A strongly complete proof system for propositional dynamic logic

Gerard Renardel de Lavalette, Barteld Kooi* Rineke Verbrugge†

Abstract

Propositional dynamic logic (PDL) is complete but not compact. As a consequence, strong completeness (the property $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$) does not hold for the standard finitary axiomatisation. In this paper, we present an infinitary proof system of PDL and prove strong completeness. The result is extended to epistemic logic with common knowledge.

1 Introduction

Dynamic logic is a modal logic that was developed to reason about computer processes. The branch of logic was started by Pratt [11]. The propositional part of his logic (PDL) became an object of study in itself. Segerberg [12] gave an axiomatisation of it, that is mostly used today, but the completeness proof was not easy to find. It took some time before several proofs were produced. The proof by Kozen and Parikh [8] is considered to be one of the most elegant. The problem is that the canonical model method for proving completeness cannot be applied successfully. The axiomatisation by Segerberg is only weakly complete, because PDL is not compact (see below). The topic of this paper is a *strongly* complete proof system for propositional dynamic logic, for which the canonical model method can be used to prove completeness.

Strong completeness (also called extended completeness) with respect to a class of frames $\mathcal{C}$ is the following property of a modal logical system S:

$$\Gamma \models_{\mathcal{C}} \varphi \text{ implies } \Gamma \vdash_S \varphi, \text{ for all formulae } \varphi \text{ and all sets of formulae } \Gamma.$$

This generalises weak completeness, where Γ is empty. Observe that weak completeness implies strong completeness whenever the logic in question is semantically compact, i.e. when $\Gamma \models_{\mathcal{C}} \varphi$ implies that there is a finite $\Gamma' \subseteq \Gamma$

*Department of Computing Science, University of Groningen, P.O. Box 800, 9700 AV Groningen,The Netherlands, {grl, barteld}@cs.rug.nl

†Department of Artificial Intelligence, University of Groningen, Grote Kruisstraat 2/1, 9712 TS Groningen, The Netherlands, rineke@ai.rug.nl

with $\Gamma' \models_{\mathcal{C}} \varphi$, hence $\models_{\mathcal{C}} \bigwedge \Gamma' \to \varphi$. This is, for example, the case in modal logics such as K and $\mathsf{S5}$.

Propositional dynamic logic is a well known example of a non-compact logic: we have for the relevant class of frames $\mathcal{C}$, that $\{[a^n]p \mid n \in \mathsf{N}\} \models_{\mathcal{C}} [a^*]p$ but there is no natural number k with $\{[a^n]p \mid n \leq k\} \models_{\mathcal{C}} [a^*]p$. As a consequence, we do not have strong completeness for any finitary axiomatisation, *a fortiori* not for its usual, weakly complete proof system (see definition 3). So strong completeness requires an infinitary proof system.

Some infinitary systems have been presented in the literature. A strongly complete infinitary proof system for *Propositional Algorithmic Logic* is presented in [10]. Although propositional algorithmic logic is related to PDL, it differs too much for these results to be applicable to PDL. In [6], an infinitary axiomatisation of PDL is given. It contains the ∞-rule *from* $\{\psi \to [\alpha^i]\varphi \mid i < \omega\}$ *infer* $\psi \to [\alpha^*]\varphi$. Weak completeness for this axiomatisation is claimed, but we suspect that the proof is incorrect. In [7] another proof of strong completeness is presented with respect to a specific model. Although we think that the result holds, we have trouble with some of the details of the paper. For example in the proof of theorem 5.10 of [7] a reference is made to Lindenbaum's theorem to extend a set of formulae to a maximally consistent one. It seems to us that this technique cannot be applied directly in the case of infinitary proof systems. In [3], Goldblatt introduces the Omega-Iteration proof rule *from* $\{\varphi \to [\beta; \alpha^n]\psi \mid n \in \omega\}$ *deduce* $\varphi \to [\beta; \alpha^*]\psi$ in the context of first-order dynamic logic, and proves weak completeness.

In [4, chapter 9] a general approach for infinitary proof systems for normal modal logics is given. Goldblatt shows that the addition of rules that satisfy certain properties to a basic proof system, yields a proof system that is strongly complete with respect to the appropriate class of models. In that framework it is not hard to make a strongly complete proof system for PDL. We take another approach in this paper. Goldblatt starts out by taking a fairly strong basic proof system. Strong necessitation (SNec), and the deduction theorem (Ded) are rules in his basic system, whereas they are deduced rules in the proof system we present in this paper: PDL_ω. This makes the proofs about PDL_ω easier than they would be in Goldblatt's setting. PDL_ω is an extension of PDL with an infinitary proof system containing a variant of the Omega-iteration proof rule. We show that it is strongly complete. This result can be generalised to other logics, such as epistemic logic with common knowledge.

The rest of the paper is structured as follows. Section 2 presents the infinitary proof system PDL_ω, as well as proofs of some derived rules, which are used in the central section 3 to prove that PDL_ω is strongly complete. In this section it is also shown that the canonical model for PDL_ω does not satisfy *program harmony*. An analogue of the proof system for epistemic logics with common knowledge is sketched in section 4. Finally section 5 contains a conclusion and ideas for further research.

2 The proof system PDL_ω

The infinitary proof system PDL_ω is an extension of the usual axiom system for PDL, with respect to the same language and the same Kripke semantics. As a reminder, we repeat the definitions of both language and semantics (for more information on PDL, see [5]).

