An extensive game model for IF-logic

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Abstract: Hintikka describes the semantical games for Independence Friendly logic (IF-logic) in terms of the game rules. In this paper we elaborate in detail how the standard extensive game model serves as a mathematical model for these games. We intend the game model to be a framework in which we can reason with mathematical rigor about strategies, hence about truth and falsity in GTS. We discuss negation normal forms, and compare the notion of Skolem function with the game theoretical notion of strategy.

1 IF-logic and Game Theoretical Semantics

In classical first order logic the scopes of quantifiers are always either nested or disjoint. In IF-logic, introduced by Hintikka in [3] and by Hintikka and Sandu in [4], other dependency patterns become possible by a slash operator. This operator can be introduced into ordinary first order formulas to remove quantifications and connectives from the scope of previous quantifications.[1] The truth or falsity of IF-sentences is determined by Game Theoretical Semantics (GTS). GTS associates with every IF-sentence φ and suitable model M a semantical game $G_M(\varphi)$. The game is played by two players: Eloïse, who starts in the role of 'verifier', and Abélard, who starts in the role of 'falsifier'. Depending on the main quantifier or connective of φ (one of $\forall$, $\exists$, $\wedge$, $\vee$ or $\sim$), one of the players makes a move and the game continues with the appropriate subformula of φ. Moves in the game are choices for domain elements as assignments for the variables bound by the quantifiers $\exists$ (move for player in the role of verifier) and $\forall$ (move for the player in the role of falsifier) and choices for 'left' or 'right', associated with the connectives $\vee$ and $\wedge$ (move for verifier and falsifier respectively). The negation sign $\sim$ does not prompt a move for one of the players, but makes the two players change roles.
In all these cases the addition of '$_{/Y}$', with Y a -possibly empty- set of variables, means that the choice made by the player in question does not

[1] In [3], the operator is only applied to existential quantifiers and disjunctions. Throughout this paper we take IF-logic to be defined as in [4], where the slash is applied to universal quantifiers and conjunctions as well.

depend on the values chosen for the variables in Y.[2] For example, in the IF-formula $\forall y \exists x_{/y}[x = y]$, the choice for the value of x by Eloïse should be the same for all values of y possibly chosen by Abélard in the previous move. Truth and falsity of a sentence φ in a model M are defined as follows:

$$\varphi \text{ is } \mathtt{true} \text{ in } M \overset{d}{\Leftrightarrow} \text{there exists a winning strategy for Eloïse in } G_M(\varphi)$$

$$\varphi \text{ is } \mathtt{false} \text{ in } M \overset{d}{\Leftrightarrow} \text{there exists a winning strategy for Abélard in } G_M(\varphi)$$

From this definition and the fact that the slash operator makes semantical games into games of imperfect information, it follows that the principle of the excluded middle does not hold for IF-logic (see e.g. [3], p. 132). We also become aware of the importance of a thorough definition of the notion of strategy.

Semantical games can be characterized by the following game theoretical terms: they are 2-player, non-cooperative finite zero-sum games, in which every run of the game is won by one of the players and lost by his opponent. In IF-logic the games can be of imperfect information, and, as we shall see in our discussion on Skolem functions as strategies, also of imperfect recall. The notation of Osborne and Rubinstein [7] allows us to formalize these games in a natural way. We can leave out the possibility of infinite courses of the game, and the so called chance moves.
The goal of our project is to capture the concept of strategy in semantical games and we hope to find a correspondence between the game theoretical notion of strategy and the notion of Skolem function, which is put forward by Hintikka as the constituting element of strategies in GTS, and the basis for the translation of IF-logic to Σ_1^1 and back.

