On the proof theory of the existence predicate

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ABSTRACT.

Keywords: Intuitionistic logic, existence predicate, exististence logics, Gentzen calculus, cut-elimination, interpolation, Skolemization, eSolemization, truth-value logics, Gödel logics, Scott logics, Kripke models.

1 Introduction

In this paper we have brought together several results on the proof theory of the existence predicate in intuitionistic logic. This predicate, E, denotes whether a term exists or not: Et is read as *t exists*. Such a predicate was first introduced by Dana Scott in [20] in 1979. In this paper the author, in his one words, advocates in a mild way an extension of intuitionistic logic allowing reference to partial terms. One example from the paper pointing out, if not the necessity, at least the usefulness of having an existence predicate available is the following. In the context of rings the statement

$$\forall x\, \varphi(x) \;\Rightarrow\; \varphi(0)$$

is unconditionally true, since 0 is an element of all rings. In contrast to this, however,

$$\forall x\, \varphi(x) \;\Rightarrow\; \varphi(1/x)$$

is not generally true as not every element in a ring has an inverse. One could of course turn the latter into a conditional statement by formulating it as

$$\forall x\, \varphi(x) \;\Rightarrow\; \forall y \big(x \cdot y = 1 \;\Rightarrow\; \varphi(y)\big).$$

Now, still following Scott in [20], using the existence predicate one could express this more succinctly and directly as

$$\forall x\, \varphi(x) \wedge E(1/x) \Rightarrow\; \varphi(1/x).$$

In classical logic there is less need for this as one can always split definitions, theorems and proof in cases: either the object exists and then ...., or it does not exist. It is shown in [20] that in the presence of equality one can define the existence predicate via

$$Et \;\Leftrightarrow\; \exists y\,(t = y),$$

for y not free in t. But still, not only allows the existence predicate for more elegant and direct formulations, also when equality is not there it can be interpreted in a meaningful way. The latter is in accordance with the fact that existence comes for equality. For example, in term rewriting systems equality presupposes existence:

$$t = s \;\Rightarrow\; t{\downarrow} \wedge s{\downarrow}\;.$$

In this paper we study the existence predicate on this basic level. We consider intuitionistic logic IQC without equality and extended with the existence predicate, i.e. we add E to the language of predicate logic and extend IQC by certain axioms capturing the notion of existence.

What caused the renewed interest in the existence predicate is that recently the predicate was put to use in intuitionistic logic by providing satisfying answers to various problems that did not seem solvable in the setting of intuitionistic logic pure. The two main examples of this, one in the context of Skolemization, the other in the context of truth-value logics, will be discussed below.

This paper is meant as a survey of recent results on the proof theory of E. No results included here are new. The survey contains the introduction of various sequent calculi capturing the notion of existence, and the proofs that they all satisfy a form of cut-elimination, interpolation and the Beth definability property. Furthermore it introduces a semantics based on Kripke models but with a slightly different forcing relation, together with the proofs that this semantics is sound and complete for the calculi.

Our systems diverge slightly from other approaches in the literature in that we mostly consider mixed systems. By this we mean that we consider languages $\mathcal{L} \subseteq \mathcal{L}'$ and for terms in $\mathcal{L}$ assume that they exist, but for terms in $\mathcal{L}'\backslash\mathcal{L}$ we do not assume this. The systems consist of a basic Gentzen calculus LJE to which we add axioms of the form Et for all t in $\mathcal{L}$. This allows us to consider various other systems in the literature as subsystems of our systems, depending on how we choose $\mathcal{L}$. Also the applications discussed at the end of the paper show why allowing such a mixture is desirable.

1.1 Brouwer and Kant

On a more philosophical level there might be a reason that the existence predicate has not played an important role in the context of intuitionistic logic up till now, relating to Kant's view on existence.

We quote below from *Kritik der reinen Vernunft*, B626–B628. The precise passage was pointed out to us by Mark van Atten. The context of the quotation are the proofs of God's existence, according to which the concept God includes all perfections of realities, and that it is more perfect of more real to have the existence property than to have it not. In contrast to this, Kant claims that existence is no predicate and therefore cannot be part of the concept God. Kant writes:

"Sein ist offenbar kein reales Prädicat, d. i. ein Begriff von irgend etwas, was zu dem Begriffe eines Dinges hinzukommen könne. Es ist bloß die Position eines Dinges oder gewisser Bestimmungen an sich selbst.

Im logischen Gebrauche ist es lediglich die Copula eines Urtheils. Der Satz: Gott ist allmächtig, enthält zwei Begriffe, die ihre Objecte haben: Gott und Allmacht; das Wörtchen: ist, ist noch nicht ein Prädicat obenein, sondern nur das, was das Prädicat beziehungsweise aufs Subject setzt. Nehme ich nun das Subject (Gott) mit allen seinen Prädicaten (worunter auch die Allmacht gehört) zusammen und sage: Gott ist, oder es ist ein Gott, so setze ich kein neues Prädicat zum Begriffe von Gott, sondern nur das Subject an sich selbst mit allen seinen Prädicaten und zwar den Gegenstand in Beziehung auf meinen Begriff. Beide müssen genau einerlei enthalten, und es kann daher zu dem Begriffe, der bloß die Möglichkeit ausdrückt, darum daß ich dessen Gegenstand als schlechthin gegeben (durch den Ausdruck: er ist) denke, nichts weiter hinzukommen.

Und so enthält das Wirkliche nichts mehr als das bloß Mögliche. Hundert wirkliche Thaler enthalten nicht das Mindeste mehr, als hundert mögliche. Denn da diese den Begriff, jene aber den Gegenstand und dessen Position an sich selbst bedeuten, so würde, im Fall dieser mehr enthielte als jener, mein Begriff nicht den ganzen Gegenstand ausdrücken und also auch nicht der angemessene Begriff von ihm sein. Aber in meinem Vermögenszustande ist mehr bei hundert wirklichen Thalern, als bei dem bloßen Begriffe derselben (d. i. ihrer Möglichkeit). Denn der Gegenstand ist bei der Wirklichkeit nicht bloß in meinem Begriffe analytisch enthalten, sondern kommt zu meinem Begriffe (der eine Bestimmung meines Zustandes ist) synthetisch hinzu, ohne daß durch dieses Sein außerhalb meinem Begriffe diese gedachte hundert Thaler selbst im mindesten vermehrt werden.

Wenn ich also ein Ding, durch welche und wie viel Prädicate ich will, (selbst in der durchgängigen Bestimmung) denke, so kommt dadurch, da ich noch hinzusetze: dieses Ding ist, nicht das mindeste zu dem Dinge hinzu."

In the english translation[1]: "Being is evidently not a real predicate, that is, a conception of something which is added to the conception of some other thing. It is merely the positing of a thing, or of certain determinations in

[1] Translation by J.M.D. Meicklejohn 1969.

it. Logically, it is merely the copula of a judgement. The proposition, God is omnipotent, contains two conceptions, which have a certain object or content; the word is, is no additional predicate - it merely indicates the relation of the predicate to the subject. Now, if I take the subject (God) with all its predicates (omnipotence being one), and say: God is, or, There is a God, I add no new predicate to the conception of God, I merely posit or affirm the existence of the subject with all its predicates - I posit the object in relation to my conception. The content of both is the same; and there is no addition made to the conception, which expresses merely the possibility of the object, by my cogitating the object - in the expression, it is - as absolutely given or existing. Thus the real contains no more than the possible. A hundred real dollars contain no more than a hundred possible dollars. For, as the latter indicate the conception, and the former the object, on the supposition that the content of the former was greater than that of the latter, my conception would not be an expression of the whole object, and would consequently be an inadequate conception of it. But in reckoning my wealth there may be said to be more in a hundred real dollars than in a hundred possible dollars - that is, in the mere conception of them. For the real object - the dollars - is not analytically contained in my conception, but forms a synthetical addition to my conception (which is merely a determination of my mental state), although this objective reality - this existence - apart from my conceptions, does not in the least degree increase the aforesaid hundred dollars.

By whatever and by whatever number of predicates - even to the complete determination of it - I may cogitate a thing, I do not in the least augment the object of my conception by the addition of the statement: This thing exists."

As was pointed out to us by Mark van Atten, Brouwer certainly knew the passage above, as in a letter[2] to his PhD supervisor he mentions having read Kritik der reinen Vernunft very thoroughly.

The above expressed viewpoint however overlooks the fact, that (conditions of) existence can be described in various sometimes surprising ways and the corresponding concepts can be interrelated via intuitionistic logic. It is the same story as with termination, another manifestation of existence: termination of a computable function with respect to a given input is a mere fact independent of the logical viewpoint. To analyze however the notion of termination itself logical frameworks are needed to represent (conditions of) termination in a general setting: this step is necessary even to state the totality of a program, i.e. the termination with respect to all potential input values.

[2] A letter from 5.11.1906 to Diederik Korteweg

1.2 Section contents

The paper is build up as follows. In Section 2 the main proof system is introduced. In Section 3 it is proved that to have a form of cut-elimination, in Section 5 it is shown that it satisfies interpolation and the Beth definability property. Section 4 discusses the relation of our Gentzen calculi to systems capturing E that have been considered before. In Section 6 semantics are introduced which in Section 7 are proved to be sound and complete with respect to the calculi. Section 8 contains the applications to Skolemization and truth-value logics mentioned above.

Acknowledgement We thank Mark van Atten for pointing out to us the particular passage of Kritik der reinen Vernunft quoted above and for the remarks on Brouwer.

2 A Gentzen calculus

In this section we define the Gentzen calculus LJE, an extension of LJ for intuitionistic predicate logic extended by the existence predicate, that covers the intuition that Et means t *exists*. Hilbert type systems for the existence predicate were first introduced by D. Scott in [20]. Natural deduction formulations were given by M. Unterhalt in [30]. The Gentzen calculi given below were first introduced by the authors in [2]. The relation between these systems will be discussed in Section 4.

2.1 Preliminaries

Given an existence predicate, terms, including variables, typically range over existing as well as non-existing objects, while the quantifiers range over existing objects only. Proofs are assumed to be trees.

We consider languages $\mathcal{L} \subseteq \mathcal{L}'$ for intuitionistic predicate logic plus the existence predicate E, without equality. $E \in \mathcal{L}$ and $\mathcal{L}'_-$ denotes $\mathcal{L}'$ without E. For convenience we assume that $\mathcal{L}$ contains at least one constant and no variables, and that $\mathcal{L}'$ contains infinitely many variables. The reason for requiring that $\mathcal{L}$ contains at least one constant is given in Remark 2. We assume that the variables belong only to the bigger language because in our main system we will assume terms in $\mathcal{L}$ to exist, and assuming variables to exist would block the free substitution of terms for variables. Moreover, we think it the more natural approach to let free variables be as free as possible, no restrictions put on them. More on this in Section 4.

The languages contain $\perp$, and $\neg A$ is defined as $A \to \perp$. $A, B, C, D, E, ..$ range over formulas in $\mathcal{L}'$, $s, t, ..$ over terms in $\mathcal{L}'$. Γ, Δ, Π range over multisets of formulas in $\mathcal{L}'$. Sequents are expressions of the form $\Gamma \Rightarrow C$, where Γ is a finite multiset. A sequent is in $\mathcal{L}$ if all its formulas are in $\mathcal{L}$. And similarly for $\mathcal{L}'$. A formula is *closed* when it does not contain free variables.

A sequent $\Gamma \Rightarrow C$ is closed if C and all formulas in Γ are closed. We often write Ax for $A(x)$.

In the final proof system $(\Rightarrow Et)$ will hold for the terms in $\mathcal{L}$, but not necessarily for the terms in $\mathcal{L}' \backslash \mathcal{L}$. $\mathcal{T}_{\mathcal{L}}$ denotes the set of terms in $\mathcal{L}$, $\mathcal{F}_{\mathcal{L}}$ denotes the set of formulas in $\mathcal{L}$, $\mathcal{S}_{\mathcal{L}}$ denotes the set of sequents in $\mathcal{L}$, and similarly for $\mathcal{L}'$.

2.2 The system LJE

Ax $\Gamma, P \Rightarrow P$ (P atomic) $\qquad$ $L\bot$ $\Gamma, \bot \Rightarrow C$

$$\text{L}\wedge \ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \text{R}\wedge \ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \text{R}\vee \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \ i = 0, 1$$

$$\text{L}\!\rightarrow \ \frac{\Gamma, A \rightarrow B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C} \qquad \text{R}\!\rightarrow \ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$\text{L}\forall \ \frac{\Gamma, \forall x Ax, At \Rightarrow C \qquad \Gamma, \forall x Ax \Rightarrow Et}{\Gamma, \forall x Ax \Rightarrow C} \qquad \text{R}\forall \ \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]} \ *$$

$$\text{L}\exists \ \frac{\Gamma, Ay, Ey \Rightarrow C}{\Gamma, \exists x Ax \Rightarrow C} \ * \qquad \text{R}\exists \ \frac{\Gamma \Rightarrow At \qquad \Gamma \Rightarrow Et}{\Gamma \Rightarrow \exists x A[x/t]}$$

$$\text{Cut} \ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}$$

Where $(*)$ denotes the condition that y does not occur free in Γ and C.

