Z

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Z
   Intuitions

Consequences
   Confluence
   Hyper-cofinality

Examples
   Braids
   Self-distributivity
   Normalising and confluent relations
   $\lambda$-calculus
   $\lambda$-calculus with explicit substitutions
   Weakly orthogonal term rewriting systems

Z vs.angle

Non-examples

Conclusions
A rewrite relation \( \rightarrow \) has the Z-property if there is a map \( \bullet \) from objects to objects such that for any step \( a \rightarrow b \) from \( a \) to \( b \) there exists a many-step reduction \( b \rightarrow a \bullet \) from \( b \) to \( a \bullet \) and there exists a many-step reduction \( a \bullet \rightarrow b \bullet \) from \( a \bullet \) to \( b \bullet \).
A rewrite relation $\rightarrow$ has the Z-property
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A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects such that for any step $a \rightarrow b$ from $a$ to $b$ there exists a many-step reduction $b \rightarrow a^\bullet$ from $b$ to $a^\bullet$. 
A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects such that for any step $a \rightarrow b$ from $a$ to $b$, there exists a many-step reduction $b \rightarrow a^\bullet$ from $b$ to $a^\bullet$ and there exists a many-step reduction $a^\bullet \rightarrow b^\bullet$ from $a^\bullet$ to $b^\bullet$. 
\[\exists \bullet : A \rightarrow A, \forall a, b \in A : a \rightarrow b \Rightarrow b \rightarrow a^\bullet, a^\bullet \rightarrow b^\bullet\]
Z intuitions
Z intuitions

\[ a \rightarrow b \]

upperbound on steps
Z intuitions

\[ a \rightarrow b \]

upperbound on steps

monotonic
definition

$\rightarrow$ confluent, if $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
confluence ⇒

- uniqueness of normal forms
- consistent, if some objects not joinable (distinct normal forms)
- decidable, if → is terminating
Theorem

*If a rewrite relation has the Z-property, then it is confluent*

Proof.
Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.
Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.
Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.
Z ⇒ confluence

Theorem

*If a rewrite relation has the Z-property, then it is confluent*

Proof.

\[
\begin{align*}
\quad & a_0 \rightarrow_Z a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_{n+1} \\
\downarrow & \quad \downarrow \quad \downarrow \\
\quad & b_0 \rightarrow a_0 \rightarrow a_1 \\
\end{align*}
\]
Theorem

If a rewrite relation has the Z-property, then it is confluent

Proof.
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If a rewrite relation has the Z-property, then it is confluent

Proof.
Theorem

*If a rewrite relation has the Z-property, then it is confluent*

Proof.
Z ⇒ –•→ strategy is hyper-cofinal

Definition (•-strategy)

\[ a \rightarrow_{\bullet} b \text{ if } a \text{ is not a normal form and } b = a^\bullet \]
$Z \Rightarrow \rightsquigarrow \text{ strategy is hyper-cofinal}$

Hyper: eventually always
$Z \Rightarrow \bullet \rightarrow \text{strategy is hyper-cofinal}$
$Z \Rightarrow \rightarrow \text{ strategy is hyper-cofinal}$
Definition
→•−→ hyper-cofinal, if for any reduction which eventually always contains a →•−→-step, any co-initial reduction can be extended to reach the first
$Z \implies \text{strategy is hyper-cofinal}$

hyper-cofinal $\implies$

- confluent
- (hyper-)normalising
- bullet-fast . . .
Theorem
\[ \text{strategy is hyper-cofinal} \]

Proof.
Theorem

→ is hyper-cofinal

Proof.

$$Z \Rightarrow \quad \text{strategy is hyper-cofinal}$$
Theorem

\[ \Rightarrow \text{ is hyper-cofinal} \]

Proof.

\[ Z \Rightarrow \quad \Rightarrow \]

\[ a_0 \Rightarrow a_{n+1} \Rightarrow a_{n+1} \]

\[ b_0 \Rightarrow a_0 \]
Theorem

→ • is hyper-cofinal

Proof.

\[ Z \Rightarrow \bullet \rightarrow \text{strategy is hyper-cofinal} \]
Theorem

\[ \Rightarrow \quad \text{is hyper-cofinal} \]

Proof.

\[
\begin{array}{c}
Z \\
\Rightarrow \\
\Rightarrow \quad \text{strategy is hyper-cofinal}
\end{array}
\]

\[
\begin{array}{c}
\text{induction}
\end{array}
\]
Theorem

→• is hyper-cofinal

Proof.

\[ Z \Rightarrow \bullet \Rightarrow \text{strategy is hyper-cofinal} \]
Theorem

$\Rightarrow$ is hyper-cofinal

Proof.