2 The general game model

We first give the definition of a general mathematical game model for extensive form games, based on [7], pp. 200-203. In this definition, we use the following notation for finite sequences: let A be a set ('the alphabet'), A^* the set of finite sequences of elements of A, $\alpha = (a_1, \ldots, a_k) \in A^*$

[2] Some notational issues: In the case Y is empty, we will omit the slash, and if Y is nonempty we say that the quantifier or connective is slashed. If Y is non-empty and written out explicitly under the slash, then we omit the braces: e.g. we write $\exists x_{/y,z}$ rather than $\exists x_{/\{y,z\}}$.

and $a \in A$. Then: $\texttt{len}(\alpha) := k$; $\alpha|m := (a_1, \ldots, a_m)$ for $m \leq \texttt{len}(\alpha)$; $\alpha \cdot a := (a_1, \ldots, a_k, a)$; $a \in \alpha := \exists j \leq \texttt{len}(\alpha)[a = a_j]$. Finally, $()$ denotes the empty sequence.

An **extensive form game of imperfect information** is defined to be a 5-tuple $\Gamma = \langle H, n, P, \overline{U}, r \rangle$, where

- H is a set of finite sequences that satisfies the following conditions:
  - $()$ is a member of H;
  - H is **prefix closed**: if $\alpha \in H$ and $m \leq \texttt{len}(\alpha)$, then $\alpha|m \in H$.

  Notation: $A := \bigcup_{h \in H}\{a | a \in h\}$ (thus: $H \subseteq A^*$). Define for $h \in H$: $A(h) := \{a \in A | h \cdot a \in H\}$. Let $Z := \{h \in H | A(h) = \emptyset\}$, and $X := H \setminus Z$.

  H is the set of *histories*. Z is the set of *terminal* histories; X is the set of non-terminal histories or *decision points*. A terminal history describes a *course of the game*. $A(h)$ is the set of allowed *actions* from history h.

- $n \in \mathbb{N}$, notation: $N = \{1, \ldots, n\}$.

  n is the number of *players*. We identify N with the set of players.

- $P : X \to N$. Notation: for $i \in N$ we write $P_i := P^{-1}(\{i\})$.

  P is the *player function*: it determines for each decision point whose turn it is. Note that one or more of the P_i might be empty.

- $\overline{U} = (U_1, U_2, \ldots, U_n)$, where for every $i \in N$, U_i is a partition of P_i such that for all $u \in U_i$ the so-called *consistency* condition is satisfied: if $h, h' \in u$ then $A(h) = A(h')$. Hence we can define $A[u] := A(h)$ (any $h \in u$). Notation: $U := \bigcup_{i \in N} U_i$.

  Elements of U are called *information sets*; $A[u]$ is the set of allowed actions from information set u.

- $r : Z \to \mathbb{R}^N$: a *payoff* function[3], giving for every terminal history the respective gain of each player in that situation.

[3]We prefer to define the last component as a function (like e.g. in [10]), rather than as tuple of preference relations like in [7].

A **strategy** for player i in the game Γ is a function $F^i : U_i \to A$ s.t. for every $u \in U_i : F^i(u) \in A[u]$.

The information partition U makes this a model for *imperfect information* games: the intended interpretation of the information sets u in U_i is that player i knows the actual history h must be one of the elements of u, but is not able to determine which one when choosing an action (an element of $A(h) = A[u]$) to continue the game. Hence, this choice is *uniform* over the set of histories u. If all information sets are singletons, Γ models a game of perfect information.

In the next section, we will specify semantical games in terms of this standard game model.

3 A game model for semantical games in IF-logic

In section 1, we described semantical games for IF-first order sentences in terms of the *game rules*, as is done in [3]. This type of description is not very mathematical of nature, and hence does not facilitate mathematical proofs over semantical games.
We specialize the general game model of the previous section to describe semantical games as extensive form games.[4] A semantical game $G_M(\varphi)$ is a 2-player game with two parameters: an IF-sentence φ and a model M suitable for the language of φ. With these parameters, we can define a set of histories H, a player function P, the set of information sets U, and an outcome function r.

First we fix some notations that we will use in the definitions. For a given IF-sentence φ and a suitable model M, we determine:

1. the set $Sub(\varphi)$ of subformulas of φ, that contains all different occurrences of syntactically identical subformulas as distinct elements. (For example: in a formula of the form $\psi \vee \sim\psi$, we want to distinguish between both occurrences of (the subformulas of) ψ.) Mostly, we will need the set $S_\varphi := \{[\psi]^+ | \psi \in Sub(\varphi)\}$, where we write $[\psi]^+$ for the formula resulting from ψ after removing initial negation signs.