We write LJE $\vdash S$ if the sequent S is derivable in LJE. For a set of sequents $\mathcal{S}$ and a sequent S, we say that S *is derivable from* $\mathcal{S}$ *in* LJE, and write $\mathcal{S} \vdash_{\mathsf{LJE}} S$, if S is derivable in LJE extended by axioms $\mathcal{S}$. We define

$$\mathsf{LJE}(\mathcal{S}) \equiv_{def} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \mathcal{S} \vdash_{\mathsf{LJE}} S\}.$$

In the system LJE no existence of any term is assumed. This implies e.g. that we cannot derive $\Rightarrow \exists x Ex$. And we cannot derive $\forall x Px \Rightarrow Pt$ either, but only $\forall x Px, Et \Rightarrow Pt$. Note however that the former is derivable in LJE from $(\Rightarrow Et)$. This is the reason why we consider derivations from extra axioms, especially axioms of the form $(\Rightarrow Et)$. Therefore, we define the following sets of sequents

$$\Sigma_{\mathcal{L}} \equiv_{def} \{\Gamma \Rightarrow Et \mid t \in \mathcal{T}_{\mathcal{L}}, \Gamma \text{ a multiset}\}.$$

Note that for all sequents $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, t is a closed term, and that because of the assumptions on $\mathcal{L}$, $\Sigma_{\mathcal{L}}$ contains at least one sequent. Given two languages $\mathcal{L} \subseteq \mathcal{L}'$, we write

$$\mathsf{LJE}(\Sigma_{\mathcal{L}}) \equiv_{def} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \Sigma_{\mathcal{L}} \vdash_{\mathsf{LJE}} \Rightarrow S\}.$$

The $\mathcal{L}'$ is not denoted in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, but most of the time it is clear what 'bigger' language $\mathcal{L}'$ of which $\mathcal{L}$ is a subset is.

We often write $\vdash_{\mathcal{L}}$ for $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})}$.

EXAMPLE 1.

$$\nvdash_{\mathsf{LJE}} \Rightarrow \exists x Ex \qquad \vdash_{\mathsf{LJE}} \Rightarrow \forall x Ex.$$

$$\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow \exists x Ex \wedge \forall x Ex.$$

In Proposition 8 the relation between LJ and LJE is explained.

REMARK 2. The requirement that $\mathcal{L}$ contains at least one constant is needed to make the construction of the reduction trees work: see Definition 18, the remark at the case R$\forall$.

2.3 Uniqueness

Observe that given another predicate E' that satisfies the same rules of LJE as E', it follows that

$$\vdash_{\mathcal{L}} Et \Rightarrow E't \;\text{ and }\; \vdash_{\mathcal{L}} E't \Rightarrow Et.$$

We namely have that $\vdash_{\mathcal{L}} (\Rightarrow \forall x Ex \wedge \forall x E'x)$, and also $\vdash_{\mathcal{L}} (\forall x Ex, E't \Rightarrow Et)$ and $\vdash_{\mathcal{L}} (\forall x E'x, Et \Rightarrow E't)$. Finally, two cuts do the trick. This shows that the existence predicate E is unique up to provable equivalence.

3 Cut elimination

We assume eigenvariables, free and bound variables to be three distinct sets of variables. The variable y in L$\exists$ and R$\forall$ is called an eigenvariable. The depth of a sequent in a proof is inductively defined as the sum of the depths of its upper sequents plus 1. Thus axioms have depth 1. The complexity $|C|$ of a formula is the number of occurrences of connectives and quantifiers in C. The rank of a cut is 1 + the complexity of the cut formula. The level of a cut is the sum of the depths of its two hypotheses. The cutrank *cr(P)* of a proof P is the maximal rank of cuts in P. The depth of a proof, *dp(P)*, is the depth of its endsequent. We write $\mathsf{LJE} \vdash_d S$ when S has a proof of depth $\leq d$ in LJE, We write $\mathsf{LJE} \vdash^c S$ when S has a proof of cutrank $\leq c$. Similarly for $\mathsf{LJE}(\Sigma_{\mathcal{L}})$. For a proof P, $P[t/y]$ denotes the result of substituting t for y everywhere in P.

3.1 Substitution, Weakening and Contraction

We start with the substitution lemma.

LEMMA 3. *For* $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$*:*
If P is a proof in L of a sequent S in $\mathcal{L}'$ in which y occurs free, and if t is a term in $\mathcal{L}'$ that does not contain eigenvariables or bound variables of P, then $P[t/y]$ is a proof of $S[t/y]$ in L. Moreover, $cr(P[t/y]) \leq cr(P)$ and $dp(P[t/y]) \leq dp(P)$.

Proof. We treat the case $\mathsf{L} = \mathsf{LJE}(\Sigma_{\mathcal{L}})$. We use induction to the depth d of P. Let $P' = P[t/y]$, $S' = S[t/y]$. First $d = 1$, the case that P is an instance of an axiom. The axioms Ax, L$\perp$ in P are replaced by instances of the same axioms in P', so these will not be violated under the transformation. For axioms $\Pi \Rightarrow Es$ in $\Sigma_{\mathcal{L}}$ it follows that s is a closed term in $\mathcal{L}$. Hence the sequent that results from the substitution, $(\Pi[t/y] \Rightarrow Es)$, belongs to $\Sigma_{\mathcal{L}}$ too. This completes the case $d = 1$.

Suppose $d > 1$. First note that because eigenvariables are distinct from free variables in a proof, y cannot be an eigenvariable in P. We distinguish by cases according to the last rule in P. The connective rules and cuts in P are replaced by instances of the same rules in P', so these will not be violated under the transformation. Thus the quantifier rules remain.

Suppose the last inference in P is a quantifier rule. In the case of L$\forall$ and R$\exists$ there are no side conditions, whence these rules will not be violated in going from P to P'. We treat R$\forall$, the case L$\exists$ is similar. Consider an application of R$\forall$ in P:

$$\dfrac{\begin{array}{c} P_1 \\ \Pi, Ez \Rightarrow Bz \end{array}}{\Pi \Rightarrow \forall u Bu}$$

Thus z is not free in Π, and $z \neq y$ and $u \neq y$, since y is no eigenvariable or bound variable. By assumption on t, u does not occur in t. Under the transformation this will become

$$\dfrac{\begin{array}{c} P_1[t/y] \\ \Pi[t/y], Ez \Rightarrow Bz[t/y] \end{array}}{\Pi[t/y] \Rightarrow \forall u Bu[t/y]}$$

To see that this a valid application of R$\forall$, it suffices to see that z is not free in $\Pi[t/y]$, which is clear from the assumption on t.

To check that $cr(P') \leq cr(P)$ and $dp(P') \leq dp(P)$ is left to the reader. ■

LEMMA 4. *For* $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathfrak{L}}), \mathsf{LJE}\}$*:* $\mathsf{L} \vdash_d^c \Gamma \Rightarrow C$ *implies* $\mathsf{L} \vdash_d^c \Gamma, A \Rightarrow C$.

Proof. Left to the reader. For the quantifier rules, use Lemma 3 to repair variable clashes. ■

LEMMA 5. *For* $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathfrak{L}}), \mathsf{LJE}\}$*:* L *has contraction. In fact:*

$$\mathsf{L} \vdash_d^c \Gamma, A, A \Rightarrow C \quad \textit{implies} \quad \mathsf{L} \vdash_d^c \Gamma, A \Rightarrow C \tag{1}$$

Proof. To show that the system has contraction we need the following claim.
CLAIM. For $d > 0$, it holds that

$$\begin{array}{lll}
\mathsf{L} \vdash_d^c \Gamma, A \wedge B \Rightarrow C & \text{implies} & \mathsf{L} \vdash_d^c \Gamma, A, B \Rightarrow C \\
\mathsf{L} \vdash_d^c \Gamma, A \vee B \Rightarrow C & \text{implies} & \mathsf{L} \vdash_d^c \Gamma, A \Rightarrow C \text{ and } \mathsf{L} \vdash_d \Gamma, B \Rightarrow C \\
\mathsf{L} \vdash_d^c \Gamma \Rightarrow A \rightarrow B & \text{implies} & \mathsf{L} \vdash_d^c \Gamma, A \Rightarrow B \\
\mathsf{L} \vdash_d^c \Gamma, \exists x A x \Rightarrow C & \text{implies} & \mathsf{L} \vdash_d^c \Gamma, Ey, Ay \Rightarrow C, \text{ for all } y.
\end{array}$$

Proof of Claim. The only detail here is the possibility of variable clashes. We only treat the case of the existential quantifier, with induction to d. If $\Gamma, \exists x Ax \Rightarrow C$ is an axiom, then so is $\Gamma, Ey, Ay \Rightarrow C$. Suppose it it not an axiom. If in the last inference in the proof of $\Gamma, \exists x Ax \Rightarrow C$, $\exists x Ax$ is not principal, then the induction hypothesis applies: for the rules without eigenvariables this is immediate. For the rules with eigenvariables, if the eigenvariable is y, we just replace it by a fresh eigenvariable not occurring in the proof, and then using the induction hypothesis we obtain a proof of $\Gamma, Ey, Ay \Rightarrow C$ of same rank and depth. If $\exists x Ax$ is principal in the last rule, the result follows immediately. This proofs the claim.

Using this claim we prove (1) with induction to the depth d of the proof of $\Gamma, A, A \Rightarrow C$ in L. If $d = 1$, the sequent is an axiom, and so $\Gamma, A \Rightarrow C$ clearly is an axiom too (also in the case of $\Sigma_{\mathfrak{L}}$). Consider the case $d+1$. If the last rule in the proof is a right rule or the principal formula is in Γ, then the induction hypothesis applies. Therefore, suppose it is a left rule and the principal formula is not in Γ. We distinguish by cases. We treat L$\wedge$ and leave the other cases to the reader. In this case the last part of the proof then looks as follows.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A \wedge B, A, B \Rightarrow C\end{array}}{\Gamma, A \wedge B, A \wedge B \Rightarrow C}$$

Assume the cutrank of the proof is n. Let P be the proof of $\Gamma, A\wedge B, A, B \Rightarrow C$. Note that P has depth d. Thus we can apply the claim and obtain a proof of $\Gamma, A, B, A, B \Rightarrow C$ of depth $\leq d$ and cutrank $\leq n$. Then we apply the induction hypothesis, first to A and then to B, and obtain a proof of $\Gamma, A, B \Rightarrow C$ of depth $\leq d$ and cutrank $\leq n$. An application of L$\wedge$ provides a proof of $\Gamma, A \wedge B \Rightarrow C$ of depth $\leq d+1$ and cutrank $\leq n$, as desired. ■

3.2 Restriction to Ecuts

THEOREM 6. *For* $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathfrak{L}}), \mathsf{LJE}\}$*:*
Every sequent in $\mathfrak{L}'$ *provable in* L *has a proof in* L *in which the only cuts are instances of the ECut rule:*

$$\textit{ECut:}\ \dfrac{\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}$$

In particular, LJE *has cut-elimination.*

Proof. For a smooth induction it is convenient to replace the Cut rule in LJE by the following generalization of it, the so-called *Mix rule*:

$$\text{Mix}\ \dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

In the Mix rule A is called the cutformula. When we speak about cuts in a proof, we refer to instances of the Cut or the Mix rule. The notions of cutrank are extended to proofs with the Mix rule in the obvious way. To prove the theorem we then show that applications of Mix can be removed from a proof, unless they are instances of EMix, which is

$$\text{EMix:}\ \dfrac{\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma', Et \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

Note that this indeed implies that all provable sequents have a proof in which the only cuts are instances of ECut: $\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}}$ implies $\Gamma' \Rightarrow Et \in \Sigma_{\mathfrak{L}}$ for all Γ', and thus the conclusion of the EMix as above can be obtained also via the ECut

$$\dfrac{\Gamma\Gamma' \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma\Gamma', Et \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

For now, we call a proof *ecutfree* if all applications of Mix are instances of EMix, and we call it *cutfree* when it contains no cuts at all. Recall that the cutrank *cr(P)* of a proof P is 1 + the maximal complexity of cutformulas in P.

The proof of the theorem consists of two claims. The first shows how to remove cuts of rank > 1 from a proof, and the second shows how cuts of rank 1 that are not instances of EMix can be removed from a proof. These two claims together imply the theorem.

CLAIM. For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: Every sequent in $\mathcal{L}'$ provable in L has a proof in L in which all cuts have rank 1.

Proof of Claim. We treat the case $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, the case LJE is similar. It suffices to show that a proof P ending in a cut

$$\dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \begin{array}{c} P_2 \\ \Gamma', A \Rightarrow C \end{array}}{\Gamma\Gamma' \Rightarrow C}$$

with $|A| > 0$ and with $cr(P_1), cr(P_2) \leq |A|$, can be transformed into a proof P' of $\Gamma\Gamma' \Rightarrow C$ such that $cr(P') < cr(P)$. Note that $cr(P) = |A| + 1 > 1$. We prove this by induction on the cutrank of P with a sub induction to the level of the lowest cut of maximal rank in P (the level of a cut is the sum of the depths of its two hypotheses). We call $\Gamma \Rightarrow A$ and $\Gamma', A \Rightarrow C$ the hypotheses of the cut and $\Gamma\Gamma' \Rightarrow C$ the conclusion. Since $|A| > 0$, A cannot be principal in an axiom, including $\Sigma_{\mathcal{L}}$. Note also that A cannot be of the form Et. Therefore, we only have to distinguish the following two cases:
(a) the cutformula is not principal in one of the hypotheses,
(b) cutformula is principal in both hypotheses, which are not axioms.

(a) Suppose the cut formula is not principal in one of the hypotheses. If this hypothesis is an instance of axioms Ax or L$\bot$, then so is the conclusion of the cut, and whence we have a cutfree proof of it. If this hypothesis is an instance of an axiom $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, then since $|A| > 0$ it has to be the right hypothesis. Observe that $(\Gamma \Rightarrow Et) \in \Sigma_{\mathcal{L}}$, implies that $(\Pi \Rightarrow Et) \in \Sigma_{\mathcal{L}}$ for all Π. Hence the conclusion of the cut is a sequent in $\Sigma_{\mathcal{L}}$, in which case we have a cutfree proof of it.

