss-judgements: the inference is an act of mediate judgement.

Thus we have two Bolzanian reductions, namely (i) that of the correctness of the judgement to that of the truth of the propositional content and (ii) that of the validity of an inference between judgements to a

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[7] Bolzano's term was *Ableitbarkeit*, WL § 155(2). The literal translation 'derivability' would prove too confusing against the background of current practice which uses the two metamathematical turnstiles |= and |—. The semantic double turnstile is the analogue of Bolzano's *Ableitbarkeit*, whereas the (modern, non-Fregean) single turnstile expresses syntactic derivability according to certain derivation rules.

[8] As a representation of Bolzano this is substantially but not literally correct: Bolzano imposed certain compatibility conditions on the antecedents in *Ableitbarkeiten* that need not detain us further in the present context.

[9] 'Über die Grundlagen der Geometrie', *Jahresberichte der Deutschen Mathematiker-Vereinigung*, 15(1906), pp. 377-403, at p. 387. (My) English translation:
> An inference … is an act of judgement that is drawn according to logical laws from judgements previously made. Each premiss is a certain proposition which has been recognised as true, and also in the conclusion-judgement a certain proposition is recognised as true.

[10] Following Per Martin-Löf, 'On the meanings of the logical constants and the justification of the logical laws', *Nordic Journal of Philosophical Logic* 1 (1996), p. 26, I explain a judgement in terms of the knowledge required for having the right to make it. Alternatively the explanation might run in terms of what one has to do (namely, acquire the knowledge in question) in order to have the right to make the judgement in question.

corresponding logical consequence among suitable propositions. From an epistemological point of view, we get the problem that the reduced notions may obtain *blindly*. This happy term was coined by Brentano for the case when an assertion without ground happens to agree with an evidenceable judgement.[11] An example would be when I hazard a guess as to the size of the fortune of a former Dutch premier and by fluke happen to hit bull's eye, even though my knowledge of the financial situation of Dutch statesmen is nil. On the Bolzano reduction, this unsubstantiated claim would be an *Erkenntnis*, in spite of its being completely unwarranted. In the same way, an act of inference between judgements whose contents happened to be true and happened to stand in the relation of logical consequence would be valid, even though no epistemic warrant had been offered.

Blind correct judgement — be it mediate or not — is not to my taste, whence I am concerned to find other explications of judgemental correctness and inferential validity that do not admit of such blindness. By the side of Bolzano, Frege is virtually the only other modern logician that is of any help in the philosophical study of the notion of inference. In my opinion his much decried view that inference starts from true, nay, known, premisses contains an important insight:

> Aus falschen Praemissen kann überhaupt nichts geschlossen werden. Ein blosser Gedanke, der nicht als wahr anerkannt ist, kann überhaupt nicht Praemisse sein. … Blosse Hypothesen können nicht als Praemissen gebraucht werden.[12]

Properly understood, this Fregean insight does not contradict Gentzen's views — when *they* are properly understood — concerning the use of assumptions within so called natural deduction derivations.[13] In general these derivations depend on open assumptions: accordingly the endfomula of a derivation-tree will express a proposition that is not true outright, but only dependently true, that is, true, *given* the truth of the propositions expressed by the assumption-formulae. Thus, the form of judgement used by Gentzen in his system of natural deduction is not

A is true,

but

C is true, provided that $A_1, \ldots, A_k$ are true.

Hence an inference effects an act of passage between known judgements of the latter dependent form, whence there is no contradiction with Frege. In Gentzen's sequential version of natural deduction, on the other hand, the form of the conclusion-judgement that is demonstrated is better thought of as being

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[11] *Wahrheit und Evidenz*, Felix Meiner, Hamburg, 1974[II] (1930), p. 135.

[12] Letter to Jourdain, *Wissenschaftliche Briefwechslung* (eds. G. Gabriel *et al*), Felix Meiner, Hamburg, 1976, p. 118. (My) English translation:

> Nothing at all can be inferred from false premises. A mere thought, that has not been recognised as true, cannot be a premiss. … Mere hypotheses cannot be premisses.

[13] Gerhard Gentzen, 'Untersuchungen über das logische Schliessen', *Mathematische Zeitschrift*, Vol. 39 (1934-1935), pp. 176-210, 405-431, and 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Mathematische Annalen* 112(1936), pp. 493-565.

S holds,

where the sequent S expresses a consequence.