2. for each $\psi \in S_\varphi$ the set D_ψ of all variables in φ under whose scope ψ occurs. Note that this set contains but is not necessarily equal to

[4] In [8] and [9], Sandu and Pietarinen follow a similar approach.

the set of free variables of ψ, and that unbound variables under the slashes also count as free variables.

3. for every $\psi \in S_\varphi$, the set V_ψ of all valuations in $Dom(M)$ with domain D_ψ. Furthermore: for $\psi \in S_\varphi$ that are non-atomic, let Y_ψ be the -possibly empty- sequence of variables occurring under the slash of the main connective or quantifier of ψ, and $\overline{V}_\psi$ the set of of valuations with domain $\overline{D}_\psi := D_\psi \setminus Y_\psi$. Let $V := \bigcup_{\psi \in S_\varphi} V_\psi$, and $\overline{V} := \bigcup_{\psi \in S_\varphi} \overline{V}_\psi$.

We are now ready to define the components of the game model: $\Gamma_M(\varphi)$.

the set of histories H
To define the set of histories for a semantical game, we first introduce for every $\psi \in S_\varphi$ a set of finite sequences H_ψ. Note that we do this by induction on the structure of φ –from outside to inside– in a way that keeps us within S_φ: the induction 'skips' the negation signs. It terminates at the atomic formulas.

- $H_{[\varphi]^+} := \{()\}$;
- Let $\psi \in S_\varphi$, H_ψ already defined. Then there are three cases:
  1. ψ is atomic: induction stops;
  2. ψ is of the form $\psi_1 \vee_{/Y} \psi_2$ or $\psi_1 \wedge_{/Y} \psi_2$: then $H_{[\psi_1]^+} := \{h \cdot \mathtt{l} | h \in H_\psi\}$ and $H_{[\psi_2]^+} := \{h \cdot \mathtt{r} | h \in H_\psi\}$.
  3. ψ is of the form $\exists x_{/Y} \psi'$ or $\forall x_{/Y} \psi'$: then $H_{[\psi']^+} := \{h \cdot a | a \in Dom(M), h \in H_\psi\}$;

Let $H := \bigcup_{\psi \in S_\varphi} H_\psi$, then H is the set of histories for $\Gamma_M(\varphi)$. If $h \in H_\psi$, we say that h leads to the subformula ψ. For $h \in H$, we write ψ_h for the unique $\psi \in S_\varphi$ such that $h \in H_\psi$.

We see that the set of actions is $A = Dom(M) \cup \{\mathtt{l}, \mathtt{r}\}$, and for every $h \in X$: either $A(h) = Dom(M)$ (if ψ_h prompts a choice for a quantifier) or $A(h) = \{\mathtt{l}, \mathtt{r}\}$ (if ψ_h prompts a choice for a connective).
Note that $S_\varphi = \{\psi_h | h \in H\}$. Let X and Z be defined as the sets of non-terminal and terminal histories respectively (cf. the general case in section 2). Now define $S_X := \{\psi_h | h \in X\}$ and $S_Z := \{\psi_h | h \in Z\}$, then S_X is the set of subformulas for which the game rules of section 1 prescribe a choice by one of the players, and S_Z is the set of atomic formulas.

Every history $h \in H$ 'contains' a valuation $v_h \in V_{\psi_h}$ for the formula ψ_h:

- $v_{()} := \emptyset$;
- if v_h has been defined and $\psi_h = \psi_1 \vee_{/Y} \psi_2$ or $\psi_h = \psi_1 \wedge_{/Y} \psi_2$, then $v_{h\cdot\mathtt{l}} = v_{h\cdot\mathtt{r}} := v_h$;
- if v_h has been defined and $\psi_h = \exists x_{/Y}\psi'$, or $\psi_h = \forall x_{/Y}\psi'$ then for every $a \in Dom(M)$: $v_{h\cdot a} := v_h \cup \{(x,a)\}$;[5]

In fact, the mapping $h \mapsto \langle \psi_h, v_h \rangle$ is a one-one correspondence of the set of histories H with the set $\bigcup_{\psi \in S_\varphi} \{\langle \psi, v \rangle | v \in V_\psi\}$, consisting of pairs of subformulas and valuations.

the set of players N
Semantical games are two-person games by definition, so $n = 2$. The players are usually called Abélard and Eloïse, hence we choose as set of players $N = \{\mathtt{A}, \mathtt{E}\}$ rather than $\{1, 2\}$.