Next suppose that the hypothesis in which A is not principal is the lower sequent of an application of one of the rules. In this case we can cut higher up. That is, suppose the cutformula is not principal in the left hypothesis, and assume this is a two hypotheses rule R, say R$\vee$. Then P looks as follows.

$$\dfrac{\mathrm{R}\lor\ \dfrac{\begin{matrix}P_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_2\\ \Gamma_2 \Rightarrow A\end{matrix}}{\Gamma \Rightarrow A} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma\Gamma' \Rightarrow C}$$

Note that by assumption $cr(P_i) < cr(P)$ for $i = 1, 2, 3$. Then we transform the proof into a proof P' as follows.

$$\mathrm{R}\ \dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma_1\Gamma' \Rightarrow C} \qquad \dfrac{\begin{matrix}P_2\\ \Gamma_2 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma_2\Gamma' \Rightarrow C}}{\Gamma\Gamma' \Rightarrow C}$$

Now we have two cuts on A, but the level of the lowest cut of maximal rank in P' is one of these cuts. Thus $cr(P') = cr(P)$, but the level of the lowest cut of maximal rank in P' is smaller than the level of the lowest cut of maximal rank in P. Therefore, we can apply the induction hypothesis and are done. The other cases are similar. Note that in the case that R is a cut, it is by assumption a cut of rank $< |A| + 1$. Hence also in this case the induction hypothesis applies to P'.

(b) In this case the cut is principal in both hypotheses, and both hypotheses are not axioms. We distinguish by cases according to the outermost logical symbol in A: the cases $\land$, $\lor$, $\to$ are treated in the same way as in the case of LJ, see e.g. [29]. We treat the quantifiers.

$\forall$: then P looks as follows:

$$\dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma, Ey \Rightarrow Ay\end{matrix}}{\Gamma \Rightarrow \forall x Ax}\, d_1 \qquad \dfrac{\begin{matrix}P_2\\ \Gamma', \forall x Ax, At \Rightarrow C\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', \forall x Ax \Rightarrow Et\end{matrix}}{\Gamma', \forall x Ax \Rightarrow C}\, d_2}{\Gamma\Gamma' \Rightarrow C}$$

Note that y is not free in Γ because of the conditions on $R\forall$, and y is not free in Γ', C and t because of the conditions on eigenvariables in a proof. By assmptions on variables, t does not contain eigenvariables or bound variables in P.

We can transform the above proof into the following proof P':

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma, Ey \Rightarrow Ay\end{matrix}}{\Gamma \Rightarrow \forall x Ax} \qquad \begin{matrix}P_3\\ \Gamma', \forall x Ax \Rightarrow Et\end{matrix}}{\Gamma\Gamma' \Rightarrow Et} \qquad \begin{matrix}P_1[t/y]\\ \Gamma, Et \Rightarrow At\end{matrix}}{\Gamma\Gamma\Gamma' \Rightarrow At} \qquad \dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma, Ey \Rightarrow Ay\end{matrix}}{\Gamma \Rightarrow \forall x Ax} \qquad \begin{matrix}P_2\\ \Gamma', \forall x Ax, At \Rightarrow C\end{matrix}}{\Gamma\Gamma', At \Rightarrow C}}{\Gamma\Gamma\Gamma\Gamma'\Gamma' \Rightarrow C}$$

Note that the endsequent of $P_1[t/y]$ indeed is $\Gamma, Et \Rightarrow At$ as y is not free Γ. By Lemma 3, $P_1[t/y]$ is a proof of $(\Gamma, Et \Rightarrow At)$ in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ such that $cr(P_1[t/y]) \leq cr(P_1) < cr(P)$. The cuts on $\forall xAx$ both have a lower level and the same rank as in P. Therefore, we can apply the induction hypothesis and obtain proofs of their conclusions of cutrank $< cr(P)$. Whence there is a proof of $\Gamma\Gamma\Gamma\Gamma'\Gamma' \Rightarrow C$ of cutrank $< cr(P)$. Application of some contractions, Lemma 5, gives a proof of $\Gamma\Gamma' \Rightarrow C$ of cutrank $< cr(P)$. This proves the case $\forall$.

$\exists$: Similar. Here P looks as follows:

$$\dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma\Rightarrow At\end{matrix}\qquad \begin{matrix}P_2\\ \Gamma\Rightarrow Et\end{matrix}}{\Gamma\Rightarrow\exists xAx}\qquad \dfrac{\begin{matrix}P_3\\ \Gamma', Ey, Ay\Rightarrow C\end{matrix}}{\Gamma', \exists xAx\Rightarrow C}}{\Gamma\Gamma'\Rightarrow C}$$

Because of the side condition that y is not free in Γ' and C we can transform this proof into the following proof P':

$$\dfrac{\begin{matrix}P_1\\ \Gamma\Rightarrow At\end{matrix}\qquad \dfrac{\begin{matrix}P_2\\ \Gamma\Rightarrow Et\end{matrix}\qquad \begin{matrix}P_3[t/y]\\ \Gamma', Et, At\Rightarrow C\end{matrix}}{\Gamma\Gamma', At\Rightarrow C}}{\Gamma\Gamma\Gamma'\Rightarrow C}$$

By Lemma 3, $cr(P_3[t/y]) \leq cr(P_3)$. Thus $cr(P') < cr(P)$. This completes (b) and thereby the proof of the claim.

CLAIM. For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: Every sequent in $\mathcal{L}'$ that has a proof in L of cutrank 1, has a proof in L in which all cuts are instances of EMix.

Proof of Claim. We treat the case $\mathsf{LJE}(\Sigma_{\mathcal{L}})$. We use induction to the depth d of a proof P of cutrank ≤ 1 of a sequent S. The case $d = 1$ is trivial, as then P consists of an axiom only. Suppose $d > 1$. If the last inference in P is not a cut or it is an application of EMix, we can apply the induction hypothesis and are done. Therefore, suppose P ends in a cut that is not an instance of EMix:

$$\dfrac{\begin{matrix}P_1\\ \Gamma\Rightarrow A\end{matrix}\qquad \begin{matrix}P_2\\ \Gamma', A\Rightarrow C\end{matrix}}{\Gamma\Gamma'\Rightarrow C}\; d$$

Thus by the induction hypothesis P_1 and P_2 are ecutfree, i.e. all cuts they contain are instances of EMix. And as P has cutrank ≤ 1, A is atomic or $\bot$ or of the form Et. Denote $\Gamma\Gamma' \Rightarrow C$ by S. We distinguish the following cases:

(c) the cutformula is principal in the rigth hypothesis,

(d) the cutformula is not principal in the right hypothesis.

(c) Assume the cutformula is principal in the right hypothesis. The form of A implies that whence the right hypothesis $\Gamma', A \Rightarrow C$ has to be an axiom. Since A is principal in it, $C = A$ or $A = \bot$. In the former case we can obtain a ecutfree proof of S by weakening the sequent $\Gamma \Rightarrow A$. If $A = \bot$, then it follows that either $\bot \in \Gamma$ or A is not principal in the left hypothesis. In the former case S is an instance of L$\bot$ and we are done. In the latter case, since A is not principal in it, $\Gamma \Rightarrow \bot$ is the conclusion of a rule R in which $\bot$ is not principal. In this case one can cut higher up, like in case (b) in the proof of the first claim: we treat the case that R is an EMix, and leave the other cases to the reader. In this case P looks as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow Et \;\in\; \Sigma_{\mathcal{L}} \qquad \begin{array}{c} P_1 \\ \Gamma', Et \Rightarrow \bot \end{array}}{\Gamma\Gamma' \Rightarrow \bot} \qquad \Gamma'', \bot \Rightarrow C}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

We transform this proof into the proof P':

$$\dfrac{\Gamma \Rightarrow Et \;\in\; \Sigma_{\mathcal{L}} \qquad \dfrac{\begin{array}{c} P_1 \\ \Gamma', Et \Rightarrow \bot \end{array} \qquad \Gamma'', \bot \Rightarrow C}{\Gamma'\Gamma'', Et \Rightarrow \bot}}{\Gamma\Gamma'\Gamma'' \Rightarrow \bot}$$

We apply the induction hypothesis to P' and are done.

(d) Assume the cutformula is not principal in the right hypothesis. If $\Gamma', A \Rightarrow C$ is an axiom, then $\bot \in \Gamma'$, $C \in \Gamma'$ or $C = Et$ for some $t \in \mathcal{T}_{\mathcal{L}}$. In all cases S is an instance of the same axiom. If the right hypothesis is an application of a rule R we proceed as follows. We treat the cases that R is a two hypothesis rule that is not a cut, and the case that it is a cut, and leave the other cases to the reader. First, suppose R is not a cut. Then P looks as follows.

$$\dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \dfrac{\begin{array}{c} P_2 \\ \Gamma_1, A \Rightarrow C_1 \end{array} \qquad \begin{array}{c} P_3 \\ \Gamma_2, A \Rightarrow C_2 \end{array}}{\Gamma', A \Rightarrow C}\,\mathrm{R}}{\Gamma\Gamma' \Rightarrow C}$$

Note that by the induction hypothesis the P_i are ecutfree. Then we transform the proof into a proof P' as follows.

$$\dfrac{\dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \begin{array}{c} P_2 \\ \Gamma_1, A \Rightarrow C_1 \end{array}}{\Gamma\Gamma_1 \Rightarrow C_1} \qquad \dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \begin{array}{c} P_3 \\ \Gamma_2, A \Rightarrow C_2 \end{array}}{\Gamma, \Gamma_2 \Rightarrow C_2}}{\Gamma\Gamma' \Rightarrow C}\,\mathrm{R}$$

Since R is not a cut we can apply the induction hypothesis to P' and are done.

Finally, we treat the case that R is a cut. By the induction hypothesis it is an instance of EMix. Hence P looks like this:

$$\dfrac{\dfrac{P_1}{\Gamma \Rightarrow A} \quad \dfrac{\Gamma', A \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \dfrac{P_2}{\Gamma'', Et, A \Rightarrow C}}{\Gamma'\Gamma'', A \Rightarrow C}}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

Then we transform the proof into a proof P' as follows:

$$\dfrac{\Gamma' \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \dfrac{\dfrac{P_1}{\Gamma \Rightarrow A} \qquad \dfrac{P_2}{\Gamma'', Et, A \Rightarrow C}}{\Gamma\Gamma'', Et \Rightarrow C}}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

To see that this is indeed a proof, note that $(\Gamma', A \Rightarrow Et) \in \Sigma_{\mathcal{L}}$ implies $t \in \mathcal{T}_{\mathcal{L}}$, which implies $(\Gamma' \Rightarrow Et) \in \Sigma_{\mathcal{L}}$. Now the induction hypothesis applies to P', and we are done. This proves the second claim.

As explained above, the two claims imply the theorem. ∎

COROLLARY 7. $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ *is consistent.*

The cut elimination theorem allows us to proof the following correspondence between LJ and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, one direction of which has already been proved above.

PROPOSITION 8. *For every sequent S in $\mathcal{L}$ not containing E:*

$$\mathsf{LJ} \vdash S \textit{ if and only if } \mathsf{LJE}(\Sigma_{\mathcal{L}}) \vdash S.$$

Proof. The direction from right to left: show with induction to the depth of the proof that for Γ and A not containing E, if $Et_1, \ldots, Et_n, \Gamma \Rightarrow A$ is derivable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ by a proof in which all cuts are instances of ECut, then $\Gamma \Rightarrow A$ is derivable in LJ. We leave the other direction to the reader. ∎

PROPOSITION 9. *For quantifier free closed sequents the relations $\vdash_{\mathsf{LJE}}$ and $\vdash_{\mathcal{L}}$ are decidable.*

Proof. Show, using the theorem on ECuts above, that when t is a term that does not occur in a quantifier free sequent $\Gamma \Rightarrow C$, not even as a subterm, then

$$\vdash_{\mathcal{L}} \Gamma, Et \Rightarrow C \ \text{ implies } \ \vdash_{\mathcal{L}} \Gamma \Rightarrow C,$$

and similarly for LJE. ∎

4 IQCE and IQCE$^+$

As remarked above, given an existence predicate, closed terms typically range over existing as well as non-existing elements, while quantifiers range over existing objects only. As to the choice of the domain for the variables, there have been different approaches. Scott in [20] introduces a system IQCE for the predicate language with the distinguished predicate E, in which variables range over all objects, like in LJE and LJE$(\Sigma_{\mathcal{L}})$. On the other hand, Beeson in [6] discusses a system in which variables range over existing objects only.

The formulation of the system IQCE in [20], where logic with an existence predicate was first introduced was in Hilbert style, where the axioms and rules for the quantifiers were the following:

$$\forall x Ax \wedge Et \to At \qquad \dfrac{\vdots \atop B \wedge Ey \to Ay}{B \to \forall x Ax}\, *$$

$$\dfrac{\vdots \atop Ay \wedge Ey \to B}{\exists x Ax \to B}\, * \qquad At \wedge Et \to \exists x Ax$$

Here $*$ are the usual side conditions on the eigenvariable y.