Definition 1 (Language of PDL) *A language $\mathcal{L}_{P,\Pi}$ of propositional dynamic logic PDL is based on a countable set P of atomic formulae p and a countable set Π of atomic programs a, and is given by the following rules, defining the set of formulae $Fma(P,\Pi)$ and the set of programs $Prog(P,\Pi)$, respectively:*

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid (\varphi \to \psi) \mid [\alpha]\varphi$$

$$\alpha ::= a \mid \alpha;\beta \mid \alpha \cup \beta \mid \alpha^* \mid \varphi?$$

Definition 2 (Kripke semantics of PDL) *A model for $\mathcal{L}_{P,\Pi}$ is a tuple $M = (W, \{R_a : a \in \Pi\}, V)$ such that:*

- *$W \neq \emptyset$; a set of states or possible worlds;*
- *R_a: a binary relation on W for each atomic program a;*
- *$V : P \to 2^W$; assigns a set of states to each propositional variable.*

The truth definition is as expected for normal modal logics. As a reminder, here follows the modal clause:

$$M, w \models [\alpha]\varphi \text{ iff } M, v \models \varphi \text{ for all } v \text{ with } wR_\alpha v$$

where R is extended in the following way

- $R_{\alpha;\beta} = R_\alpha \circ R_\beta$;
- $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$;
- $R_{\alpha^*} = R_\alpha^*$ = reflexive transitive closure of R_α;
- $R_{\varphi?} = \{(w,w) \mid M, w \models \varphi\}$.

It is sometimes relevant to abstract from the valuation V and consider Kripke frames $(W, \{R_a : a \in \Pi\})$ satisfying the conditions on W and R_a above. Let $\mathcal{C}$ be the class of frames corresponding to this class of models. We show that PDL_ω is complete with respect to this class of frames $\mathcal{C}$:

$$\Gamma \models_{\mathcal{C}} \varphi \text{ implies } \Gamma \vdash_{\mathsf{PDL}_\omega} \varphi, \text{ for all formulae } \varphi \text{ and all sets of formulae } \Gamma.$$

Below $\models_{\mathcal{C}}$ and $\vdash_{\mathsf{PDL}_\omega}$ is abbreviated with $\models$ and $\vdash$ respectively. By $\Gamma \models \varphi$ we mean the local consequence relation, i.e. $\Gamma \models \varphi$ iff for every model M such that the corresponding frame is in $\mathcal{C}$, $M, w \models \psi$ for every $\psi \in \Gamma$ implies that $M, w \models \varphi$.

Definition 3 (Axioms for PDL) *Here follows the usual set of axioms for PDL.*

Taut	all instantiations of propositional tautologies
Distr	$[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
?AX	$[\varphi?]\psi \leftrightarrow (\varphi \to \psi)$
;AX	$[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
$\cup$AX	$[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$
*AX	$[\alpha^*]\varphi \leftrightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi)$

In the following we extend the system PDL to an infinitary proof system PDL_ω by inductively defining a derivation relation $\Gamma \vdash \varphi$ (φ a formula, Γ a set of formulae). Notice that in the following definition, the language remains finitary and only the rule Inf^* is non-standard.

Besides the usual shorthand notation Γ, φ for $\Gamma \cup \{\varphi\}$, Γ, Δ for $\Gamma \cup \Delta$, $\vdash \varphi$ for $\emptyset \vdash \varphi$, and $\varphi_1, \ldots, \varphi_n \vdash \psi$ for $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi$, we shall also write:

$\Gamma \vdash \Delta$	for	$\Gamma \vdash \varphi$ for all $\varphi \in \Delta$
$[\alpha]\Gamma$	for	$\{[\alpha]\varphi \mid \varphi \in \Gamma\}$
$\varphi \to \Gamma$	for	$\{\varphi \to \psi \mid \psi \in \Gamma\}$

Definition 4 (Infinitary derivation relation for PDL_ω) $\Gamma \vdash \varphi$ *is defined as the smallest relation closed under the following rules:*

AX	$\vdash \varphi$ if φ is an axiom of PDL	
MP	$\varphi, \varphi \to \psi \vdash \psi$	(modus ponens)
Inf^*	$\{[\alpha;\beta^n]\varphi \mid n \in \mathsf{N}\} \vdash [\alpha;\beta^*]\varphi$	(infinitary $*$-introduction)
Nec	if $\vdash \varphi$ then $\vdash [\alpha]\varphi$	(necessitation)
W	if $\Gamma \vdash \varphi$ then $\Gamma, \Delta \vdash \varphi$	(weakening)
Cut	if $\Gamma \vdash \Delta$ and $\Gamma, \Delta \vdash \varphi$ then $\Gamma \vdash \varphi$	

Observe the formulation of Inf^*: it contains Omega-Iteration (via ?AX) , i.e. the version with formulae $\psi \to [\alpha;\beta^n]\varphi, \psi \to [\alpha;\beta^*]\varphi$.

It is not hard to verify that these rules are sound with respect to the semantics of PDL (i.e. $\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$). We shall show in section 3.1 that the system PDL_ω is also strongly complete with respect to these semantics. For this, we shall use some derived rules that we introduce now.

The most important of these are *deduction* (Ded) and *strong necessitation* (SNec), while the other rules are only used to prove Ded and SNec. Ded will be used in the Lindenbaum lemma and both Ded and SNec in the Truth lemma. We remark that the fact that SNec holds while only Nec is part of the proof system, is essential in our proof of strong completeness.

Lemma 1 (Derived rules of $\mathbf{PDL}_\omega$) *We can prove the following derived rules:*

SCut	if $\Gamma \vdash \Delta$ and $\Gamma', \Delta \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$	(strong cut)
Det	if $\Gamma \vdash \varphi \to \psi$ then $\Gamma, \varphi \vdash \psi$	(detachment)
Cond	if $\Gamma, \Delta \vdash \varphi$ then $\Gamma, (\psi \to \Delta) \vdash \psi \to \varphi$	(conditionalisation)
Ded	if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$	(deduction)
SNec	if $\Gamma \vdash \varphi$ then $[\alpha]\Gamma \vdash [\alpha]\varphi$	(strong necessitation)

Proof Notice that the structure of the proof below is an infinitary induction over derivations; this is not a problem because of the well-foundedness of derivations.

SCut is easy to prove using W and Cut. Det is also easy, with MP and SCut.