$Z \Rightarrow \Rightarrow$ strategy is hyper-cofinal

$\Rightarrow$ induction

$\Rightarrow a_1 \Rightarrow a_{n+1} \Rightarrow a_n \Rightarrow a_{n+1}$

$\Rightarrow b \Rightarrow a_1 \Rightarrow a_n \Rightarrow a_{n+1}$

induction

$\Rightarrow Z \Rightarrow \Rightarrow$ induction $\Rightarrow Z$
Examples
Example: braids

Definition
Braid rewriting: cross adjacent strands, right over left.
Example: braids

Definition

Braid rewriting: cross adjacent strands, right over left.

Example:
Example: braids

Definition
Braid rewriting: cross adjacent strands, right over left.

Example:

Up to topological equivalence:
Example: braids

Theorem

*Braid rewriting has the Z-property, for • full crossing*

Example
Example: braids

Theorem

\textit{Braid rewriting has the Z-property, for \textbullet full crossing}

Proof.
Example: braids

Theorem

*Braid rewriting has the Z-property, for full crossing*

Proof.
Example: braids

Theorem

*Braid rewriting has the Z-property, for • full crossing*

Proof.
Example: braids

Theorem

*Braid rewriting has the Z-property, for • full crossing*

Proof.
Example: self-distributivity

Definition
Self-distributivity, rewrite relation generated by $xyz \rightarrow xz(yz)$
Example: self-distributivity

Definition
Self-distributivity, rewrite relation generated by $xyz \rightarrow xz(yz)$

Some models:
- ACI operations
- take middle of points in space
- substitution

In depth: Braids and Self-distributivity (Dehornoy 2000)
Example: self-distributivity

**Definition**
Self-distributivity, rewrite relation generated by $xyz \rightarrow xz(yz)$

Some models:
- ACI operations
- take middle of points in space
- substitution

In depth: Braids and Self-distributivity (Dehornoy 2000)
Example: self-distributivity

**Theorem**

*Self-distributivity has the Z-property, for • full distribution:*

\[ x^\bullet = x \quad (ts)^\bullet = t^\bullet[s^\bullet] \]

*with \( t[s] \) uniform distribution of \( s \) over \( t \):*

\[ t[x_1:=x_1s, x_2:=x_2s, \ldots] \]
Example: self-distributivity

Theorem

*Self-distributivity has the Z-property, for full distribution:*

\[ x^\bullet = x \quad (ts)^\bullet = t^\bullet[s^\bullet] \]

*with t[s] uniform distribution of s over t:*

\[ t[x_1:=x_1s, x_2:=x_2s, \ldots] \]

Example

- \((xy)^\bullet = x[y] = x[x:=xy] = xy;\)
- \((xyz)^\bullet = (xy)[x:=xz, y:=yz] = xz(yz).\)
Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for full distribution:

\[ x^\bullet = x \quad (ts)^\bullet = t^\bullet[s^\bullet] \]

with \( t[s] \) uniform distribution of \( s \) over \( t \):

\[ t[x_1:=x_1s, x_2:=x_2s, \ldots] \]

Proof.

By induction on \( t \):

\[ \text{(Sequentialisation)} \quad ts \rightarrow t[s] \]

\[ \text{(Substitution)} \quad t[s][r] \rightarrow t[r][s[r]] \]

\[ \text{(Self)} \quad t \rightarrow t^\bullet \]

\[ \text{(Z)} \quad s \rightarrow t^\bullet \rightarrow s^\bullet, \text{ if } t \rightarrow s \]

Example: self-distributivity

**Theorem**

Self-distributivity has the Z-property, for • full distribution:

\[ x^\bullet = x \quad (ts)^\bullet = t^\bullet [s^\bullet] \]

with \( t[s] \) **uniform distribution of** \( s \) **over** \( t \):

\[ t[x_1 := x_1 s, x_2 := x_2 s, \ldots] \]

**Proof.**

By induction on \( t \):

- (Sequentialisation) \( ts \rightarrow t[s] \);
Example: self-distributivity

Theorem
Self-distributivity has the Z-property, for \( \bullet \) full distribution:
\[
x^\bullet = x \quad (ts)^\bullet = t^\bullet[s^\bullet]
\]
with \( t[s] \) uniform distribution of \( s \) over \( t \):
\[
t[x_1:=x_1s, x_2:=x_2s, \ldots]
\]

Proof.
By induction on \( t \):

- (Sequentialisation) \( ts \rightarrow t[s] \);
- (Substitution) \( t[s][r] \rightarrow t[r][s[r]] \).
Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for • full distribution:

\[ x\cdot = x \quad (ts)\cdot = t\cdot[s\cdot] \]

with \( t[s] \) uniform distribution of \( s \) over \( t \):

\[ t[x_1:=x_1s, x_2:=x_2s, \ldots] \]

Proof.