The following formulation of IQCE in natural deduction style was first given in [30]. We recall the system as given in [28]. We call the system NDE (Natural Deduction Existence). It consists of the axioms and quantifier rules of the standard natural deduction formulation of IQC (as e.g. given in [28]), where the quantifier rules are replaced by the following rules:

$$\forall \mathrm{I}\ \dfrac{\begin{matrix}[Ey]\\ \vdots \\ Ay\end{matrix}}{\forall x Ax}\, * \qquad \forall \mathrm{E}\ \dfrac{\begin{matrix}\vdots \\ \forall x Ax\end{matrix} \quad \begin{matrix}\vdots \\ Et\end{matrix}}{At}$$

$$\exists \mathrm{I}\ \dfrac{\begin{matrix}\vdots \\ At\end{matrix} \quad \begin{matrix}\vdots \\ Et\end{matrix}}{\exists x Ax} \qquad \exists \mathrm{E}\ \dfrac{\begin{matrix}\vdots \\ \exists x Ax\end{matrix} \quad \begin{matrix}[Ay][Ey]\\ \vdots \\ C\end{matrix}}{C}\, *$$

Again, the $*$ are the usual side conditions on the eigenvariable y. It is easy to see that the following holds.

FACT 10. $\forall A \in \mathcal{F}_{\mathcal{L}'}$: $\vdash_{\mathsf{IQCE}} A$ if and only if $\vdash_{\mathsf{NDE}} A$ if and only if $\vdash_{\mathsf{LJE}} \Rightarrow A$.

Existence logic in which terms range over all object while quantifiers and variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by M. Beeson in [6]. The logic is the result of leaving out Ey in the two rules for the quantifiers in IQCE given above and adding Ex as axioms for all variables x. A formulation in natural deduction style is obtained from NDE by replacing the $\forall\,$I and $\exists\,$E by their standard formulations for IQC and adding Ex as axioms for all variables x. We call the system NDE^+. There are some details concerning substitutions for these systems, but we will not discuss them here, but only remark in how far these systems are equivalent:

FACT 11. $\forall A \in \mathcal{F}_{\mathcal{L}}$:
$\vdash_{\mathsf{IQCE}^+} A$ iff $\vdash_{\mathsf{NDE}^+} A$ iff $\{\Gamma \Rightarrow Ex \mid x$ a variable, Γ a multiset$\} \vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow A$.

M. Unterhalt in [30] thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+. Section 6 discusses his and our completeness results.

5 Interpolation

Recall that we say that a single conclusion Gentzen calculus L has *interpolation* if whenever $\mathsf{L} \vdash \Gamma_1, \Gamma_2 \Rightarrow C$, there exists an I in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

$$\Gamma_1 \vdash_{\mathsf{L}} I \quad \text{and} \quad I, \Gamma_2 \vdash_{\mathsf{L}} C.$$

In the context of existence logics, the *common language* of two multisets Γ_1 and Γ_2, denoted by $\mathcal{L}(\Gamma_1, \Gamma_2)$, consists of all variables, $\top$, $\bot$ and E, and all predicates and non-variable terms that occur both in Γ_1 and Γ_2.

We say that a Gentzen calculus L satisfies the *Beth definability property* if whenever $A(R)$ is a formula with R an n-ary relation symbol in a language $\mathcal{L}$, and R', R'' are two relation symbols not in $\mathcal{L}$ such that

$$\mathsf{L} \vdash A(R') \wedge A(R'') \Rightarrow \forall \bar{x}(R'\bar{x} \ \leftrightarrow \ R''\bar{x}),$$

then there is a formula S in $\mathcal{L}$ such that

$$\mathsf{L} \vdash \Rightarrow \forall \bar{x}(S\bar{x} \ \leftrightarrow \ R'\bar{x}).$$

In this section we prove that the calculus LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ have interpolation. To this end we use a calculus LJE' that is equivalent to LJE but in which the structural rules are not hidden.

The system LJE'

$Ax \;\; \Gamma, P \Rightarrow P \;\; (P \text{ atomic})$ $\qquad$ $L\bot \;\; \Gamma, \bot \Rightarrow C$

$$\text{LW} \; \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad \text{LC} \; \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

$$\text{L}\wedge \; \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \text{R}\wedge \; \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \; \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \text{R}\vee \; \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \; i = 0, 1$$

$$\text{L}\rightarrow \; \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C} \qquad \text{R}\rightarrow \; \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$\text{L}\forall \; \frac{\Gamma, At \Rightarrow C \qquad \Gamma \Rightarrow Et}{\Gamma, \forall x Ax \Rightarrow C} \qquad \text{R}\forall \; \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]} \; *$$

$$\text{L}\exists \; \frac{\Gamma, Ay, Ey \Rightarrow C}{\Gamma, \exists x A[x/y] \Rightarrow C} \; * \qquad \text{R}\exists \; \frac{\Gamma \Rightarrow At \qquad \Gamma \Rightarrow Et}{\Gamma \Rightarrow \exists x Ax}$$

$$\text{ECut:} \; \frac{\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}$$

The calculus $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ is the system LJE' extended by the axioms $\Sigma_{\mathfrak{L}}$ (Section 2).

LEMMA 12. *For al formulas A in $\mathfrak{L}'$:*

$$\mathsf{LJE} \vdash A \;\Leftrightarrow\; \mathsf{LJE}' \vdash A \qquad \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash A \;\Leftrightarrow\; \mathsf{LJE}'(\Sigma_{\mathfrak{L}}) \vdash A.$$

Proof. Use Theorem 6 and Lemma's 4 and 5. ■

Recall that we write $\mathfrak{L}(\Gamma_1, \Gamma_2)$ for the common language of Γ_1 and Γ_2, i.e. the language consisting of the predicates and non-variable terms that occur both in Γ_1 and Γ_2, plus $\top$, $\bot$ and E and the variables.

THEOREM 13. LJE' *and* $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ *have interpolation.*

Proof. We first prove the theorem for LJE' and then for $\mathsf{LJE}'(\Sigma_{\mathcal{L}})$ by showing how this case can be reduced to the LJE' case. We write $\vdash$ for $\vdash_{\mathsf{LJE}'}$ in this proof. Assume $\vdash \Gamma_1, \Gamma_2 \Rightarrow C$. We look for a formula I in the common language $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{C\})$ of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

(2) $\vdash \Gamma_1 \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C.$

We prove the theorem with induction to the depth d of P. Recall that the depth of a sequent in a proof is inductively defined as the sum of the depths of its upper sequents plus 1. Thus axioms have depth 1. The depth of a proof is the depth of its endsequent.

$d = 1$: P is an instance of an axiom. When the axiom is Ax we have $\Gamma_1\Gamma_2, Q \Rightarrow Q$, where Q is an atomic formula. There are two cases: we look for interpolants I and J such that

$$\vdash \Gamma_1, Q \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow Q \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, Q, \Gamma_2 \Rightarrow Q.$$

This case is trivial: take $I = Q$ and $J = \top$. The case that P is an instance of L$\bot$ is equally simple: again there are two possibilities, like above, and the interpolants are $\top$ and $\bot$.

$d > 1$. We distinguish by cases according to the last rule applied in P. If it is a LC, P looks as follows.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1\Gamma_2, A, A \Rightarrow C\end{array}}{\Gamma_1\Gamma_2, A \Rightarrow C}$$

Again there are several cases: we look for interpolants

$$\vdash \Gamma_1, A \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, A, \Gamma_2 \Rightarrow C.$$

By the induction hypothesis there are interpolants I' and J' such that the sequents $\Gamma_1, A, A \Rightarrow I'$ and $I', \Gamma_2 \Rightarrow C$, and $\Gamma_1 \Rightarrow J'$ and $J', A, A, \Gamma_2 \Rightarrow C$ are derivable. Moreover, I' is in $\mathcal{L}(\Gamma_1 \cup \{A\}, \Gamma_2 \cup \{C\})$, and J' is in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A, C\})$. Hence taking $I = I'$ and $J = J'$ and applying contraction gives the desired result. The case LW is equally trivial.

The connective cases are completely straightforward. For completeness sake we treat $\to$. Suppose the last rule is L$\to$:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1\Gamma_2 \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma_1\Gamma_2, B \Rightarrow C\end{array}}{\Gamma_1\Gamma_2, A \to B \Rightarrow C}$$

We have to find $I \in \mathcal{L}(\Gamma_1 \cup \{A \to B\}, \Gamma_2 \cup \{C\})$ and $J \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A \to B, C\})$ such that

$$\vdash \Gamma_1, A \to B \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, A \to B, \Gamma_2 \Rightarrow C.$$

For I, note that by the induction hypothesis there are $I' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{A\})$ and $J' \in \mathcal{L}(\Gamma_1 \cup \{B\}, \Gamma_2 \cup \{C\})$ such that

$$(3) \quad \vdash \Gamma_2 \Rightarrow I' \quad \vdash I', \Gamma_1 \Rightarrow A \qquad \text{and} \qquad \vdash \Gamma_1, B \Rightarrow J' \quad \vdash J', \Gamma_2 \Rightarrow C.$$

By applying LW with I' to the 3rd sequent in (3), then applying L$\to$ to this and the 2nd sequent, and then applying R$\to$, gives

$$\vdash \Gamma_1, A \to B \Rightarrow I' \to J'.$$

Applying L$\to$ to the 1st and 4th sequents in (3) gives $I' \to J', \Gamma_2 \Rightarrow C$. Hence we can take $I = I' \to J'$. Note that I indeed is in the common language $\mathcal{L}(\Gamma_1 \cup \{A \to B\}, \Gamma_2 \cup \{C\})$.

For J, observe that by the induction hypothesis there are $I'' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A\})$ and $J'' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{B, C\})$ such that

$$(4) \quad \vdash \Gamma_1 \Rightarrow I'' \quad \vdash I'', \Gamma_2 \Rightarrow A \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J'' \quad \vdash J'', B, \Gamma_2 \Rightarrow C.$$

From this it follows that

$$\vdash \Gamma_1 \Rightarrow I'' \wedge J'' \quad \vdash I'' \wedge J'', A \to B, \Gamma_2 \Rightarrow C.$$

Hence we can take $J = I'' \wedge J''$ in this case. Note that J indeed is in the common language $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A \to B, C\})$.

Suppose the last rule is R$\to$:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1\Gamma_2, A \Rightarrow B\end{array}}{\Gamma_1\Gamma_2 \Rightarrow A \to B}$$

By the induction hypothesis there is a interpolant $I \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A, B\})$ for the upper sequent: $\vdash \Gamma_1 \Rightarrow I$ and $\vdash I, A, \Gamma_2 \Rightarrow B$. I is an interpolant for the lower sequent too.

We treat the universal quantifier and leave the existential quantifier to the reader. Suppose the last rule is R$\forall$:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1\Gamma_2, Ey \Rightarrow A(y)\end{array}}{\Gamma_1\Gamma_2 \Rightarrow \forall x A[x/y]}$$

By the induction hypothesis there is a interpolant $I \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{Ey, A(y)\})$ for the upper sequent: $\vdash \Gamma_1 \Rightarrow I$ and $\vdash I, Ey, \Gamma_2 \Rightarrow A(y)$. In case y is not free in I the sequent $I, \Gamma_2 \Rightarrow \forall x A[x/y]$ is derivable too. Hence we can take I as an interpolant of the lower sequent and are done. Therefore, suppose y occurs free in I. By the side conditions y is not free in $\Gamma_1\Gamma_2$. Hence we have the following derivation:

$$\dfrac{\vdots \atop \Gamma_1, Ey \Rightarrow I}{\Gamma_1 \Rightarrow \forall z I[z/y]}$$

Thus the following derivation shows that $\forall z I[z/y]$ is an interpolant for the lower sequent:

$$\dfrac{\dfrac{\vdots \atop I, Ey, \Gamma_2 \Rightarrow A(y) \qquad Ey, \Gamma_2 \Rightarrow Ey}{\forall z I[z/y], Ey, \Gamma_2 \Rightarrow A(y)}}{\forall z I[z/y], \Gamma_2 \Rightarrow \forall x A[x/y]}$$

Finally, we treat L$\forall$:

$$\dfrac{\vdots \atop \Gamma_1\Gamma_2, A(t) \Rightarrow C \qquad \vdots \atop \Gamma_1\Gamma_2 \Rightarrow Et}{\Gamma_1\Gamma_2, \forall x A(x) \Rightarrow C}$$

We have to find $I \in \mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ and $J \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$ such that

$$\vdash \Gamma_1, \forall x A(x) \Rightarrow I \;\; \vdash I, \Gamma_2 \Rightarrow C \quad \text{and} \quad \vdash \Gamma_1 \Rightarrow J \;\; \vdash J, \forall x A(x), \Gamma_2 \Rightarrow C.$$

First we treat the case J. Note that by the induction hypothesis there are three formulas $I' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A(t), C\})$, $J' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{Et\})$ and $H' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ such that

$$(5) \quad \vdash \Gamma_1 \Rightarrow I' \;\; \vdash I', A(t), \Gamma_2 \Rightarrow C \quad \text{and} \quad \vdash \Gamma_1 \Rightarrow J' \;\; \vdash J', \Gamma_2 \Rightarrow Et$$

$$\vdash \Gamma_2 \Rightarrow H' \;\; \vdash H', \Gamma_1 \Rightarrow Et.$$

Note that I', J' and H' may contain t. If t does not occur in I' and J' or it occurs in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$, then $I', J' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$. Moreover, (5) implies

$$\vdash \Gamma_1 \Rightarrow I' \wedge J' \quad \vdash I' \wedge J', \forall x A(x), \Gamma_2 \Rightarrow C.$$

Thus in this case we can take $J = I' \wedge J'$.