However, in order to find further genuinely relevant views one has to turn to the Scholastics. Towards the end of the 13th century tracts entitled *De Consequentiis* begin to appear, by such authors as William of Ockham, Walter Burleigh, Richard Billingham, Ralph Strode, John Buridan, Marsilius of Inghen, Paul of Venice, ... A *consequence* is a hypothetical proposition (in the old sense) which can be recognised through the use of certain indicator words:

Indicator	Example	Modern analogue
Si (if)	If A, then B	conditional
Sequitur (follows)	From A follows B	consequence
Quia (because)	B because A	causal grounding
Igitur (therefore)	A. Therefore B	inference

These were all variants of one and the same notion. Thus, where today we would formulate four different theories with various and sometimes conflicting principles, the scholastics sought for principles that covered all four (modern) notions. An example of such a principle is, of course, *modus ponens*, which from the premisses A and the consequence of A and B draws the conclusion B.

Today one would say that
- a conditional is a proposition that may be true;
- a consequence is a relation between propositions that may hold;[14]
- causal grounding is a relation (between states of affairs) that may obtain;
- an inference is an act of passage from judgement(s) to judgement that may be valid.

The task I set myself is to elucidate relationship between the second and fourth notions among these four alternatives.

One can discern two views concerning *consequentia* and their validity (holding) in the medieval logical tradition:[15]

(i) the *containment theory* which was adumbrated by Peter Abelard and advocated by "English" logicians at Padua from 1400 onwards;

(ii) the *incompatibility theory*, which is of Stoic origin and was advocated by Parisian logicians around 1400.

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[14] *Tenere* is the term that the scholastics applied to a *consequentia*.

[15] The distinction was drawn by Christopher J. Martin, 'William's Machine', *Journal of Philosophy*, 83(1986), pp. 564-572, and used by Ivan Boh, *Epistemic Logic in the Later Middle Ages*, Routledge, London, 1993.

Aristotle held that in a valid syllogism, when the premises are true, necessarily the conclusion must be true.
The Stoics refined this into:

[A. **Therefore** B] is *valid*
if and only if
A is true and B is false are *incompatible*.

Using elementary modal logic and Boolean combinations,

[A. **Therefore** B] is valid **iff**
¬◊(A is true **and** B is false) **iff**
□ ¬ (A is true **and** B is false) **iff**
□ (**if** A is true, **then** B is true).

When the necessity □ is read as "holds in every variant", or "in all terms", ordinary (Bolzano) logical consequence is the result. Thus on the *Incompatibility Theory,* inferential validity is reduced to the logical holding, that is, holding in all alternatives, of the consequence from A to B:

The inference [A. **Therefore** B] is valid
when the consequence A |=B holds formally (*in omnibus terminis*).

Essentially, this is the theory that we found in Bolzano, Tarski, and Quine: the theory from the passage is an intellectual descendant of the medieval incompatibility theory thus construed. I am not satisfied with this reduction, though, since the above difficulties concerning blindly valid inference remain unresolved. Logic is an epistemological tool for obtaining new knowledge from known premises. The incompatibility theory does not fully acknowledge this epistemic aspect of logic: the (logical) holding of a consequence is, as well as propositional truth, will (in general) be "evidence transcendent".[16] In modern terms the incompatibility theory pertains not so much to the validity of inferences as to the (logical) holding of consequences.

Inference, like judgement, is primarily an act: one draws an inference and makes a judgement.[17] We have the diagram:

_____act_____ object.

The object, however, is not the only objective correlate of the act. Coupled to the exercised act, the subject(ive) process, there is also the objective signified act, that is, the trace, or track, of the subjective act:
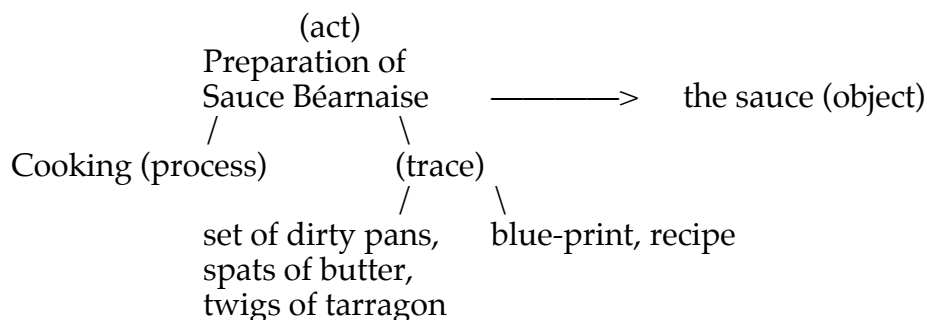
act ————> object
/ \
subjective objective
process trace.[18]