Cond : Proof by induction over a derivation of $\Gamma, \Delta \vdash \varphi$. By W, we may assume without loss of generality that $\Gamma \cap \Delta = \emptyset$. The cases below are named by the rule applied last:

- **AX** If φ is an axiom, then $\vdash \varphi$; also $\vdash \varphi \to (\psi \to \varphi)$ for it is a tautology. By MP we have $\varphi, \varphi \to (\psi \to \varphi) \vdash \psi \to \varphi$. Now apply SCut twice to obtain $\vdash \psi \to \varphi$.
- **MP** There are four cases, corresponding with Δ equals $\emptyset, \{\varphi_1\}, \{\varphi_1 \to \varphi_2\}$, or $\{\varphi_1, \varphi_1 \to \varphi_2\}$; they all follow via tautologies and Det.
- **Inf*** We have to show

  $$\{[\alpha;\beta^n]\varphi \mid n \in I\}, \{\psi \to [\alpha;\beta^n]\varphi \mid n \in J\} \vdash \psi \to [\alpha;\beta^*]\varphi$$

  for I, J with $I \cup J = \mathsf{N}, I \cap J = \emptyset$. By Inf, we have

  $$\{[\psi?;\alpha;\beta^n]\varphi \mid n \in \mathsf{N}\} \vdash [\psi?;\alpha;\beta^*]\varphi$$

  so, by Cut and ?AX, it suffices to show

  $$\{[\alpha;\beta^n]\varphi \mid n \in I\}, \{\psi \to [\alpha;\beta^n]\varphi \mid n \in J\} \vdash \{[\psi?;\alpha;\beta^n]\varphi \mid n \in \mathsf{N}\}$$

  and this is correct if $[\alpha;\beta^n]\varphi \vdash [\psi?;\alpha;\beta^n]\varphi$ for $n \in I$, and $\psi \to [\alpha;\beta^n]\varphi \vdash [\psi?;\alpha;\beta^n]\varphi$ for $n \in J$. Both are easily obtained using ?AX.
- **Nec** Analogously to AX.

W Easy application of the induction hypothesis.

Cut Now there is a Θ with $\Gamma, \Delta \vdash \Theta$ and $\Gamma, \Delta, \Theta \vdash \varphi$. With the induction hypothesis, we get $\Gamma, \psi \to \Delta \vdash \psi \to \Theta$ and $\Gamma, \psi \to \Delta, \psi \to \Theta \vdash \psi \to \varphi$, so with Cut we obtain $\Gamma, \psi \to \Delta \vdash \psi \to \varphi$.

Ded : if $\Gamma, \varphi \vdash \psi$, then (by Cond) $\Gamma, \varphi \to \varphi \vdash \varphi \to \psi$; with Cut we remove the tautology $\varphi \to \varphi$ and obtain $\Gamma \vdash \varphi \to \psi$.

SNec : induction over a derivation of $\Gamma \vdash \varphi$.

AX Easy, with Nec.

MP Take Distr and apply Det twice.

Inf* We have $\{[\gamma; \alpha; \beta^n]\varphi \mid n \in \mathsf{N}\} \vdash [\gamma; \alpha; \beta^*]\varphi$; via ;AX and SCut we obtain

$$[\gamma]\{[\alpha; \beta^n]\varphi \mid n \in \mathsf{N}\} \vdash [\gamma][\alpha; \beta^*]\varphi$$

Nec Easy, via ;AX and Cut.

W,Cut These cases follow directly by the induction hypothesis.

An immediate consequence of Ded is that if Γ, φ is PDL_ω-inconsistent, then $\Gamma \vdash \neg\varphi$.

3 The canonical model for PDL_ω

In this section, we consider a fixed language $\mathcal{L}_{P,\Pi}$. We shall prove strong completeness of PDL_ω via the Henkin construction of a canonical model in section 3.1. There are two steps: first we show that every PDL_ω-consistent set can be extended to a maximal PDL_ω-consistent set, then we construct a Kripke model consisting of maximal PDL_ω-consistent sets. In section 3.2 we show that this model does not satisfy program harmony, by giving a countermodel.

3.1 Strong completeness of PDL_ω

We recall the obvious fact that a collection of formulae Γ is maximal PDL_ω-consistent iff it is PDL_ω-consistent (i.e. $\Gamma \nvdash \bot$) and Γ contains exactly one from $\varphi, \neg\varphi$ for every formula φ in the language $\mathcal{L}_{P,\Pi}$. In the remainder of this section, we will omit the prefix PDL_ω before "consistent".

Lemma 2 (Lindenbaum lemma for PDL_ω) *Every consistent set can be extended to a maximal consistent set.*

Proof Let Δ be a consistent set, i.e. $\Delta \not\vdash \bot$. Let $\{\varphi_n \mid n \in \mathsf{N}\}$ be an enumeration of all formulae in $\mathcal{L}_{P,\Pi}$. We shall inductively define an increasing sequence $\{\Gamma_n \mid n \in \mathsf{N}\}$ of formula sets extending Δ, and show that $\Gamma =_{\text{def}} \bigcup\{\Gamma_n \mid n \in \mathsf{N}\}$ is maximal consistent.

$$
\begin{array}{lll}
\Gamma_0 & =_{\text{def}} & \Delta \\
\Gamma_{n+1} & =_{\text{def}} & \left\{ \begin{array}{ll}
\Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \vdash \varphi_n \\
\Gamma_n \cup \{\neg\varphi_n\} & \text{if } \Gamma_n \not\vdash \varphi_n \text{ and } \varphi_n \text{ is not} \\
 & \text{of the form } [\alpha;\beta^*]\psi \\
\Gamma_n \cup \{\neg\varphi_n, \neg[\alpha;\beta^k]\psi\} & \text{otherwise, where } k \text{ is the} \\
 & \text{least natural number} \\
 & \text{such that } \Gamma_n \not\vdash [\alpha;\beta^k]\psi \\
 & \text{(and } \varphi_n \text{ is of the form} \\
 & [\alpha;\beta^*]\psi)
\end{array} \right.
\end{array}
$$

We observe that the k in the last case always exists: for if $\Gamma_n \vdash [\alpha;\beta^k]\psi$ for all $k \in \mathsf{N}$, then (by Inf^* and Cut) $\Gamma_n \vdash [\alpha;\beta^*]\psi$, contradicting $\Gamma_n \not\vdash \varphi_n$. So the definition of Γ_n is correct.
Now we claim the following for all formulae φ, ψ; from these claims, especially from (3) and (6), it follows immediately that Γ is maximal consistent:

1. every Γ_n is consistent;
2. $\vdash \varphi \Rightarrow \varphi \in \Gamma$;
3. $\varphi \notin \Gamma \Leftrightarrow \neg\varphi \in \Gamma$;
4. $(\varphi \to \psi) \in \Gamma \Leftrightarrow (\varphi \in \Gamma \Rightarrow \psi \in \Gamma)$;
5. $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$;
6. $\Gamma \not\vdash \bot$.