On the other hand, if t does occur in I' or J' and not in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\}$ we proceed as follows. Either t occurs not in Γ_1 or t does not occur in $\Gamma_2 \cup \{\forall x A(x), C\}$. In the first case, it follows that t does not occur in I' and not in J', contradicting our assumptions. Thus t occurs in Γ_1 but not in $\Gamma_2 \cup \{\forall x A(x), C\}$. Hence t does not occur in H'. Note that we have a derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ H', \Gamma_1 \Rightarrow I' \wedge J'\end{array} \qquad \begin{array}{c}\vdots\\ H', \Gamma_1 \Rightarrow Et\end{array}}{H', \Gamma_1 \Rightarrow \exists x (I' \wedge J')[x/t]}}{\Gamma_1 \Rightarrow \big(H' \to \exists x (I' \wedge J')[x/t])\big)}$$

Now note something important: because t does not occur in $\forall x A(x)$, this implies that $\forall x A(x) = \forall x A[x/t]$ (for the difference between $A(x)$ and $A[x/t]$ see the preliminaries, Section 2.1). Thus also $\forall x (A[y/t])[x/y] = \forall x A(x)$. And because t does not occur in Γ_2 or C, by the substitution lemma, Lemma 3, we also have a derivation for a variable y not occurring in P of

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_2 \Rightarrow H'\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ (I' \wedge J')[y/t], Ey, A[y/t], \Gamma_2 \Rightarrow C\end{array} \qquad \begin{array}{c}\vdots\\ Ey, (I' \wedge J')[y/t], \Gamma_2 \Rightarrow Ey\end{array}}{Ey, (I' \wedge J')[y/t], \forall x A(x), \Gamma_2 \Rightarrow C}}{\exists x (I' \wedge J')[x/t], \forall x A(x), \Gamma_2 \Rightarrow C}}{\big(H' \to \exists x (I' \wedge J')[x/t]\big), \forall x A(x), \Gamma_2 \Rightarrow C}$$

Hence we can take $J = \big(H' \to \exists x (I' \wedge J')[x/t]\big)$ and are done.

The last case we have to treat is the one where we look for the interpolant $I \in \mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ such that

(6) $\vdash \Gamma_1, \forall x A(x) \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C.$

Note that by the induction hypothesis there are $I' \in \mathcal{L}(\Gamma_1 \cup \{A(t)\}, \Gamma_2 \cup \{C\})$, $J' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ and $H' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ such that

$$\vdash \Gamma_1, A(t) \Rightarrow I' \quad \vdash I', \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_2 \Rightarrow J' \quad \vdash J', \Gamma_1 \Rightarrow Et$$

$$\vdash \Gamma_1 \Rightarrow H' \quad \vdash H', \Gamma_2 \Rightarrow Et.$$

Observe that whence we have $\vdash (J' \to I'), \Gamma_2 \Rightarrow C$. Furthermore, we have a derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ J', A(t), \Gamma_1 \Rightarrow I'\end{array} \qquad \begin{array}{c}\vdots\\ J', \Gamma_1 \Rightarrow Et\end{array}}{J', \forall x A(x), \Gamma_1 \Rightarrow I'}}{\forall x A(x), \Gamma_1 \Rightarrow J' \to I'}$$

Thus, in case t belongs to the common language $\mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ we can take $I = (J' \to I')$ and are done. Therefore, assume t does not belong to the common language. In case it does not belong to $\Gamma_2 \cup \{C\}$, it follows that both I' and J' cannot contain t and we can again take $I = (J' \to I')$. Therefore, assume t does not belong to $\Gamma_1 \cup \{\forall x A(x)\}$. Hence H' does not contain t. But then we can infer, by Lemma 3, for a fresh variable y, from $\vdash \forall x A(x), \Gamma_1 \Rightarrow J' \to I'$ above, that we have the following derivation

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall x A(x), Ey, \Gamma_1 \Rightarrow (J' \to I')[y/t]}}{\forall x A(x), \Gamma_1 \Rightarrow \forall z (J' \to I')[z/t]} \qquad \dfrac{\vdots}{\Gamma_1 \Rightarrow H'}}{\forall x A(x), \Gamma_1 \Rightarrow \forall z (J' \to I')[z/t] \wedge H'}$$

On the other hand we also have

$$\dfrac{\dfrac{\dfrac{\vdots}{H', J' \to I', \Gamma_2 \Rightarrow C} \qquad \dfrac{\vdots}{H', J' \to I', \Gamma_2 \Rightarrow Et}}{H', \forall z (J' \to I')[z/t], \Gamma_2 \Rightarrow C}}{H' \wedge \forall z (J' \to I')[z/t], \Gamma_2 \Rightarrow C}$$

Hence we take $I = H' \wedge \forall z (J' \to I')[z/t]$ as the interpolant.

It is interesting to note that (6) also holds for $I = (Et \to I') \wedge H'$. But in this case I belongs in general not to the common language.

Finally, we show that $\mathsf{LJE}'(\Sigma_{\mathcal{L}})$ has interpolation too, by reducing this case to the case LJE' in the following way. Given a proof P of $\Gamma_1\Gamma_2 \Rightarrow C$ in $\mathsf{LJE}'(\Sigma_{\mathcal{L}})$ we consider all axioms of the form $\Pi \Rightarrow Et \in \Sigma_{\mathcal{L}}$ that occur in P. Suppose there are n of them: $\Pi_1 \Rightarrow Et_1, \ldots, \Pi_n \Rightarrow Et_n$. Note that all t_i have to be closed. Clearly, there is a proof of $Et_1, \ldots, Et_n, \Gamma_1\Gamma_2 \Rightarrow C$ in LJE' by replacing the axioms $\Pi_i \Rightarrow Et_i$ by the logical axioms $\Pi_i, Et_i \Rightarrow Et_i$. Now we consider the following partition $\Gamma_1'\Gamma_2' \Rightarrow C$ of $Et_1, \ldots, Et_n, \Gamma_1\Gamma_2 \Rightarrow C$:

$$\Gamma_1' = \Gamma_1 \cup \{Et_j \mid j \leq n, \ \ t_j \text{ occurs in } \Gamma_1 \text{ or not in } \Gamma_1 \cup \Gamma_2\}.$$

$$\Gamma_2' = \Gamma_2 \cup \{Et_j \mid j \leq n, \ \ t_j \text{ occurs in } \Gamma_2\}.$$

By the interpolation theorem for LJE' there exists an interpolant I such that $\vdash_{\mathsf{LJE}'} \Gamma_1' \Rightarrow I$ and $\vdash_{\mathsf{LJE}'} I, \Gamma_2' \Rightarrow C$ where I is in the common language of Γ_1' and $\Gamma_2' \cup \{C\}$. It is not difficult to see that whence I is in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ too. By cutting on the Et_i's we obtain

$$\vdash_{\mathsf{LJE}'(\Sigma_{\mathcal{L}})} \Gamma_1 \Rightarrow I \qquad \vdash_{\mathsf{LJE}'(\Sigma_{\mathcal{L}})} I, \Gamma_2 \Rightarrow C.$$

This proves that $\mathsf{LJE}'(\Sigma_{\mathcal{L}})$ has interpolation too. ■

COROLLARY 14. LJE *and* LJE($\Sigma_{\mathcal{L}}$) *have interpolation.*

5.1 Beth's theorem

Following standard proofs for the Beth definability property of LJ, it is easy to prove the following theorem.

THEOREM 15. LJE *and* LJE($\Sigma_{\mathcal{L}}$) *satisfy the Beth definability property.*

6 Two notions of forcing

In [30] Unterhalt proved the completeness of IQCE^+ with respect to a certain Kripke semantics that is similar to the semantics defined below. Here we define a semantics via Kripke models equipped with a slightly different notion of forcing, called *eforcing* and denoted by $\Vdash^e$. We then show that LJE is sound and complete with respect to $\Vdash^e$.

As we will define various kinds of Kripke models and various kinds of forcing, let us start with listing the notions we are going to define:

> *Kripke models*, *models* for short: standard Kripke models,
>
> *Kripke existence models*, *emodels* for short: Kripke models with constant domains in which the existence predicate plays a special role as it is assumed to be nonempty,
>
> *total emodels*: emodels in which the image of a function on arguments that exist always exist,
>
> *forcing*: the standard notion of forcing,
>
> *existence forcing*, *eforcing* for short: a notion of forcing in which predicates and connectives are forced in the usual way, but for which the quantifiers range over existing objects only. This notion of forcing is only defined for emodels.

6.1 Kripke models and Kripke existence models

A *classical structure for* $\mathcal{L}'$ is a pair (D_w, I_w) such that D_w is a nonempty set and I_w is a map from $\mathcal{L}'_{D_w}$ such that

> for every n-ary predicate P in $\mathcal{L}'$, $I_w(P)$ is an n-ary predicate on D_w,
>
> for every n-ary function f in $\mathcal{L}'_D$, $I_w(f)$ is an n-ary function on D_w (constants are considered as 0-ary functions),
>
> $I_w(a) = a$ for every constant $a \in D_w$.

A *classical existence structure for* $\mathcal{L}'$ is a classical structure for $\mathcal{L}'$ satisfying the extra requirement

$I_w(E)$ is a *nonempty* unary predicate on D_w.

Note that in a classical structure the existence predicate plays no special role, while in a classical existence structure.

For any closed $\mathcal{L}'_D$-term t, $I_w(t)$ denotes the interpretation of t under I_w in D, which is defined as usual. $I_w(t_1,\ldots,t_n)$ is short for $I_w(t_1),\ldots,I_w(t_n)$. For $\mathcal{L}'_D$-sentences A, let $(D,I_w)\models A$ denote that A holds in the structure (D,I_w), which is defined as usual for classical structures. Note that the interpretation of any closed term in $\mathcal{L}'_D$ is an element of and the same in all domains.

A *frame* is a pair $(W,\preccurlyeq)$ where W is a nonempty set and $\preccurlyeq$ is a partial order on W with a root. A *Kripke model*, *model* for short, on a frame $F=(W,\preccurlyeq)$ is a triple $K=(F,D,I)$, where $D=\{D_w \mid w\in W\}$ is a collection of nonempty sets and I is a collection $\{I_w \mid w\in W\}$, such that the (D_w,I_w) are classical structures for $\mathcal{L}'$ that satisfy the persistency requirements:

$$w\preccurlyeq v \;\Rightarrow\; D_w\subseteq D_v,$$

and for all predicates $P(\bar{x})$ in $\mathcal{L}$ and for all closed $\mathcal{L}'_D$-terms $\bar{t}$,

$$w\preccurlyeq v \;\Rightarrow\; \big((D,I_w)\models P(\bar{t}) \;\Rightarrow\; (D,I_v)\models P(\bar{t})\big),$$

$$w\preccurlyeq v \;\Rightarrow\; I_w(\bar{t})=I_v(\bar{t}).$$

In particular, $I_w(t)=I_v(t)$ for all closed terms in $\mathcal{L}'_{D_w}$. Therefore, we sometimes write $I(t)$ instead of $I_w(t)$. A *Kripke existence model*, *emodel* for short, is a Kripke model in which the (D_w,I_w) are classical existence structures and in which for all nodes w and v: $D_w=D_v$, i.e. Kripke existence model have constant domains. Therefore, we denote emodels often by $K=(F,D,I)$, where D is now a non empty set, and not a collection of sets as in the case of models. We call an emodel *total* when for all its nodes k and for all functions $f(x_1,\ldots,x_n)$ in $\mathcal{L}'$

$$\forall\, a_1,\ldots,a_n\in D : k\Vdash \bigwedge_{i=1}^{n} Ea_i\to Ef(a_1,\ldots,a_n).$$

6.2 Forcing and existence forcing

Given a Kripke model the notion of *forcing*, $\Vdash$, is defined as usual. Given a Kripke existence model $K=(D,\preccurlyeq,I)$, the *existence forcing relation* $\Vdash^e$, *eforcing* for short, is defined as follows, and denoted by $\Vdash^e$ to distinguish it from the standard forcing relation. For our purposes it suffices to define the eforcing relation $K,w\Vdash^e A$ at node w inductively only for *sentences* in $\mathcal{L}'_D$. For predicates $P(\bar{x})$ in $\mathcal{L}'$ (including E) and closed $\mathcal{L}'_D$-terms t, we put

$$K,w\Vdash^e P(\bar{t})\equiv_{def}(D,I_w)\models P(\bar{t}),$$

and extend $K, w \Vdash^e A$ to all sentences in $\mathcal{L}'_D$ in the usual way for connectives, but differently for the quantifiers:

$$\begin{array}{lll}
k \nVdash^e \bot & & \\
k \Vdash^e A \wedge B & \text{iff} & k \Vdash^e A \text{ and } k \Vdash^e B \\
k \Vdash^e A \vee B & \text{iff} & k \Vdash^e A \text{ or } k \Vdash^e B \\
k \Vdash^e A \to B & \text{iff} & \forall k' \succcurlyeq k : \; k' \Vdash^e A \Rightarrow k' \Vdash^e B \\
k \Vdash^e \exists x A(x) & \text{iff} & \exists d \in D \; k \Vdash^e Ed \wedge A(d) \\
k \Vdash^e \forall x A(x) & \text{iff} & \forall d \in D : \; k \Vdash^e Ed \to A(d).
\end{array}$$

Note that the upwards persistency requirement

$$k \preccurlyeq l \;\wedge\; k \Vdash^e A \;\Rightarrow\; l \Vdash^e A.$$

is fulfilled. Moreover, note that

$$k \Vdash^e \forall x A(x) \;\Leftrightarrow\; \forall l \succcurlyeq k \forall d \in D \, l \Vdash^e Ed \to Ad.$$

When K is clear from the context we write $k \Vdash^e A$ instead of $K, k \Vdash^e A$. We call an emodel K an $\mathcal{L}$-emodel when

$$\forall k \forall t \in \mathcal{T}_{\mathcal{L}} : \; k \Vdash^e Et.$$

A *valuation on* K is a map α from variables to the domain at the root. Thus in the case of existence models it is a map from variables to D. For a formula A in $\mathcal{L}'_D$ we write $A[\alpha]$ for the formula that is the result of substituting $\alpha(x)$ for x, for every variable x in A. For formulas A we write $K \Vdash^e A$ and say that A *is eforced in* K if for all nodes k, for all valuations α on K, $K, k \Vdash^e A[\alpha]$. We say that A is $\mathcal{L}$*-eforced*, written $\Vdash^e_{\mathcal{L}} A$, when $K \Vdash^e A$ for all $\mathcal{L}$-emodels K. We define similar notions for sequents $\Gamma \Rightarrow C$, considering them as formulas $\bigwedge \Gamma \to C$. We say that a collection of sequents $\mathcal{S}$ ($\mathcal{L}$-)eforces S when for all ($\mathcal{L}$-)models K, if $K \Vdash^e_{\mathcal{L}} S'$ for all $S' \in \mathcal{S}$, then $K \Vdash^e_{\mathcal{L}} S$. Similar notions are defined for forcing, reading forcing everywhere for eforcing and model for emodel.