Before we define the player function P and the outcome function R, we define for $\psi \in S_\varphi$ its **polarity in** φ: ψ is **positive** in φ if it occurs in the scope of an even number of negation signs in φ, and **negative** if it occurs in the scope of an odd number of negation signs in φ.

the player function P
The player function $P : X \to N$ is determined for each $h \in X$ by the main 'constructor' (quantifier or connective) of $\psi_h \in S_X$, and its polarity in φ:

$$\begin{cases} P(h) = \mathtt{E} & \text{if either:} \\ & \psi_h \text{ is positive and of the form } \psi_1 \vee_{/Y} \psi_2 \text{ or } \exists x_{/Y}\psi', \\ & \text{or:} \\ & \psi_h \text{ is negative and of the form } \psi_1 \wedge_{/Y} \psi_2 \text{ or } \forall x_{/Y}\psi'; \\ P(h) = \mathtt{A} & \text{in the other cases} \end{cases}$$

Thus P is constant on each H_ψ, $\psi \in S_X$. If $H_\psi \subseteq P_i$, we say that subformula ψ is assigned to player i.

the outcome function r
In win/loss games like the semantical games of GTS, we can simplify the payoff function of section 2. We will define r as what could be called an

[5] Throughout this paper we assume: if $Qx\psi$ is a subformula of φ, then ψ does not contain a second quantification binding x. We do not consider this to be a restriction. By this assumption, $v_{h\cdot a}$ as defined here is well defined as a function.

outcome function: r assigns to each terminal history $h \in Z$ a winner in N. This outcome is fixed by the polarity of ψ_h in φ, and the truth value of ψ_h in $\langle M, v_h \rangle$:

$$\begin{cases} r(h) = \mathtt{E} & \text{if either: } \psi_h \text{ is positive and } \langle M, v_h \rangle \models \psi_h \\ & \text{or: } \psi_h \text{ is negative and } \langle M, v_h \rangle \not\models \psi_h \\ r(h) = \mathtt{A} & \text{in the other cases} \end{cases}$$

the information partition $\overline{U} = (U_{\mathtt{A}}, U_{\mathtt{E}})$

The information sets in the game model for semantical games are induced by the slash notation in IF-logic: it is the slash notation that makes semantical games for IF-formulas into games of imperfect information. Informally, the effect of a slash at a quantifier or connective in a sentence φ, is that the player making the choice associated with it, cannot base it on the values chosen for the variables under the slash. In other words: the player has to make the same choice if other values had been chosen for those variables.

We define the information sets formally using the following equivalence relation on valuations for a subformula $\psi \in S_X$: let $v, v' \in V_\psi$, then:

$$v' \sim_\psi v \overset{d}{\Leftrightarrow} v'(x) = v(x) \text{ for all } x \in \overline{D}_\psi$$

In other words: $v' \sim_\psi v$ if and only if v and v' coincide outside the values they assign to the variables in Y_ψ.

The information sets in the semantical game for φ can now be defined as the equivalence classes $u[h]$ of the following equivalence relation on X:

$$h' \sim_U h \overset{d}{\Leftrightarrow} \psi_{h'} = \psi_h \text{ and: } v_{h'} \sim_{\psi_h} v_h$$

In words: two non-terminal histories are considered to be indistinguishable for the player assigned to them, if they lead to the same subformula ψ of φ *and* if the associated valuations assign the same values to the variables in $\overline{D}_\psi$.

Remark: if ψ is not slashed, meaning that Y_ψ is empty, then $D_\psi = \overline{D}_\psi$, and hence all histories leading to ψ are distinguishable. In the special case that φ is a classical first order formula, all information sets $u[h]$ as defined above are singletons. Semantical games for classical first order formulas are hence games of perfect information.