The proofs of (2), (3), (4) and (6) are as usual. We give the proofs in the two unusual cases:

1. Induction over n. For $n = 0$, consistency of Δ is given. Now assume that Γ_n is consistent. If, in the definition of Γ_{n+1}, the first or second case applies, it is clear that Γ_{n+1} is consistent. If the last case applies and Γ_{n+1} were inconsistent, then $\Gamma_n \vdash [\alpha;\beta^*]\psi \vee [\alpha;\beta^k]\psi$ via Ded, so, by using $\mathsf{*AX}$ k times, $\Gamma_n \vdash [\alpha;\beta^k]\psi$, contradicting the definition of k.

5. We shall prove a more general statement with induction over a derivation of $\Gamma' \vdash \varphi$: if $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \varphi$, then $\varphi \in \Gamma$.

- φ is an axiom: by (2).
- MP: by (4).

- Inf^*: Let $\{[\alpha;\beta^n]\varphi \mid n \in \mathsf{N}\} \subseteq \Gamma$. To show that $[\alpha;\beta^*]\varphi \in \Gamma$, assume using contraposition that this is not the case: then by (3), $\neg[\alpha;\beta^*]\varphi \in \Gamma$. Let n be the index with $\varphi_n = [\alpha;\beta^*]\varphi$, then $\Gamma_n \not\vdash \varphi_n$ (for otherwise, by the first case in the definition of Γ_{n+1}, $[\alpha;\beta^*]\varphi \in \Gamma_{n+1} \subseteq \Gamma$), so $\neg[\alpha;\beta^k]\varphi \in \Gamma_{n+1} \subseteq \Gamma$ for some k by the last case of the definition of Γ_{n+1}. But also, by assumption, $[\alpha;\beta^k]\varphi \in \Gamma$, so $\{\neg[\alpha;\beta^k]\varphi, [\alpha;\beta^k]\varphi\} \in \Gamma_m$ for some $m > n$, contradicting the consistency of Γ_m (1).
- Nec: by (2).
- W: direct consequence of the induction hypothesis.
- Cut: so $\Gamma' \vdash \Gamma''$ and $\Gamma' \cup \Gamma'' \vdash \varphi$ for some Γ''. By the induction hypothesis, we get $\Gamma'' \subseteq \Gamma$, so $\Gamma' \cup \Gamma'' \subseteq \Gamma$; by applying the induction hypothesis again we obtain $\varphi \in \Gamma$.

Now we can define the canonical model needed for strong completeness.

Definition 5 (Canonical model) *We define the canonical Kripke model* $M = (W, \{R_a : a \in \Pi\}, V)$ *by*

- $W =_{\text{def}} \{\Gamma \mid \Gamma \textit{ maximal consistent}\}$
- $R_a =_{\text{def}} \{(\Gamma, \Delta) \in W^2 \mid \varphi \in \Delta \textit{ for all } \varphi \textit{ such that } [a]\varphi \in \Gamma\}$
- $V(p) =_{\text{def}} \{\Gamma \in W \mid p \in \Gamma\}$

The truth lemma shows that $M, \Gamma \models p \Leftrightarrow p \in \Gamma$ extends to all formulae of the language:

Lemma 3 (Truth lemma) *For all* $\Gamma \in W$ *and all formulae* φ*, we have*

$$M, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof Induction over φ. The atomic and propositional cases are standard. We will prove the case $\varphi = [\alpha]\psi$, by induction over α; the cases for complex programs α of the forms $\chi?$, $\beta;\gamma$ and $\beta \cup \gamma$ are easy, so we only give the proofs of the remaining two unusual cases. Note that the proof as a whole has the form of an induction over a well-ordering of formulae, where $[\alpha^n]\varphi$ is considered to be a subformula of $[\alpha^*]\varphi$.

1. $\alpha = a$, atomic. Using the definition of the truth relation and the induction hypothesis $M, \Delta \models \psi \Leftrightarrow \psi \in \Delta$ for all $\Delta \in W$, we see that $M, \Gamma \models [a]\psi$ is equivalent to

$$\text{for all } \Delta \in W (\text{for all } \chi([a]\chi \in \Gamma \Rightarrow \chi \in \Delta) \Rightarrow \psi \in \Delta) \tag{A}$$

It is evident (A) follows from $[a]\psi \in \Gamma$. To see that (A) implies $[a]\psi \in \Gamma$ as well, we argue via contraposition. So assume $[a]\psi \notin \Gamma$, i.e. (by maximal consistency) $\neg[a]\psi \in \Gamma$. We shall show that there is a maximal consistent Δ with $\theta \in \Delta$ for all θ such that $[\alpha]\theta \in \Gamma$, and $\neg\psi \in \Delta$. By the Lindenbaum lemma, it suffices to show that $\{\chi \mid [a]\chi \in \Gamma\} \cup \{\neg\psi\}$ is consistent. Assume it is not, i.e. $\{\chi \mid [a]\chi \in \Gamma\} \cup \{\neg\psi\} \vdash \bot$, then $\{\chi \mid [a]\chi \in \Gamma\} \vdash \psi$ via Ded. Thus, with SNec: $\{[a]\chi \mid [a]\chi \in \Gamma\} \vdash [a]\psi$. Hence *a fortiori* $\Gamma \vdash [a]\psi$ and $[a]\psi \in \Gamma$, contradicting the assumption $[a]\psi \notin \Gamma$.