7 Soundness and completeness

For the soundness and completeness proof of LJE with respect to $\Vdash^e$ to come, it will be convenient to work in Gentzen calculus LJE^∞ that is equivalent to LJE for finite sequents but that can deal with sequents of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ may be infinite and Δ may contain more than one formula. It is similar to LJE, and in the case of R$\forall$ and R$\to$ the antecedent may still contain only one formula, like in LJE. Furthermore, it has structural rules weakening and contraction. Because of this, in L$\to$ and L$\forall$ the principal formula does not have to occur in the hypotheses.

Important: From now on Γ, Π, Δ and Λ range over countably infinite multisets of formulas, and sequents may be infinite from now on. Except in the setting of LJE or $\vdash_{\mathcal{L}}$ that only apply to finite sequents: in these cases we tacitly assume the sequents to be finite.

The system LJE^{∞}

$Ax\ \ \Gamma, P \Rightarrow P, \Delta\ \ (P \text{ atomic})$ $\qquad$ $L\bot\ \ \Gamma, \bot \Rightarrow \Delta$

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{RC}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\text{L}\!\to\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} \qquad \text{R}\!\to\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$\text{L}\forall\ \ \frac{\Gamma, At \Rightarrow \Delta \qquad \Gamma \Rightarrow Et, \Delta}{\Gamma, \forall x Ax \Rightarrow \Delta} \qquad \text{R}\forall\ \ \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]}\ *$$

$$\text{L}\exists\ \ \frac{\Gamma, Ay, Ey \Rightarrow \Delta}{\Gamma, \exists x A[x/y] \Rightarrow \Delta}\ * \qquad \text{R}\exists\ \ \frac{\Gamma \Rightarrow At, \Delta \qquad \Gamma \Rightarrow Et, \Delta}{\Gamma \Rightarrow \exists x Ax, \Delta}$$

$$\text{Cut}\ \ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We write $\mathsf{LJE}^{\infty}(\Sigma_{\mathcal{L}})$ for the system obtained from LJE^{∞} by adding the sequents $\Sigma_{\mathcal{L}}$ as axioms. We say that LJE^{∞} derives $\Gamma \Rightarrow \Delta$, $\vdash_{\mathsf{LJE}^{\infty}} \Gamma \Rightarrow \Delta$, when there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{\mathsf{LJE}^{\infty}} (\Gamma' \Rightarrow \Delta')$. We say that a set of sequents $\mathcal{S}$ derives a sequent S in LJE^{∞}, when there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \Rightarrow \Delta'$ is derivable in the system LJE^∞ to which the sequents in $\mathcal{S}$ are added as axioms. We have similar notions for $\mathsf{LJE}^\infty(\Sigma_\mathcal{L})$. We often write $\vdash^\infty_\mathcal{L}$ for $\mathsf{LJE}^\infty(\Sigma_\mathcal{L})\vdash$.

We leave it to the reader to verify that the following holds, using the fact that LJE has weakening and contraction, Lemma's 4 and 5:

LEMMA 16. *For finite* Γ *and* Δ*:* $\vdash^\infty_\mathcal{L} \Gamma \Rightarrow \Delta$ *if and only if* $\vdash_\mathcal{L} \Gamma \Rightarrow \bigvee \Delta$.

7.1 Soundness

THEOREM 17. *For all sets of closed sequents* $\mathcal{S}$ *and all closed sequents* S *in* $\mathcal{L}'$*:*

$$\mathcal{S} \vdash^\infty_\mathcal{L} S \textit{ implies } \mathcal{S} \Vdash^e_\mathcal{L} S.$$

Proof. We only consider the case that $\mathcal{S}$ is empty and that S is a sequent with at most one formula in the succedent and leave the other cases to the reader. For a smooth induction we prove that all axioms of $\mathsf{LJE}^\infty(\Sigma_\mathcal{L})$ are $\mathcal{L}$-forced, and that for all its rules, if the hypotheses of the rule are $\mathcal{L}$-forced, then so is the conclusion. The case of the axioms is simple and so are most of the rules. We treat the axiom $\Sigma_\mathcal{L}$ and the rules $R\forall$ and $R\exists$. Let K be an $\mathcal{L}$-model. Recall that we write $k \Vdash^e \Gamma$ meaning that $k \Vdash^e A$ for all $A \in \Gamma$.

Consider a sequent $\Gamma \Rightarrow Et$ in $\Sigma_\mathcal{L}$. Hence t is a closed term in $\mathcal{T}_\mathcal{L}$. Let $\bar{x}$ be all the free variables that occur in Γ. By assumption on $\mathcal{L}$-models it follows that $K \Vdash^e_\mathcal{L} (\Gamma \Rightarrow Et)[\bar{a}/\bar{x}]$ for all $\bar{a} \in D$.

For R$\forall$ suppose $\Vdash^e \Pi, Ey \Rightarrow Ay$ and y not free in Π. Consider k in K, suppose that the free variables in Π and Ay are among $\bar{x}y$, let $\bar{a} \in D$, and assume $k \Vdash^e \Pi[\bar{a}/\bar{x}]$. We have to show that

$$\forall\, d \in D: \; k \Vdash^e (Ed \to Ad)[\bar{a}/\bar{x}].$$

Therefore, consider $l \succcurlyeq k$ and $d \in D$ such that $l \Vdash^e Ed$. We have to show that $l \Vdash^e Ad[\bar{a}/\bar{x}]$. As the side condition on R$\forall$ implies that y does no occur free in Π, we have $l \Vdash^e (\Pi \wedge Ey)[\bar{a}d/\bar{x}y]$. As $\Vdash^e \Pi, Ey \Rightarrow Ay$, this implies $l \Vdash^e Ay[\bar{a}d/\bar{x}y]$, that is, $l \Vdash^e Ad[\bar{a}/\bar{x}]$.

For $R\exists$ suppose $\Vdash^e \Pi \Rightarrow At$, $\Vdash^e \Pi \Rightarrow Et$, and let all free variables of Π and At be among $\bar{x}$, pick $\bar{a} \in D$ and assume $k \Vdash^e \Pi[\bar{a}/\bar{x}]$. We have to show that

$$\exists\, d \in D: \; k \Vdash^e (Ed \wedge Ad)[\bar{a}/\bar{x}].$$

Since $\Vdash^e \Pi \Rightarrow Et$ and $k \Vdash^e \Pi[\bar{a}/\bar{x}]$, this gives $k \Vdash^e Et[\bar{a}/\bar{x}]$. Similarly, $k \Vdash^e At[\bar{a}/\bar{x}]$. Let $d = t[\bar{a}/\bar{x}]$. Then we have $k \Vdash^e (Ed \wedge Ad)[\bar{a}/\bar{x}]$, as desired. ■

7.2 Completeness

As mentioned above, the completeness proof given follows the pattern of the completeness proof for LJ as given in [26]. The idea is that if a sequent is underivable we apply the inference rules in the reversed order as long as possible, resulting is a so-called reduction tree with at least one branch along which all sequents are underivable. This branch will be a node in the Kripke model that we obtain by repeating this process, and that will refute the sequent we started with. Therefore, we first have to introduce the notion of a reduction tree, a notion similar to that of a Beth tableau.

DEFINITION 18. Given a (possibly infinite) sequent S, the *reduction tree for S* is inductively defined as follows. Recall that we assumed that $\mathcal{L}$ contains at least one constant and no variables, and that $\mathcal{L}'$ has an infinite set of variables. Furthermore, we assume that at every stage of the construction we have infinetely many fresh variables of $\mathcal{L}'$ available, i.e. variables that do not occur in the sequents constructed so far.

The construction of the reduction tree for $S = (\Gamma \Rightarrow \Delta)$ consists of repeated application of steps 0,1, 2, …, 7, which correspond to inference rules of LJE without the structural rules, R$\forall$ and R$\rightarrow$. We leave it to the reader to check that at every stage of the construction we deal with countably infinite sequents only, i.e. with sequents for which the antecedent and succedent contain countably inifnite many formulas only.

Step $n = 0$: write S at the bottom of the tree.

Step $n > 0$: if every leave is an axiom of LJE or a sequent in $\Sigma_{\mathcal{L}}$, then stop. If this is not the case, then this stage is defined according to $n \equiv 0, 1, \ldots, 8$ mod 9. Let $\Pi \Rightarrow \Lambda$ be any leave of the tree defined at stage $n-1$.

$n \equiv 0$: L$\wedge$ reduction. Let α be a set such that $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol $\wedge$ to which no reduction has yet been applied. Then above S write the sequent

$$\Pi, \{A_{i0}, A_{i1} \mid i \in \alpha\} \Rightarrow \Lambda.$$

$n \equiv 1$: R$\wedge$ reduction. Let α be a set such that $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Λ with outermost logical symbol $\wedge$ to which no reduction has yet been applied. Then above S write all sequents of the form

$$\Pi \Rightarrow \{A_{if(i)} \mid i \in \alpha\}, \Lambda$$

for any map $f : \alpha \rightarrow \{0, 1\}$.

$n \equiv 2$: L$\vee$ reduction. Defined in a similar way as R$\wedge$ reduction.

$n \equiv 3$: R$\vee$ reduction. Defined in a similar way as L$\wedge$ reduction.

$n \equiv 4$: L$\rightarrow$ reduction. Let α be a set such that $\{A_i \rightarrow B_i \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol $\rightarrow$ to which no reduction has yet been applied. Then for all $f : \alpha \to \{0,1\}$, write above S the sequent

$$\Pi, \{B_i \mid f(i) = 1\} \Rightarrow \{A_i \mid f(i) = 0\}, \Lambda.$$

$n \equiv 5$: L$\forall$ reduction. Let α be a set such that $\{\forall x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol $\forall$. Let $\mathcal{T}$ consists of all terms t for which Et occurs in Π. Above S write the sequent

$$\Pi, \{A_i(t) \mid i \in \alpha, t \in \mathcal{T}\} \Rightarrow \Lambda.$$

Note that if $\{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \subseteq \Pi$ we can always carry out this step, since there is at least one constant in $\mathcal{L}$, which implies there is at least one expression of the form Et in $\{Et \mid t \in \mathcal{T}_{\mathcal{L}}\}$, and thus in Π.

$n \equiv 6$: L$\exists$ reduction. Let α be a set such that $\{\exists x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol $\exists$ to which no reduction has yet been applied. Introduce fresh variables $\{y_i \mid i \in \alpha\}$ of $\mathcal{L}'$, and above S write the sequent

$$\Pi, \{A_i(y_i), Ey_i \mid i \in \alpha\} \Rightarrow \Lambda.$$

$n \equiv 7$: R$\exists$ reduction. Defined in a similar way as L$\forall$ reduction.

$n \equiv 8$: if $\Pi \Rightarrow \Lambda$ is an axiom of LJE or a sequent in $\Sigma_{\mathcal{L}}$, then stop. If this is not the case write the same sequent $\Pi \Rightarrow \Lambda$ above it.

This completes the definition of reduction trees.

The follwing is straightforward.

LEMMA 19. *If all leaves of the reduction tree of a sequent S are axioms of $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, then S is provable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$.*

The following lemma, Lemma 21, is non-trivial and crucial in the completeness proof. It is an analogue of a lemma in [26] for LJ, and its main ingredient is the following generalization of König's Lemma.

PROPOSITION 20. *(A generalized König's Lemma, Takeuti [26]) Let X be any set. Let $*(\cdot)$ be a property on partial functions $f : X \to \{0,1\}$. If*

1. *$*(f)$ holds if and only if there is a finite subset $Z \subseteq X$ such that $*(f \upharpoonright Z)$ (here $f \upharpoonright Z$ is the restriction of f to Z), and*

2. *$*(f)$ holds for all total functions f on X,*

then there exists a finite set $X' \subseteq X$ such that $*(f)$ for any f with $X' \subseteq dom(f)$ ($dom(f)$ is the domain of f).