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[16] The felicitous term 'evidence transcendent' derives from the realism/anti-realism debate: cf. Crispin Wright, *Realism, Meaning and Truth*, Blackwell, Oxford, 1987, p. 2.
[17] Cf. the quote from Frege offered at f. n. 9.
[18] I am indebted to Per Martin-Löf for drawing my attention to this notion of an act-trace. He spoke about it in an as yet unpublished lecture in Paris, April 1992.
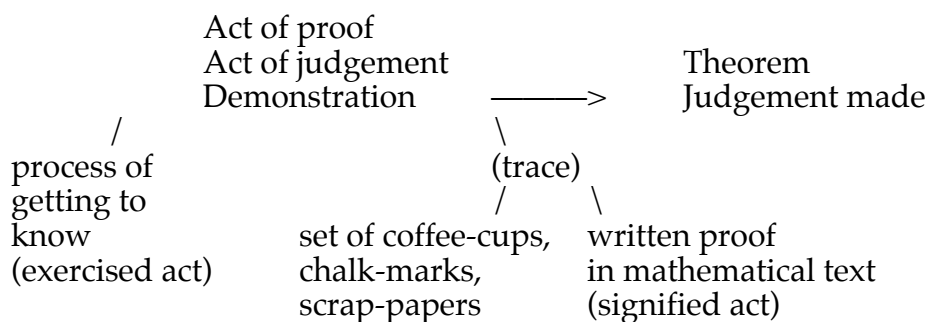
When applied to a concrete example, for instance, the preparation of a Sauce Béarnaise, this abstract scheme becomes concrete as:

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                    (act)
                    Preparation of
                    Sauce Béarnaise        ———————>      the sauce (object)
                    /            \
       Cooking (process)         (trace)
                              /         \
                    set of dirty pans,    blue-print, recipe
                    spats of butter,
                    twigs of tarragon
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As we see the act-trace can be taken in two senses:
      (i) as the actual (concrete) trace of the exercised act, and
      (ii) as the blue-print of the signified act.[19]

This battery of distinctions can now be applied to the act of demonstration (judgement):

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                    Act of proof
                    Act of judgement           Theorem
                    Demonstration      ———>     Judgement made
              /                          \
       process of                        (trace)
       getting to                       /        \
       know                 set of coffee-cups,   written proof
       (exercised act)      chalk-marks,          in mathematical text
                            scrap-papers          (signified act)
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The object (product) of an act of judgement (demonstration) is the judgement made (theorem proved). Also an act of inference, though, has a theorem (judgement) as its product. An inference-figure is not so much the product as the trace of an act of inference. An inference, be it immediate or not, is a mediate act of judgement. Inferences are discursive (acts of) judgement. Immediate, or intuitive, acts of judgement, on the other hand, have axioms as products, that is, known judgements that rest upon no other knowledge. Following Martin-Löf, a judgement is actually true when it is known (evident) and potentially true when it can be made evident (is evidenceable, justifiable, warrantable, demonstrable, knowable, etc.).[20] This notion of potential truth of a judgement corresponds to the "objective correctness" of a statement or assertion that is familiar from the anti-realist literature.[21]

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[19] For 'The distinction *actus exercitus/actus significatus* in medieval semantics', see the article of that title by G. Nuchelmans in: Kretzmann, N. (ed.), *Meaning and Inference in Medieval Philosophy*, Kluwer, Dordrecht, 1988, pp. 57-90.
[20] 'Truth and knowability: on the principles C and K of Michael Dummett', forthcoming in H. G. Dales and G. Oliveri (eds.), *Truth in Mathematics*, Clarendon Press, Oxford.
[21] Michael Dummett, 'What is a theory of meaning? II', in G. Evans and J. McDowell, *Truth and Meaning*, Oxford U.P., 1976, pp. 119-120.

With these distinctions at our disposal we can now deal with the other proposal for inferential validity, namely the *Containment Theory*:

> An inference is valid when the conclusion is "contained" (in some suitable sense) in the premisses.

Already Aristotle used an idea of this kind when he wished to ground the validity of a syllogism in the existence of a chain of linking terms.

It is often said that a valid inference is a truth-preserving one. What kind of truth has to be preserved? True propositions? Actually true judgements? Objectively correct judgements? Preservation of propositional truth can hardly be what is at issue here: that gives us not the validity of an inference, but the holding of a consequence. Preservation of actual truth for judgements is also ruled out as an explication of inferential validity. On such an account the completely general inference I above would be valid when the premisses $J_1,\ldots, J_k$ are *unknown*.