2. $\alpha = \beta^*$: $M, \Gamma \models [\beta^*]\psi \Leftrightarrow$ for all $n \in \mathsf{N}$ $M, \Gamma \models [\beta^n]\psi \Leftrightarrow ([\beta^n]\psi \in \Gamma$ for all $n \in \mathsf{N}) \Leftrightarrow [\beta^*]\psi \in \Gamma$, using the induction hypothesis in the second step, and $\mathsf{*AX}$, Inf^* in the last step.

Note that in the truth lemma, we do not prove the dual property for programs $\Gamma R_\alpha \Delta$ iff $\varphi \in \Delta$ for all φ such that $[\alpha]\varphi \in \Gamma$ (it holds by definition for atomic programs a). In section 3.2 we elaborate on this lack of "full harmony".

Theorem 1 (Strong completeness of PDL_ω) *Let $\mathcal{C}$ be the class of all Kripke frames for the language $\mathcal{L}_{P,\Pi}$. Then for all formulae φ and all sets of formulae Φ, $\Phi \models_{\mathcal{C}} \varphi$ implies $\Phi \vdash \varphi$.*

Proof By contraposition. Suppose $\Phi \nvdash \varphi$, then $\Phi \cup \{\neg\varphi\}$ is consistent. By lemma 2, $\Phi \cup \{\neg\varphi\}$ is extended to a maximal consistent set Γ with $\neg\varphi \in \Gamma$. Now by lemma 3, we conclude that in the canonical Kripke model, $M, \Gamma \not\models \varphi$, as desired.

Note that the completeness proof immediately gives a canonical standard model, contrary to the early proofs of weak completeness for PDL as they appear in [8, 5], which use nonstandard models.

3.2 Program disharmony

Somewhat surprisingly, the *formula harmony* property $M, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma$ proved in lemma 3 for the canonical model, is not matched by *program harmony*, i.e. the property

$$\Gamma R_\alpha \Delta \Leftrightarrow \{\varphi \mid [\alpha]\varphi \in \Gamma\} \subseteq \Delta \tag{2}$$

for all programs α and all maximal consistent sets Γ, Δ. For atomic programs, this holds by definition. The program constructors $(;, \cup)$ preserve (2), and it also holds for test programs. It fails, however, for programs with iterations. In [8], program disharmony (i.e. failure of program harmony) for finite canonical models for PDL was claimed without proof.

We shall sketch a counterexample for (2) with the program b^*. The maximal consistent sets will be defined by $\Gamma = \{\varphi \mid M, \eta \models \varphi\}, \Delta = \{\varphi \mid M, \omega \models$

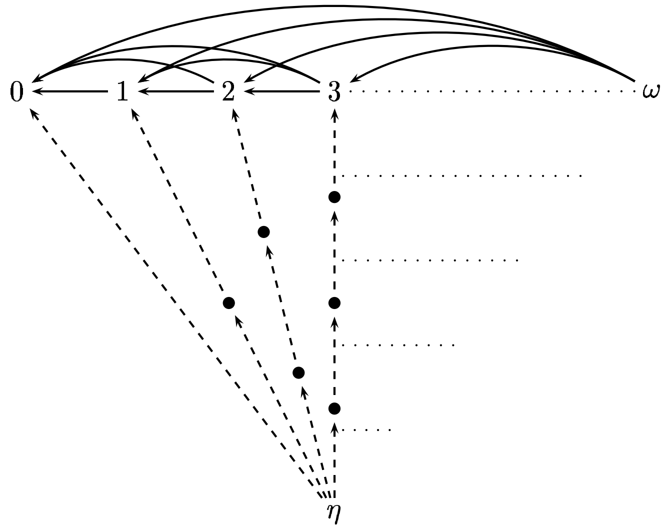

Figure 1: A picture of the countermodel M. The solid lines indicate the accessibility relation of the atomic program a, the dashed lines for b.

$\varphi\}$, where ω, η are two worlds in some model M to be constructed. More precisely, we shall show

$$\forall\varphi(M, \eta \models [b^*]\varphi \Rightarrow M, \omega \models \varphi) \tag{3}$$

$$\forall n \exists \varphi_n(M, \eta \models [b^n]\varphi_n \;\&\; M, \omega \models \neg\varphi_n) \tag{4}$$

Now (3) immediately implies $\{\varphi \mid [b^*]\varphi \in \Gamma\} \subseteq \Delta$, and (4) implies *not* $\Gamma R_{b^*} \Delta$, where $R_{b^*} = (R_b)^*$ is given by the canonical model (see definition 5).

We define M in two steps: first we study the model $N = \langle \mathsf{N}, > \rangle$, then we extend N to M. In N, we start with interpreting only the language $\mathcal{L}_{\emptyset,a}$ (no propositional atoms besides $\bot$, and a as the only atomic program); when going to M, we add the atomic program b. The extension of R_a is the strict order relation $>$ on N. We write $[\![\varphi]\!]$ for $\{n \in \mathsf{N} \mid N, n \models \varphi\}$, and $[\![\alpha]\!]$ for $R_\alpha \subseteq \mathsf{N}^2$. See Figure 1.

We say that $X \subseteq \mathsf{N}$ is *representable* iff $X = [\![\varphi]\!]$ for some φ. The crucial property of N is

$$\text{for every } \varphi \text{ in } \mathcal{L}_{\emptyset,a}, [\![\varphi]\!] \text{ is finite or cofinite} \tag{5}$$

($X \subseteq \mathsf{N}$ is cofinite iff $\mathsf{N} - X$ is finite). So there is no formula φ with e.g. $[\![\varphi]\!] = \{2x \mid x \in \mathsf{N}\}$. The hard part in proving (5) lies in finding the corresponding property for programs: which relations on N are representable as $[\![\alpha]\!]$, for some program α in $\mathcal{L}_{\emptyset,a}$? We shall define this property using $\mathsf{ST}(n)$, the collection of n-*stable* subsets of N:

$$X \in \mathsf{ST}(n) \text{ iff } \forall p \geq n(p \in X \Rightarrow [p, \omega) \subseteq X)$$

It is rather obvious that $\mathsf{ST}(n)$ is closed under $\cup$ and $\cap$. More surprising is that $\mathsf{ST}(n)$ is closed under *infinite* conjunctions. This is a consequence of the fact that $\supset$ (strict inclusion) is a wellordering on $\mathsf{ST}(n)$, hence every infinite conjunction $\bigcup_{i\in I} X_i$ of n-stable sets is equal to some finite subconjunction $X_{i_1} \cup \ldots \cup X_{i_k}$.