Concluding, we define U to be $\{u[h] | h \in H\}$, and $U_i := \{u[h] | h \in P_i$. (Note that the U_i are well defined, because for $i \in N$: if $h \in P_i$, then

$u[h] \subseteq H_{\psi_h} \subseteq P_i$.)

In analogy to the correspondence we had between histories h and pairs $\langle \psi_h, v_h \rangle$, we can characterize the information sets $u[h]$ by the pairs $\langle \psi_h, \overline{v_h} \rangle$, where $\overline{v_h} \in \overline{V}_{\psi_h}$ is the restriction of v_h to $\overline{D}_{\psi_h}$. This is then a one-one correspondence of the set U of information sets with the set: $\bigcup_{\psi \in S_X} \{\langle \psi, \overline{v} \rangle | \overline{v} \in \overline{V}_\psi\}$.

strategies and strategy functions

We claim that the 5-tuple $\Gamma_M(\varphi) = \langle H, N, P, r, \overline{U} \rangle$, as defined above, captures the semantical game $G_M(\varphi)$ as described by the game rules in section 1. But the central issue was the formalization of the concept of strategy. We conclude this section describing the notion of strategy following the game theoretical definition of section 2. In the next section, this notion will be compared with the approach of regarding Skolem-functions as strategies.

Following the definition of section 2, a strategy for a player i in game Γ, is a function F^i assigning to each information set $u \in U_i$ an action $a \in A[u]$. For semantical games, it is more natural to see a strategy as a *set* of **strategy functions** $f^i_\psi : \overline{V}_\psi \to A_\psi$, one for each subformula of φ that is assigned to player i. Here A_ψ is either $Dom(M)$ or $\{\mathtt{l}, \mathtt{r}\}$, depending on the kind of move prompted by the structure of ψ.

Due to the correspondence between U and pairs $\langle \psi, \overline{v} \rangle$ with $\psi \in S_X, \overline{v} \in \overline{V}_\psi$ (see the previous subsection), both approaches can be seen to be equivalent:

- If $F^i : U_i \to A$ is a strategy for player i we can define the corresponding strategy functions $f^i_\psi : \overline{V}_\psi \to A$ (ψ assigned to player i) by

  $$f^i_\psi(\overline{v}) := F^i(u),$$

  where u is the information set corresponding to the pair $\langle \psi, \overline{v} \rangle$.

- If $\{f^i_\psi | \psi \in S_X, \psi \text{ assigned to player } i\}$ is a set of strategy functions, we can form the strategy $F^i : U_i \to A$ by defining for each $h \in P_i$:

  $$F^i(u[h]) = f^i_{\psi_h}(\overline{v_h}).$$

We choose to describe strategies as sets of strategy functions. Informally, a strategy function f^i_ψ gives us a choice for player i at formula ψ on the basis of the values previously chosen for the variables in $\overline{D}_\psi$.

4 Discussion of the model

Semantical games defined for sentences only:
Note that our approach is non-compositional. All the building blocks are defined in the context of a sentence φ. Open formulas occur in the context of an IF-sentence φ only. It is much more complicated to give a sensible game theoretical interpretation of open IF-formulas. The papers [5], [1] and [6] show alternative systems that allow for quantifier independences by the same syntactical construction, but do give interpretations to open formulas.

Invariance of the game model for negation normal form:
In the definition of GTS in game rules (see section 1), negation appears to be a dynamic element in the game: during the game, the roles of the two players are reversed. But in fact, the 'meaning' of the negation signs is completely covered in both the player function and the payoff function. This is reflected by the following claim:
Let φ be an IF-first order sentence. If a formula φ' results from sentence φ by application of either De Morgan's Laws or cancelling of double negation, then in every model M: $\Gamma_M(\varphi)$ and $\Gamma_M(\varphi')$ have the same set of histories H, information partition U, player function P and outcome function r.
This implies that φ and φ' cannot be distinguished game theoretically, and hence are (strongly) equivalent in GTS in the sense that for *both* players holds: there exists a winning strategy for that player in $\Gamma_M(\varphi)$ iff there exists a winning strategy for that player in $\Gamma_M(\varphi')$. Even stronger: a winning strategy for a player in $\Gamma_M(\varphi)$ *is* a winning strategy in $\Gamma_M(\varphi')$, and vice versa.[6]
It follows that every sentence φ has a strongly equivalent negation normal form φ'. Hence, Hintikka's claim that we may assume IF-formulas to be in negation normal form ([3], p. 52) is justified by our model.[7]