Proof. For completeness sake we repeat Takeuti's proof from [26]. Let Y be the product of $|X|$ times $\{0,1\}$. Give $\{0,1\}$ the discrete topology and Y the product topology. Since $\{0,1\}$ is compact, so is Y by Tychonoff's theorem. For maps f and g call g an *extension* of f, when $dom(f) \subseteq dom(g)$ and f and g are equal on $dom(f)$. For every f with finite domain, let

$$\mathcal{N}_f \equiv_{def} \{g \mid g \text{ is total and an extension of } f\}.$$

Furthermore, let

$$\mathcal{C} \equiv_{def} \{\mathcal{N}_f \mid \ dom(f) \text{ is finite and } *(f)\}.$$

$\mathcal{C}$ is an open cover of Y. Therefore, $\mathcal{C}$ has a finite subcover, say $\mathcal{N}_{f_1}, \ldots, \mathcal{N}_{f_n}$. Let

$$X' = dom(f_1) \cup \ldots \cup dom(f_n).$$

Then X' satisfies the conditions of the theorem: assume $Z \subseteq dom(f)$. Let f' be a total extension of f. Then $*(f')$ by 2. and $f \in \mathcal{N}_{f_1} \cup \ldots \cup \mathcal{N}_{f_n}$, say $f \in \mathcal{N}_{f_i}$. Whence f is an extension of f_i and $*(f_i)$. Therefore, $*(f)$ by 1. ■

LEMMA 21. *If a sequent S is not provable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, then its reduction tree has a branch along which all sequents are underivable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$.*

Proof. In this proof provable will always mean provable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, $\vdash$ stands for $\mathsf{LJE}(\Sigma_{\mathcal{L}}) \vdash$. We prove the lemma by proving the following: if in a reduction tree, for some set α, $\Gamma_\beta \Rightarrow \Delta_\beta$ $(\beta = 1, 2 \ldots, \alpha)$ are all the immediate successors of $\Gamma \Rightarrow \Delta$, then if all these successors are provable, then so is $\Gamma \Rightarrow \Delta$. Recall that a sequent $\Pi \Rightarrow \Lambda$ is provable when there are finite $\Pi' \subseteq \Pi$ and $\Lambda' \subseteq \Lambda$ such that $\Pi' \Rightarrow \Lambda'$ is provable.

We distinguish by cases acceding to the rule that is applied to $\Gamma \Rightarrow \Delta$ resulting in the immediate successors $\Gamma_\beta \Rightarrow \Delta_\beta$.

L$\wedge$ reduction: then $\Gamma \Rightarrow \Delta$ has one upper sequent, which is of the form $\Gamma, \{A_{i0}, A_{i1} \mid i \in \alpha\} \Rightarrow \Delta$, where $A_{i0} \wedge A_{i1}$ are all the sequents in Γ with outermost logical symbol $\wedge$. By assumption there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ and $B_i \in \{A_{i0}, A_{i1}\}$ for $i \leq n$ such that $\vdash \Gamma, B_1, \ldots, B_n \Rightarrow \Delta'$. Hence $\vdash \Gamma', \{A_{i0}, A_{i1} \mid i \leq n\} \Rightarrow \Delta'$, which again implies $\vdash \Gamma', \{A_{i0} \wedge A_{i1} \mid i \leq n\} \Rightarrow \Delta'$. Thus $\vdash \Gamma \Rightarrow \Delta$.

R$\wedge$ reduction: then $\Gamma \Rightarrow \Delta$ has immediate successors $\Gamma \Rightarrow \{A_{if(i)} \mid i \in \alpha\}, \Delta$ for any map $f : \alpha \to \{0,1\}$, where $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Δ with outermost logical symbol $\wedge$. By assumption for all $f : \alpha \to \{0,1\}$ there are finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and $n_f \in \omega$ such that $\vdash \Gamma' \Rightarrow \{A_{if(i)} \mid i \leq n_f\}, \Delta'$. Now we are going to use the generalized König's

Lemma. We define a property $*(\cdot)$ on the partial functions $f:\alpha\rightarrow\{0,1\}$ as follows ($dom(f)$ denotes the domain of f):

$$*(f)\equiv\exists\, m\exists\, a_1\dots a_m\in dom(f)\exists \text{ finite } \Gamma'\subseteq\Gamma,\Delta'\subseteq\Delta: \\ \vdash\Gamma'\Rightarrow\{A_{\alpha_i f(a_i)}\mid i\leq m\},\Delta'.$$

Then conditions 1. and 2. of the generalized König's Lemma 20 are satisfied. Hence there is a finite subset $\beta\subseteq\alpha$ such that $*(f)$ whenever $\beta\subseteq dom(f)$. Let $\mathcal{F}$ be the collection of f for which $dom(f)=\beta$. Thus for all $f\in\mathcal{F}$ there are finite $\Gamma^f\subseteq\Gamma$ and $\Delta^f\subseteq\Delta$ such that

$$\vdash\Gamma^f\Rightarrow\{A_{if(i)}\mid i\in\beta\},\Delta^f.$$

Hence by weakening and repeated application of R$\wedge$, one obtains

$$\vdash\{\Gamma^f\mid f\in\mathcal{F}\}\Rightarrow\{A_{i0}\wedge A_{i1}\mid i\in\beta\},\{\Delta^f\mid f\in\mathcal{F}\}.$$

This implies that $\vdash\Gamma\Rightarrow\Delta$.

The case R$\vee$ is similar to R$\wedge$, and L$\vee$ and L$\rightarrow$ are similar to R$\wedge$.

L$\exists$ reduction: then $\Gamma\Rightarrow\Delta$ has immediate successor

$$\Gamma,\{A_i(y_i),Ey_i\mid i\in\alpha\}\Rightarrow\Delta,$$

where $\{\exists\, x_iA_i(x_i)\mid i\in\alpha\}$ consists exactly of all formulas in Δ with outermost logical symbol $\exists$. By assumption there are finite $\Gamma'\subseteq\Gamma$ and $\Delta'\subseteq\Delta$ and $n\in\omega$ such that

$$\vdash\Gamma',\{A_i(y_i),Ey_i\mid i\leq n\}\Rightarrow\Delta'.$$

Applications of L$\exists$ imply that then $\Gamma\Rightarrow\Delta$ is provable too.

The cases R$\exists$ and L$\forall$ are similar. This proves the lemma. ■

THEOREM 22. *For all sets of closed sequents $\mathcal{S}$ and all closed sequents S in $\mathcal{L}'$:*

$$\mathcal{S}\Vdash^e_{\mathcal{L}} S \text{ implies } \mathcal{S}\vdash^{\infty}_{\mathcal{L}} S.$$

Proof. We treat the case that $\mathcal{S}$ is empty and leave the other case to the reader. The proof we give is similar to the elegant completeness proof for LJ in [26]. Recall that $\vdash^{\infty}_{\mathcal{L}}$ stands for $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})}$. In the proof we will write $\vdash$ for $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})}$. Let $S=(\Gamma\Rightarrow\Delta)$ be a closed sequent and assume that $\nvdash S$. We will construct a $\mathcal{L}$-model K such that $K\nVdash^e S$ in the following way.

K will be defined in ω many steps using reduction trees, which will be the nodes of K. We assume that $\mathcal{L}'$ contains infinitely many variables that do not occur in S.

Step 0: Let T_0 be the reduction tree for $\Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta$. Call this node 0. Since $\nvdash \Gamma \Rightarrow \Delta$, also

$$\nvdash \Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta.$$

By Lemma 21 there is a branch b_0 in T_0 containing only unprovable sequents. We proceed with b_0 and T_0 to the next step 1.

Step $i{+}1$. For any reduction tree T with branch b along which all sequents are unprovable constructed at step i, we consider Π and Λ, which are the respective unions of the formulas in the antecedents and succedents along b. Note that thus $\nvdash \Pi \Rightarrow \Lambda$. Let k range over all formulas in Λ with outermost logical symbol $\rightarrow$ or $\forall$. We proceed in the following way.

If k is a formula of the form $A \rightarrow B$. Then construct the reduction tree T_k for $\Pi, A \Rightarrow B$. This tree will be an immediate successor of T. Note that $\nvdash \Pi, A \Rightarrow B$. Thus by Lemma 21 there is a branch b in T containing only unprovable sequents. We proceed with b and T to the next step $i + 2$.

If k is a formula $\forall x A(x)$. Then construct the reduction tree T_k for $\Pi, Ey \Rightarrow A(y)$, where y is a variable in $\mathcal{L}$ that has not yet occurred in the construction of K. This tree will be an immediate successor of T. Observe that if $\Pi, Ey \Rightarrow A(y)$ is derivable, then so is $\Pi \Rightarrow \forall x Ax$, since y does not occur in Π. Thus $\nvdash \Pi, Ey \Rightarrow A(y)$, and whence by Lemma 21 there is a branch b in T containing only unprovable sequents. We proceed with b and T to the next step $i + 2$.

This process is continued ω times. Let W be the union of 0 and all k's in the construction and let $\preccurlyeq$ be the reflexive transitive closure of the immediate successor relation constructed at the stages. Define D to be the set of all terms appearing in the construction. Given a k in the construction, let T_k be the reduction tree at k and let Π_k and Λ_k be the respective unions of the formulas in the antecedents and succedents along the chosen infinite branch b_k in T_k. Then define an interpretation I as follows:

$$I_k(R) \equiv_{def} \{\bar{d} \in D \mid R(\bar{d}) \in \Pi_k\},$$

and I_k is the identity on function symbols: $I_k(f)(\bar{a}) = f(\bar{a}) \in D$. Since $\mathcal{L}$ contains at least one constant c, it also implies that $I_k(E)$ is nonempty. Note that $K = ((W, \preccurlyeq), D, I)$ indeed is a Kripke existence model. The fact that we started with the sequent $\Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta$ implies that $Et \in \Pi_k$ for all k and all terms t in $\mathcal{L}$. Hence $K \Vdash^e Et$ for all terms $t \in \mathcal{T}_{\mathcal{L}}$, and thus K is an $\mathcal{L}$-model. It is not difficult to show with formula induction that we have

$$A \in \Pi_k \;\Rightarrow\; k \Vdash^e A$$
$$A \in \Lambda_k \;\Rightarrow\; k \nVdash^e A$$

we treat the case $A = B \to C$ and leave the other cases to the reader. This will complete the theorem.

First assume $B \to C \in \Pi_k$. We have to show that $k \Vdash^e B \to C$. Therefore, consider $l \succcurlyeq k$ such that $l \Vdash^e B$. Thus by the induction hypothesis $B \in \Pi_l$. By the construction of the reduction tree, $C \in \Pi_l$ or $B \in \Lambda_l$. Since $B \notin \Lambda_l$, otherwise the branch would be derivable, it follows that $C \in \Pi_l$, and thus $l \Vdash^e C$.

Second, assume $B \to C \in \Lambda_k$. By the construction of the model, there is a node $l \succcurlyeq k$ such that $B \in \Pi_l$ and $C \in \Lambda_l$. This implies that $l \Vdash^e B$ and $l \nVdash^e C$. Hence $k \nVdash^e B \to C$. ■

By Lemma 16 it follows that :

COROLLARY 23. *For all sets of finite closed sequents* $\mathcal{S}$ *and all finite closed sequents* S *in* $\mathcal{L}'$*:*

$$\mathcal{S} \vdash_{\mathcal{L}} S \textit{ if and only if } \mathcal{S} \Vdash^e_{\mathcal{L}} S.$$

COROLLARY 24. *For all sets of closed sequents* $\mathcal{S}$ *and all closed sequents* S *in* $\mathcal{L}'$*:*
$\mathcal{S} \vdash_{\mathcal{L}} S$ *if and only if* $K \Vdash^e S$ *for all* $\mathcal{L}$*-models* K *based on frames that are conversely well-founded trees that force* $\mathcal{S}$*.*

Proof. Immediate from Lemma 16 and the proof of Theorem 22. ■

8 Applications

8.1 Skolemization

One use of the existence predicate is in the setting of Skolemization. Recall that the Skolemization of a formula is the result of replacing strong quantifiers, i.e. positive universal and negative existential quantifiers, by fresh function symbols, thus obtaining a formula without strong quantifiers that is equiconsistent with the original formula. As is well-known, Skolemization is not complete with respect to IQC. That is, there are formulas that are underivable, but for which their Skolemized version is derivable in IQC. For example,

$$\mathsf{IQC} \nvdash \forall x(Ax \vee B) \to (\forall x Ax \vee B) \qquad \mathsf{IQC} \vdash \forall x(Ax \vee B) \to (Ac \vee B).$$

In [1] an alternative Skolemization method called *eSkolemization* is introduced and is shown to be sound and complete with respect to IQC for a large class of formulas, including all formulas in which every strong quantifier is existential or of the form $\forall x \neg\neg Ax$. This class is much larger than the class of formulas for which the standard Skolemization method is sound and complete. This eSkolemization method makes use of the existence predicate. It replaces negative occurrences of existential quantifiers $\exists x Bx$ by $(Ef(\bar{y}) \wedge Bf(\bar{y}))$, and positive occurrences of universal quantifiers $\forall x Bx$ by $(Ef(\bar{y}) \rightarrow Bf(\bar{y}))$. For example, the eSkolemization of the displayed formula above is

$$\mathsf{IQCE} \nvdash \forall x(Ax \vee B) \rightarrow ((Ec \rightarrow Ac) \vee B).$$

The eSkolemization method is extended to sequents $\Gamma \Rightarrow C$ by considering them as formulas $\bigwedge \Gamma \rightarrow C$. The eSkolemization of a sequent S is denoted by S^s. Clearly,

$$\mathsf{LJE} \vdash A \Rightarrow A^s.$$

Then it is shown in [1] that

THEOREM 25. *[1] For each closed sequent S in $\mathcal{L}'$ in which all strong quantifiers are existential: $\vdash_{\mathcal{L}} S$ if and only if $\vdash_{\mathcal{L}} S^s$.*

[1] By Lemma 8 this implies that

COROLLARY 26. *[1] For each closed sequent S in $\mathcal{L}\backslash E$ in which all strong quantifiers are existential: $\vdash_{\mathsf{LJ}} S$ if and only if $\vdash_{\mathcal{L}} S^s$.*

There is an extension of the main result to a larger class of formulas than the one occurring in the theorem above. This class of formulas is not syntactically defined and therefore less useful. However, it contains a syntactically defined class of formulas strictly larger than the class of formulas in which all strong quantifiers are existential: the class of formulas in which all strong universal quantifiers are of the form $\forall x \neg\neg Ax$. Hence the result, Theorem 29, that eSkolemization is sound and complete for this class of formulas is a genuine extension of Theorem 25.