We extend stability to relations: $\mathsf{STR}(n)$ is defined by

$$R \in \mathsf{STR}(n) \text{ iff } \forall xy(xRy \to y \leq x \wedge y < n \wedge \{z \mid zRy\} \in \mathsf{ST}(n))$$

and claim: $\mathsf{STR}(n)$ is closed under finite conjunctions and arbitrary disjunctions, and under $\circ$ (composition).

Now we can characterise the representable relations: $R = [\![\alpha]\!]$ for some program α in $\mathcal{L}_{\emptyset,a}$ iff $R = R_1 \cup R_2 \cup R_3$ where, for some n, $R_1 = \emptyset$ or $R_1 = \{(x,x) \mid x \geq n\}$, $R_2 = \emptyset$ or $R_2 = \{(x,y) \mid m \leq x, n \leq y \leq x-k\}$ for some k, and $R_3 \in \mathsf{STR}(n)$. The 'only if' part is proved together with (5) in simultaneous induction over φ and α.

We extend N to $M = \langle \mathsf{M}, R_a, R_b \rangle$, where

$$\begin{array}{rcl} \mathsf{M} & = & \mathsf{N} \cup \{\omega, \eta\} \cup \{\langle x,y\rangle \in \mathsf{N}\times\mathsf{N} \mid 0 < y \leq x\} \\ R_a & = & \{(x,y) \in \mathsf{N}^2 \mid x > y\} \cup \{(\omega, x) \mid x \in \mathsf{N}\} \\ R_b & = & \{(\eta,0)\} \cup \{(\eta,\langle x,1\rangle), (\langle x,x\rangle, x) \mid x \in \mathsf{N}-\{0\}\} \\ & & \cup\, \{(\langle x,y\rangle, \langle x,y+1\rangle) \mid x,y \in \mathsf{N}, 0 < y < x\} \end{array}$$

See Figure 1. Observe that $R_a \cap \mathsf{N}^2 = \{(x,y) \mid x > y\}$, the extension of R_a in N. We claim: for all φ in $\mathcal{L}_{\emptyset,a}$

$$M, \omega \models \varphi \text{ iff } [\![\varphi]\!] \text{ is cofinite.} \tag{6}$$

(recall that $[\![\cdot]\!]$ denotes interpretation in N; we shall use R_α to denote the interpretation of α in M). This is proved via simultaneous induction over φ and α in $\mathcal{L}_{\emptyset,a}$, together with the statements $(\omega,x) \in R_\alpha \Leftrightarrow \{y \mid (y,x) \in [\![\alpha]\!]\}$ cofinite, and $(\omega,\omega) \in R_\alpha \Leftrightarrow \{(x,x) \mid (x,x) \in [\![\alpha]\!]\}$ cofinite.

Also, for all $x \in \mathsf{N}$:

$$M, x \models \varphi \Leftrightarrow M, x \models \varphi^{-b} \Leftrightarrow N, x \models \varphi^{-b} \tag{7}$$

where φ^{-b} is defined as φ with all occurrences of b replaced by $\perp?$. The first equivalence in (7) also holds for $x = \omega$.

As a consequence of (6) and (7), we have, for all φ in $\mathcal{L}_{\emptyset,a}$

$$M, \omega \models \varphi \Leftrightarrow \exists x \in \mathsf{N} \forall y \in \mathsf{N}(M, y \models \varphi)$$

and this entails (3): for if $M, \eta \models [b^*]\varphi$, then $M, x \models \varphi$ for every $x \in \mathsf{N}$, hence $M, x \models \varphi^{-b}$ for every $x \in \mathsf{N}$, so $M, \omega \models \varphi^{-b}$ and $M, \omega \models \varphi$. In order to verify (4), we put $\varphi_n =_{\text{def}} [a^{n+1}]\perp$: then $\forall x \in \mathsf{N}(M, x \models \varphi_n \Leftrightarrow N, x \models \varphi_n \Leftrightarrow x \leq n)$ and $\forall x, y(\langle x,y\rangle \in \mathsf{M} \Rightarrow M, \langle x,y\rangle \models \varphi_n)$, so indeed $M, \eta \models [b^n]\varphi_n$ and $M, \omega \models \neg\varphi_n$ (using (6)). Therefore both (3) and (4) hold, and program disharmony is demonstrated.

4 Epistemic logics with common knowledge

In this section, we extend the strong completeness result for PDL_ω to epistemic logic with common knowledge. First we repeat the definitions language and semantics. (For more on epistemic logic, see [9, 2]; we loosely base the axioms below on the version of [2].).

Definition 6 (Language of epistemic logic) *A language $\mathcal{L}_{P\mathcal{A}}$ of epistemic logic is based on a countable set P of atomic formulae p and a finite set $\mathcal{A}$ of agents a, and is given by the following rule:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box_a \varphi \mid E\varphi \mid C\varphi$$

Definition 7 (Kripke semantics for epistemic logic) *A Kripke model for $\mathcal{L}_{P\mathcal{A}}$ is a pair (M, w) such that $M = (W, \{R_a : a \in \mathcal{A}\}, V)$ and:*

- *$W \neq \emptyset$; a set of states or possible worlds;*
- *$R_a \subseteq W \times W$; an accessibility relation, for each agent a;*
- *$V : P \to 2^W$; assigns a set of states to each propositional variable.*

The truth definition is as usual; we only give the clauses for the modal operators.

$$\begin{array}{lll}
(M,w) \models \Box_a \varphi & \text{iff} & (M,v) \models \varphi \text{ for all } v \text{ such that } wR_av \\
(M,w) \models E\varphi & \text{iff} & (M,v) \models \varphi \text{ for all } v \text{ such that } wR_av \text{ for any } a \\
(M,w) \models C\varphi & \text{iff} & (M,v) \models \varphi \text{ for all } v \text{ such that } v \text{ is reachable} \\
& & \text{from } w
\end{array}$$

Here, "v is reachable from w" iff there is a path of length ≥ 1 in the Kripke model from w to v along accessibility arrows R_a associated with members a of $\mathcal{A}$.