Independence vs uniformity: Skolem functions as strategy functions?
Perfect recall is informally the property that at every decision point for a player, he knows what he has known and done previously. For a formal definition: see [7], p. 203. It is quite usual to presuppose perfect recall (see

[6]In the light of the discussion of the next paragraph, we note that this holds for any notion of strategy defined in terms of $\Gamma_M(\varphi)$.

[7]For the strong equivalence however, we need the applicability of the slash operator to universal quantifiers and conjunctions as well as to existential quantifiers and disjunctions. In [3] Hintikka restricts IF-logic to the latter, but in [4] both are allowed.

also [10], p. 104). However, the following example proposed by Hodges in [5] (p. 548), shows that this presupposition conflicts with Hintikka's intended semantics:

$$\forall x \exists z \exists y_{/x}[x = y]$$

For the standard game theoretical definition of strategy described at the end of the previous section, Eloïse (E) has a winning strategy: the uniformity restrictions on strategy functions do not forbid her to choose the value for z equal to the value assigned to x, and then choose the value for y equal to the value assigned to z. The semantics of both [5] and [1] correspond with this, and make this formula `true`.

The following citation ([3], p. 63) shows that the uniformity restriction on strategies does not satisfy Hintikka's goal to formalize independence:[8] "At this point, it is in order to look back at the precise way the information sets of different moves are determined in semantical games with IF-sentences. The small extra specification that is needed, is that moves connected with existential quantifiers are always independent of earlier moves with existential quantifiers. [...] The reason for this provision is that otherwise "forbidden" dependencies of existential quantifiers on universal quantifiers could be created through the mediation of intervening existential quantifiers."

We follow Janssen ([6], which paper proposes alternative semantics specifically designed to formalize *independent choices*) in referring to this 'specification' as the 'slashing convention'.

Note that Hintikka discusses choices for Eloïse in a game $G_M(\varphi)$, where φ is assumed to be in negation normal form. In this situation, Skolem functions can be conceived as strategy functions. Examples throughout [3] show that Skolem functions are the strategy functions intended by Hintikka.[9]

We can let our model comply to the 'slashing convention' for an arbitrary IF-formula φ, so that Skolem functions come out as strategy functions, by the following simple procedure: first, rewrite φ in negation normal form (which is no problem, by the previous subsection), and subsequently: for all subformulas ψ of the form $\exists\psi'_{/Y}$ or $\psi_1 \vee_{/Y} \psi_2$, add the existentially quantified variables in D_ψ to Y. Finally, build the model $\Gamma_M(\varphi')$ for the resulting formula φ'.

[8] The start of this quote is remarkable, because generally Hintikka seems to carefully avoid terminology from extensive game models.

[9] This use of Skolem functions seems to originate from the interpretation of infinitistic and branching quantifier-formulas: see Henkin [2], p. 179.

5 Conclusions and further work

Hintikka's work on IF-logic and Game Theoretical Semantics, contains few mathematically exact definitions and theorems. This makes it hard to check claims made by Hintikka, or prove intuitions one has.
Therefore we found it useful to elaborate in this paper a mathematical, game theoretical model for the semantical games in IF-logic. This model made it possible to verify the claim that IF-sentences can be written in a (strongly) equivalent IF-formula in NNF, but it also made clear that the the game theoretical notion of strategy leads to different semantics than Hintikka's strategies built from Skolem functions.
Interesting questions and challenges that come to mind are to try include a game theoretical treatment of implication, to investigate results on extensive games for their applicability in GTS, e.g. to relate game equivalence to logical equivalence. Also, one could try to relate constraints on information sets (as for example the Von Neumann condition) to syntactical restrictions on IF-formulas. Finally, would it make sense to formalize a strategic form game model for semantical games?

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