DEFINITION 27. For a formula A that occurs in a sequent S, $S[B/A]^p$ (p for positive) denotes the result of replacing every positive occurrence of A in S by B. Note that we do not put restrictions on the possible occurrences of free variables in A or S. We say that *all strong quantifiers in S are almost existential* if for every subformula $\forall x Ax$ of S, it holds that

$$S[\neg\exists x \neg Ax/\forall x Ax]^p \vdash_{\mathcal{L}} \ S.$$

Note that we always have

$$S \vdash_{\mathcal{L}} S[\neg\exists x \neg Ax/\forall x Ax]^p.$$

Thus almost existential sequents are sequents that, as a formula, are equivalent to a formula in which all strong quantifiers are existential.

REMARK 28. Clearly, all strong quantifiers in S are almost existential, if no strong universal quantifiers occur in S. But the class of sequents in which all strong quantifiers are almost existential also contains the formulas in which all strong universal quantifiers are of the form $\forall x \neg\neg Ax$. But the class contains more: for example, $\bot \Rightarrow \forall x Ax$ does not belong to the mentioned classes but every quantifier in this formula is almost existential.

THEOREM 29. *[1] For each closed sequent S in $\mathcal{L}'$ in which all strong quantifiers are almost existential:*

$$\vdash_{\mathcal{L}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

COROLLARY 30. *[1] For all closed sequents S in $\mathcal{L}\backslash E$ in which all strong quantifiers are almost existential:*

$$\vdash_{\mathsf{LJ}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

As was first proved by Mints in [**?**] we have the following corollary when using Proposition 9.

COROLLARY 31. *[1] For the fragment of sentences without weak quantifiers and in which all strong quantifiers are almost existential, derivability on* IQC *is decidable.*

In [1] also an analogue of Herbrand's theorem is provided, which together with eSkolemization links derivability in intuitionistic predicate logic to derivability in intuitionistic propositional logic, at least for formulas in which all strong quantifiers are almost existential.

8.2 Truth-value logics

Another application of the existence predicate is in the context of truth-value logics. These are logics based on truth-value sets V, i.e. closed subsets of the unit interval $[0,1]$, also called *Gödel sets*. One can, for a given Gödel set V, interpret formulas by mapping them to elements of V. The logical symbols receive a meaning via restrictions on these interpretations, e.g. by stipulating that the interpretation of $\wedge$ is the infimum of the interpretations of the respective conjuncts, or that the interpretation of $\exists x Ax$ is the supremum of the values of Aa for all elements a in the domain. Given these interpretations, one can associate a logic with such a Gödel set V: the logic of all sentences that are mapped to 1 under any interpretation on V. Here we work only with the languages $\mathcal{L}'$ and $\mathcal{L}'\_ = \backslash E$.

We define the following frame logics:

$$
\begin{array}{rcl}
L_F \Vdash A & \equiv_{def} & K \Vdash A \text{ for all models } K \text{ on } F \\
L_F & \equiv_{def} & \{A \mid \ A \text{ a sentence in } \mathcal{L}'_-,\ F \Vdash A\} \\
L_F^{cd} & \equiv_{def} & \{A \mid \ A \text{ a sentence in } \mathcal{L}'_-, \\
 & & \quad K \Vdash A \text{ for all models } K \text{ on } F \text{ with constant domain}\} \\
L_F^{cdte} & \equiv_{def} & \{A \mid \ A \text{ a sentence in } \mathcal{L}'_-, \\
 & & \quad K \Vdash A \text{ for all total emodels } K \text{ on } F \text{ with constant domain}\}
\end{array}
$$

Gödel logics are a famous example of truth value logics. Given a Gödel set V and a nonempty set D, a *Gödel logic interpretation* I is defined as follows.

$$
\begin{array}{rcl}
I(P\bar{t}) & = & I(P)(I(\bar{t})) \\
I(A \wedge B) & = & \inf(I(A), I(B)) \\
I(A \vee B) & = & \sup(I(A), I(B)) \\
I(A \to B) & = & \begin{cases} 1 & \text{if } I(A) \leq I(B) \\ I(B) & \text{otherwise,} \end{cases} \\
I(\exists\, x A x) & = & \sup\{I(Aa) \mid a \in D\} \\
I(\forall\, x A x) & = & \inf\{I(Aa) \mid a \in D\}.
\end{array}
$$

The Gödel logic G_V consists of those sentences in $\mathcal{L}'_-$ that receive value 1 under all such Gödel logic interpretations I, for all possible domains D. A. Beckmann and N. Preining in [5] proved that Gödel logics correspond to the logics of linear frames with constant domain in the following way.

THEOREM 32. *(A. Beckmann and N. Preining [5]) For any countable linear frame F there exists a Gödel set V such that*

(7) $\quad G_V = L_F^{cd},$

and vice versa: for every Gödel set V there exists a countable linear frame F such that (7).

Based on these ideas, in [13] an analogue was found for the case of linear frames without the extra restriction to constant domains. So-called *Scott logics* S_V were defined, where S_V consists of all sentences A in $\mathcal{L}'_-$ that receive the value 1 for any domain assignment and any Scott logic interpretation on V. Here a domain assignment is a pair (D, e) where D is a nonempty set and e is a function $e : D \to V$ satisfying

$$\exists\, a \in D \ e(a) = 1.$$

Given a domain assignment (D, e), a *Scott logic interpretation* I interprets terms and predicate symbols on D, satisfies

$$\inf_i e(a_i) \leq e(I(f)(\bar{a}))$$

for all n-ary function symbols f in the language and all sequences $\bar{a} = a_1, \ldots, a_n$ in D^n, and extends to all formulas as follows:

$$\begin{array}{lcl} I(P\bar{t}) & = & I(P)(I(\bar{t})) \\ I(A \wedge B) & = & \inf(I(A), I(B)) \\ I(A \vee B) & = & \sup(I(A), I(B)) \\ I(A \to B) & = & \begin{cases} 1 & \text{if } I(A) \leq I(B) \\ I(B) & \text{otherwise,} \end{cases} \\ I(\exists\, x A x) & = & \sup\{e(a) \wedge I(Aa) \mid a \in D\} \\ I(\forall\, x A x) & = & \inf\{e(a) \to I(Aa) \mid a \in D\}. \end{array}$$

Note that a Gödel logic interpretation is a Scott logic interpretation where e is the constant 1 function. Then one can show that Scott logics correspond to the logics of linear frames:

THEOREM 33. *[13] For every countable linear frame F there exists a Gödel set V such that*

(8) $S_V = L_F$,

and vice versa: for every countable Gödel set V there exists a countable linear frame F such that (8).

Note that this correspondence is not quite as strong as in the case of Gödel logics where every V is linked to a frame. In the case of Scott logics we could establish this only for countable V. We do not know whether the stronger form also holds, but conjecture it to be the case.

In the same paper [13] it has been shown that there is a natural and faithful translation $(\cdot)^e$ from Scott logics into Gödel logics that makes use of the existence predicate. Roughly, we extend the notion of Gödel logics to the language $\mathcal{L}'$, and then we let the E in the Gödel logic of V correspond to the e in the Scott logic of V. Thus in this setting we first have to extend the notion of a Gödel logics to the language $\mathcal{L}'$, i.e. to E.

A *Gödel existence logic interpretation* I on (V, D) is a Gödel logic interpretation on (V, D) that satisfies the extra requirements that

$$\exists\, a \in D \; I(Ea) = 1,$$

and for all functions h in $\mathcal{L}$, for all $\bar{a} = a_1, \ldots, a_n \in D$,

$$I(\bigwedge_{i \leq n} Ea_i) \leq I(Eh(\bar{a})).$$

The Gödel existence logic G_V^e of a Gödel set V consists of all $\mathcal{L}'$-sentences A such that receive value 1 under all Gödel existence logic interpretations on all domain assignments.

Given these definitions, $(\cdot)^e$ is defined as follows.

$$\big(P(\bar{t})\big)^e = P(\bar{t}) \text{ for atomic } P \text{ and terms } \bar{t},$$

$$(\cdot)^e \text{ commutes with the connectives,}$$

$$(\exists\, xA(x))^e = \exists\, x\big(Ex \wedge (A(x))^e\big),$$

$$(\forall\, xA(x))^e = \forall\, x\big(Ex \rightarrow (A(x))^e\big).$$

Given this translation we then have the following theorem.

THEOREM 34. *[13] For any Gödel set V, $(\cdot)^e$ is a faithful translation of S_V into G_V, i.e. for all sentences A in $\mathcal{L}'_-$:*

$$S_V \models A \;\Leftrightarrow\; G_V^e \models A^e.$$

Furthermore, we have: (in [13] L_F^{cdte} is denoted L_F^{cde})

PROPOSITION 35. *[13] For any frame F, $(\cdot)^e$ is faithful translation of L_F into L_F^{cdte}, i.e. for all sentences A in $\mathcal{L}'_-$:*

$$L_F \Vdash A \;\Leftrightarrow\; L_F^{cdte} \Vdash A^e.$$

PROPOSITION 36. *[13] For every frame F, $L_F^{cdte} = G_V^e$.*

Note the similarity between the different applications of the existence predicate: the translation $(\cdot)^e$ does a similar thing to quantifiers as eSkolemization does.

BIBLIOGRAPHY

[1] Baaz, M. and Iemhoff, R., Skolemization in intuitionistic logic, *Submitted*, 2005.

[2] Baaz, M. and Iemhoff, R., Gentzen calculi for the existence predicate, *Studia Logica*, to appear.

[3] Baaz, M. and Iemhoff, R., On interpolation in existence logics, *Submitted*, 2005.

[4] Baaz, M. and Leitsch, A., On Skolemization and proof complexity, *Fundamenta Informaticae* 20, pp. 353-379,1994.

[5] Beckmann, A. and Preining, N., Linear Kripke Frames and Gödel Logics, *Submitted*, 2005.

[6] Beeson, M., Foundations of Constructive Mathematics, Springer, Berlin, 1985.

[7] Corsi, G., A cut-free calculus for Dummett's LC quantified, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 35, pp. 289-301, 1989.

[8] Corsi, G., A logic characterized by the class of connected models with nested domains, *Studia Logica* 48(1), pp. 15-22, 1989.

[9] Corsi, G., Completeness theorem for Dummett's LC quantified and some of its extensions, *Studia Logica* 51(2), pp. 317-335, 1992.

[10] Dyckhoff, R. and Pinto, L., Proof search in constructive logics, *Sets and proofs*, Lond. Math. Soc. Lect. vol. 258, Cambridge University Press (1999), pp. 53-65.

[11] Fitting, M., Herbrand's theorem for a modal logic, *Logic and foundations of mathematics. Selected contributed papers of the 10th international congress of logic, methodology and philosophy of science*, Kluwer (1999), pp. 219-225.

[12] Herbrand, J., *Recherches sur la théorie de la demonstration*, PhD thesis University of Paris (1930).

[13] Iemhoff, R., A note on linear Kripke models, *Journal of Logic and Computation*, 2005, to appear.

[14] Mints, G.E., An analogue of Hebrand's theorem for the constructive predicate calculus, *Sov. Math. Dokl.* 3 (1962), pp. 1712-1715.

[15] Mints, G.E., Hebrand's theorem for the predicate calculus with equality and function symbols, *Sov. Math.,Dokl.* 7 (1966), pp. 911-914

[16] Mints, G.E., The Skolem method in intuitionistic calculi, *Proc. Steklov Inst. Math.* 121 (1972), pp. 73-109

[17] Mints, G.E., Resolution strategies for the intuitionistic predicate logic, *Constraint Programming. Proceedings of the NATO Advanced Study Institute*, Comput. Syst. Sci. 131, Springer (1994), pp. 289-311.

[18] Mints, G.E., Axiomatization of a Skolem function in intuitionistic logic, *Formalizing the dynamics of information*, Faller, M. (ed.) et al., CSLI Lect. Notes 91, pp. 105-114, 2000.

[19] Preining, N., *Complete Recursive Axiomatizability of Gödel Logics*, PhD-thesis, Technical University Vienna, 2003.

[20] Scott, D.S., Identity and existence in intuitionistic logic, *Applications of sheaves, Proc. Res. Symp. Durham 1977*, Fourman (ed.) et al., Lect. Notes Math. 753, pp. 660-696, 1979.

[21] Shankar, N., Proof search in the Intuitionistic Sequent Calculus, *Automated deduction-CADE-11*, Lect. Notes Comput. Science 607 (1992), pp. 522-536.

[22] Skolem, T., Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen , *Skrifter utgitt av Videnskapsselskapet i Kristiania, I, Mat. Naturv. Kl.* 4 (1920), pp. 1993-2002.

[23] Smoryński, C., Elementary intuitionistic theories, *Journal of Symbolic Logic* 38 (1973), pp. 102-134.

[24] Smoryński, C., On axiomatizing fragments, *Journal of Symbolic Logic* 42 (1877), pp. 530-544.

[25] Takano, M., Another proof of the strong completeness of the intuitionistic fuzzy logic, *Tsukuba J. Math.* 11(1), pp. 101-105, 1987.

[26] Takeuti, G. , *Proof Theory*, Studies in Logic and the Foundations of Mathematics 81, North-Holland Publishing Company, 1987.

[27] Takeuti, G. AND Titani, M., Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, *Journal of Symbolic Logic* 49, pp. 851-866, 1984.

[28] Troelstra, A.S., and van Dalen, D., *Constructivism in Mathematics*, vol. I North-Holland, 1988.

[29] Troelstra, A.S., and Schwichtenberg, H.,, *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press, 1996.

[30] Unterhalt, M., *Kripke-Semantik mit partieller Existenz*, PhD-thesis, University of Münster, 1986.

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