The similarity between PDL and epistemic logic with common knowledge has long been noted. In fact, strong completeness of the latter immediately reduces to the former by the following embedding $'$. Suppose the set of agents $\mathcal{A} = \{a_1, \ldots, a_n\}$, and define:

$$\begin{array}{lll}
(\Box_{a_i}\varphi)' & =_{\text{def}} & [a_i]\varphi' \\
(E\varphi)' & =_{\text{def}} & [a_1 \cup \ldots \cup a_n]\varphi' \\
(C\varphi)' & =_{\text{def}} & [(a_1 \cup \ldots \cup a_n)^*]\varphi'
\end{array}$$

However, we can prove something stronger if we take a direct approach. It turns out that the Henkin method can, in addition to K, also be used for systems like T, $\mathsf{S4}$ and $\mathsf{S5}$ with common knowledge. In order to give the direct proof, we fix the axioms and give an infinitary derivation relation.

Definition 8 (Axioms for $\mathsf{KEC}_{P\mathcal{A}}$) *The axiom system for basic epistemic logic with common knowledge $\mathsf{KEC}_{P\mathcal{A}}$ contains the following axioms:*

Taut	all instantiations of propositional tautologies
Distr	$\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$
EAX	$E\varphi \leftrightarrow \bigwedge_{a\in\mathcal{A}} \Box_a\varphi$
Mix	$C\varphi \leftrightarrow E(\varphi \wedge C\varphi)$

Definition 9 (Axioms for $\mathsf{TEC}_{P\mathcal{A}}$, $\mathsf{S4EC}_{P\mathcal{A}}$, and $\mathsf{S5EC}_{P\mathcal{A}}$) *The axiom systems for stronger epistemic logics with common knowledge $\mathsf{TEC}_{P\mathcal{A}}$, $\mathsf{S4EC}_{P\mathcal{A}}$ and $\mathsf{S5EC}_{P\mathcal{A}}$ contain the axioms of $\mathsf{KEC}_{P\mathcal{A}}$. In addition, $\mathsf{TEC}_{P\mathcal{A}}$ contains* A3*;* $\mathsf{S4EC}_{P\mathcal{A}}$ *contains* A3 *and* A4*;* $\mathsf{S5EC}_{P\mathcal{A}}$ *contains* A3*,* A4*, and* A5*, as defined below:*

A3	$\Box_a\varphi \to \varphi$	knowledge
A4	$\Box_a\varphi \to \Box_a\Box_a\varphi$	positive introspection
A5	$\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$	negative introspection

We extend all four axiom systems to infinitary proof systems $\mathsf{KEC}_{P\mathcal{A}\omega}$, $\mathsf{TEC}_{P\mathcal{A}\omega}$, $\mathsf{S4EC}_{P\mathcal{A}\omega}$, and $\mathsf{S5EC}_{P\mathcal{A}\omega}$ by adding a fixed set of derivation rules. We first introduce some notation in order to describe the infinitary introduction rule for C. We want this rule to contain all instances of the form

$$\{\varphi_1 \to \Box_a(\varphi_2 \to \Box_b(\ldots \to E^n\psi)) \mid n \in \mathsf{N}, n \geq 1\} \vdash \\ \varphi_1 \to \Box_a(\varphi_2 \to \Box_b(\ldots \to C\psi)),$$

where $E^n\psi$ is the obvious abbreviation defined inductively by $E^0\psi = \psi$ and $E^{n+1}\psi = EE^n\psi$. The neat way to formulate the infinitary rule is to introduce finite sequences $\pi = (\pi_1, \ldots, \pi_n)$ where the π are either formulae or modalities $\Box_a$ for $a \in \mathcal{A}$, with

$$\begin{array}{lll} ()\varphi & =_{\text{def}} & \varphi \\ (\psi;\pi)\varphi & =_{\text{def}} & \psi \to (\pi)\varphi \\ (\Box_a;\pi)\varphi & =_{\text{def}} & \Box_a((\pi)\varphi) \end{array}$$

The infinitary rule may then be formulated as $\{(\pi)E^n\varphi \mid n \in \mathsf{N}, n \geq 1\} \vdash (\pi)C\varphi$. We give the derivation rules for the infinitary systems.

Definition 10 (Infinitary derivation relations) *Let* S *be any of* $\mathsf{KEC}_{P\mathcal{A}\omega}$, $\mathsf{TEC}_{P\mathcal{A}\omega}$, $\mathsf{S4EC}_{P\mathcal{A}\omega}$*, and* $\mathsf{S5EC}_{P\mathcal{A}\omega}$. $\Gamma \vdash_{\mathsf{S}} \varphi$ *is defined as the smallest relation closed under the following rules:*

AX	$\vdash_{\mathsf{S}} \varphi$ if φ is an axiom of S	
MP	$\varphi, \varphi \to \psi \vdash_{\mathsf{S}} \psi$	(modus ponens)
InfC	$\{(\pi)E^n\varphi \mid n \in \mathsf{N}, n \geq 1\} \vdash_{\mathsf{S}} (\pi)C\varphi$	(infinitary C-introduction)
Nec	if $\vdash_{\mathsf{S}} \varphi$ then $\vdash_{\mathsf{S}} \Box_a\varphi$	(necessitation)
W	if $\Gamma \vdash_{\mathsf{S}} \varphi$ then $\Gamma, \Delta \vdash_{\mathsf{S}} \varphi$	(weakening)
Cut	if $\Gamma \vdash_{\mathsf{S}} \Delta$ and $\Gamma, \Delta \vdash_{\mathsf{S}} \varphi$ then $\Gamma \vdash_{\mathsf{S}} \varphi$	