Preservation of objective correctness, that is, potential truth for judgements, is the only viable option. The question remains *how* such truth is going to be preserved from the premisses to the conclusion of a valid inference. Scholastic logic proves helpful also here. Robert Kilwardby (c.1215-1279) writes:

> Consequence is twofold, namely essential or natural,
> as when a consequent is naturally understood in its antecedent,
> and accidental consequence.[22]

This, I take it, is an early formulation of the reduction of valid inference to analytic containment: when the premisses of the inference are understood and known, and the conclusion is understood, that is, one knows the definitions of the essences of the terms that occur in the conclusion, nothing more is called for in order to come to know the conclusion. It is analytically contained in the premisses. We have then an instance of an inference *per se nota*, whose evidence is not founded upon anything but the knowledge of the terms out of which the judgements of the inference has been put together: the inference accordingly rests upon *evidentia ex terminis*.

In his attack upon the notion of analyticity, Quine remarked that

> meaning is what essence becomes when it is divorced from the object of reference and wedded to the word.[23]

This linguistic turn transforms the evidence conferred through the understanding of natures (essences) into "self-evidence in virtue of meaning".[24] Not every inference, though, will be conceptually self-evident

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[22] Quoted from I. M. Bochenski, *A History of Formal Logic*, Chelsea Publishing Co., New York, 1970[II], § 30.07, at p. 190. Latin text in W. Kneale and M. Kneale, *The Development of Logic*, Clarendon Press, Oxford, 1961, at p. 275.

[23] W. V. Quine, 'Two dogmas of empiricism', in: *From a Logical Point of View* (rev. ed.), Harper and Row, 1963, p.22.

[24] *Evidence* is here taken in the sense of *the property of being evident* and not in the sense *support for the truth of a proposition.*

from meaning. Only an immediate inference, that is, an inference that is not supportable further by other inferences has this character. Examples are the standard introduction and elimination rules for the intuitionistic logical constants.[25]

Consider the completely general inference-figure I once more:

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[25] See Martin-Löf's treatment in *op. cit.*, f. n. 10. The justification of the elimination rules in terms of the introduction rules does not constitute a *derivation* of the former from the latter. To know the meaning of an intuitionistic propositional connective C is to know how canonical, that is, introductory, *proof-objects* for propositions of C-form may be put together (and when two such introductory proofs are equal). That knowledge is enough to make plain the validity of the (immediate) elimination *inferences.* Note further that the introduction-/elimination-rule distinction operates on two different levels. On the one hand, on the level of propositions, it concerns how propositional proof-objects may be put together; for instance when a is a proof-object for A and when b is a proof-object for b, then &I(A, B, a, b) is a proof-object for A&B. On the other hand, at the epistemic level of judgements and inferences, it concerns for instance the inference rules

       A is true, B is true. Therefore: A&B is true
and
       A&B is true. Therefore: B is true,

or, when we use the fully explicit for of judgement including the proof-objects:

       <u>a is a proof of A   b is a proof of B</u>
            &&I(A, B, a, b) is a proof of A&B.

$$\frac{J_1,\ldots,J_k}{J.}$$

What does it mean for I to be valid?[26] We consider how an inference according to I is used. In such use one takes it for granted that the premisses $J_1, \ldots, J_k$ are known and goes on to obtain knowledge of J. Thus, under the epistemic assumption that the judgements $J_1, \ldots, J_k$ are all known, one has to make the judgement J known.[27] Given the knownness of $J_1, \ldots, J_k$, the knowability of J is secured through a chain of immediately evident axioms and inferences that begins in the premisses and ends in the conclusion. In order to have the right to infer according to I, one must posses the chain in question. When such a chain can be found, the inference-schema (as signified act) is potentially valid. For the exercised act this is not enough: then one needs the actual validity. One must actually possess the chain of immediate evidences, be they axiomatic or inferential, and actually carry out each of the immediate component steps thereof.[28]

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[26] Martin-Löf's notion of validity in his 'Truth of a proposition, evidence of a judgement, validity of a proof', *Synthese* 73(1987), pp. 191-212, is different from the validity of an inference. The former notion results from applying the notion of *rightness* to proofs: a valid (right, real, true, conclusive, …) proof is one in which each axiom really is true and each inference really is valid.

[27] Note the difference between alethic assumptions that propositions are true and epistemic assumptions that judgements are known (knowable). The former are used in natural-deduction consequences between propositions. The latter are used when making evident the validity of inferences.

[28] The picture outlined in the present paper is presented in more detail in my 'Implicit epistemic aspects of constructive logic', *Journal of Logic, Language, and Information* 6 (1997), pp. 191-212;
'Inference, consequence, implication', forthcoming in *Philosophia Mathematica*;
'Proofs as acts and proofs as objects: some questions for Dag Prawitz', forthcoming in *Theoria*.