Now the reader may check that for these systems, the derived rules SCut, Det, Cond, and Ded of lemma 1 can be proved, as well as the following analogue of strong necessitation SNec:

SNecK if $\Gamma \vdash_{\mathsf{S}} \varphi$ then $\Box_a\Gamma \vdash_{\mathsf{S}} \Box_a\varphi$ (strong necessitation for knowledge)
It is immediate that all four systems are sound with respect to the appropriate semantics: $\mathsf{KEC}_{P\mathcal{A}\omega}$ for all Kripke frames, $\mathsf{TEC}_{P\mathcal{A}\omega}$ for reflexive ones, $\mathsf{S4EC}_{P\mathcal{A}\omega}$ for reflexive transitive ones, and $\mathsf{S5EC}_{P\mathcal{A}\omega}$ for equivalence relations.

Theorem 2 *Let* S *be any of the systems* $\mathsf{KEC}_{P\mathcal{A}\omega}$, $\mathsf{TEC}_{P\mathcal{A}\omega}$, $\mathsf{S4EC}_{P\mathcal{A}\omega}$, *and* $\mathsf{S5EC}_{P\mathcal{A}\omega}$. *Then* S *is strongly complete with respect to the appropriate set of frames.*

Proof sketch By a Henkin construction of a canonical model, analogously as in section 3.1. The presence of the appropriate axioms from A3, A4, and A5 in the maximal consistent sets induces the appropriate properties of the accessibility relations in the canonical model. In the analogue of the Lindenbaum lemma, the last clause for Γ_{n+1} should be "$\Gamma_{n+1} =_{\text{def}} \Gamma_n \cup \{\neg\varphi_n, \neg(\pi)E^k\psi\}$ otherwise, where k is the least natural number ≥ 1 such that $\Gamma_n \nvdash (\pi)E^k\psi$ (and φ_n is of the form $(\pi)C\psi$)". The definition of the canonical model is as usual for epistemic logics. The main difference from the proof of the Truth lemma 3 is the induction step for operator C, which works very smoothly: $M, \Gamma \models C\psi \Leftrightarrow$ for all $n \in \mathsf{N}, n \geq 1$ $M, \Gamma \models E^n\psi \Leftrightarrow (E^n\psi \in \Gamma$ for all $n \in \mathsf{N}, n \geq 1) \Leftrightarrow C\psi \in \Gamma$, using the induction hypothesis in the second step, and Mix, InfC in the last step.

It is clear from the proof sketch that all four epistemic logics with common knowledge are canonical: on their canonical frames, all their axioms are valid [1].

Example To show that the infinitary proof systems are stronger than the usual weakly complete ones, let us look at the well-known example of the Byzantine generals. "Consider a situation of two army regiments on two hills on both sides of a valley, in which a hostile army is situated. If the two regiments attack simultaneously, they will conquer the enemy. If only one of the regiments attacks, it will be defeated by the enemy. There is no initial battle plan available which the two regiments agree upon. Neither general of the regiments would decide to attack without knowing for sure that the other one will also attack" (from [9]). "The commanding general of the first division wishes to coordinate a simultaneous attack (at some time the next day). The generals can communicate only by means of messengers. Normally, it takes a messenger one hour to get from one encampment to the other. However, it is possible that he will get lost in the dark, or worse yet, be captured by the enemy. Fortunately, on this particular night, everything goes smoothly. How long will it take them to coordinate an attack?" (from [2]). Unfortunately, it turns out that normally, common knowledge among the two generals cannot be reached in this way: at each point in time, only $E^n\varphi$ for some $n \in \mathsf{N}, n \geq 1$ is achieved, where φ is "at least one message was delivered". (See [2, 13] for explanations of this and similar phenomena).

We have a suggestion for the unfortunate generals. It follows immediately by InfC that $\{E^n\varphi \mid n \in \mathsf{N}, n \geq 1\} \vdash_{\mathsf{S}} C\varphi$ for any of the four infinitary epistemic proof systems S (unlike for the usual finitary proof systems). Now the generals only need to hire a messenger who runs twice as fast on every new round as on the previous one; the first round takes 2 hours, the second round 1 hour, and so on, ad infinitum. Summing the series, after 4 hours all the $\{E^n\varphi \mid n \in \mathsf{N}, n \geq 1\}$ are achieved, whereby $C\varphi$ holds. We admit that the suggestion cannot be modeled as a multi-agent system according to [2] because the time steps are not structured as the natural numbers ω but as $\omega + 1$. Moreover, some practical problems still need to be solved, which we leave to physicists.

5 Conclusion

In this paper we have presented a proof system for propositional dynamic logic which is strongly complete. It can also be applied to epistemic logic with common knowledge.

We suspect that the reason that the canonical model method works for this axiomatisation, is that the infinitary *-introduction rule is much closer to the semantics than the usual induction axiom or rule. The latter links up with the idea of the Kleene star as a fixed point, whereas our rule links up with the idea that it is an infinitary conjunction. However, as we showed in section 3.2, there is no complete harmony between the proof system and the semantics. The countermodel used in the completeness proof has *formula harmony*. This is shown in the truth lemma. *Program harmony* is unattainable. To our surprise it was not needed for the completeness proof. To our astonishment it was not even true. Although we came up with a countermodel rather quickly, the subtlety of the arguments involved, was also unexpected. It would be interesting to try to construct a fully harmonious model for PDL_ω. As of yet we did not find one in the literature.

There still remain some issues that need to be investigated further. Propositional dynamic logic and epistemic logic with common knowledge are examples where the introduction of an infinitary rule can be used to attain strong completeness, although the logics are not semantically compact. It should be investigated how to characterise the class of non-compact logics where the introduction of such an infinitary rule can also lead to a strong completeness result. The general approach of Goldblatt [4] seems to be a good starting point.

Another interesting issue is whether the relation $\Gamma \vdash \varphi$ between recursively enumerable sets of formulae Γ and formulae φ is decidable.

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