By induction on \( t \):

\begin{itemize}
  \item (Sequentialisation) \( ts \rightarrow t[s] \);
  \item (Substitution) \( t[s][r] \rightarrow t[r][s[r]] \);
  \item (Self) \( t \rightarrow t\cdot \);
\end{itemize}
Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for \( \bullet \) full distribution:
\[
x^\bullet = x \quad (ts)^\bullet = t^\bullet s^\bullet
\]
with \( t[s] \) uniform distribution of \( s \) over \( t \):
\[
t[x_1 := x_1 s, x_2 := x_2 s, \ldots] \]

Proof.

By induction on \( t \):

- (Sequentialisation) \( ts \rightarrow t[s] \);
- (Substitution) \( t[s][r] \rightarrow t[r][s][r] \);
- (Self) \( t \rightarrow t^\bullet \);
- (Z) \( s \rightarrow t^\bullet \rightarrow s^\bullet \), if \( t \rightarrow s \)
Example: normalising and confluent relations

Theorem

*Normalising and confluent relations have the Z-property, for \( \bullet \) the **full** reduction map (map to normal form).*
Theorem

Normalising and confluent relations have the Z-property, for • the full reduction map (map to normal form).

Proof.
If \( a \rightarrow b \), then \( b \rightarrow a^\bullet \rightarrow b^\bullet \) since \( b \) reduces to its normal form \( b^\bullet \) (normalisation) which is the same as the normal form \( a^\bullet \) of \( a \) (confluence).
Example: normalising and confluent relations

**Theorem**

*Normalising and confluent relations have the Z-property, for • the full reduction map (map to normal form).*

**Proof.**

If \( a \rightarrow b \), then \( b \rightarrow a^\bullet \rightarrow b^\bullet \) since \( b \) reduces to its normal form \( b^\bullet \) (normalisation) which is the same as the normal form \( a^\bullet \) of \( a \) (confluence).

**Corollary**

*Z-property for typed \( \lambda \)-calculi (by confluence and termination)*
Example: normalising and confluent relations

Theorem
Normalising and confluent relations have the Z-property, for • the full reduction map (map to normal form).

Proof.
If $a \rightarrow b$, then $b \rightarrow a^* \rightarrow b^*$ since $b$ reduces to its normal form $b^*$ (normalisation) which is the same as the normal form $a^*$ of $a$ (confluence).

Corollary
Z-property for typed λ-calculi (by confluence and termination)
Here reverse: use Z-property to establish meta-theory
Example: $\lambda$-calculus

Theorem

$(\lambda x. M)N \to M[x:=N]$ has the Z-property, for • full development contracting all redexes present:

\[
\begin{align*}
    x^\bullet & = x \\
    (\lambda x. M)^\bullet & = \lambda x. M^\bullet \\
    (MN)^\bullet & = M'[x:=N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x. M' \\
    & = M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]
Example: $\lambda$-calculus

**Theorem**

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for • full development contracting all redexes present:

$x^* = x$

$(\lambda x. M)^* = \lambda x. M^*$

$(MN)^* = M'[x:=N^*]$ if $M$ is an abstraction, $M^* = \lambda x. M'$

$= M^* N^*$ otherwise

**Example**

- $I^* = I; (I = \lambda x.x)$
- $(I(II))^* = I, (III)^* = II$;
- $((\lambda xy.x)zw)^* = (\lambda y.z)w$;
- $((\lambda xy.lyx)zl)^* = (\lambda y.yz)l$;
Example: \( \lambda \)-calculus

Theorem

\((\lambda x. M)N \rightarrow M[x := N]\) has the Z-property, for • full development contracting all redexes present:

\[
\begin{align*}
\mathcal{P}^\bullet &= \mathcal{P} \\
(\lambda x. M)^\bullet &= \lambda x. M^\bullet \\
(MN)^\bullet &= M'[x := N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x. M' \\
&= M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]

Proof.

By induction on \( M \):

- (Substitution) \( M[y := P][x := N] = M[x := N][y := P[x := N]] \).
Example: $\lambda$-calculus

Theorem

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for • full development contracting all redexes present:

$x^\bullet = x$

$(\lambda x. M)^\bullet = \lambda x. M^\bullet$

$(MN)^\bullet = M'[x:=N^\bullet]$ if $M$ is an abstraction, $M^\bullet = \lambda x. M'$

$= M^\bullet N^\bullet$ otherwise

Proof.

By induction on $M$:

▶ (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;

▶ (Self) $M \rightarrow M^\bullet$;
Example: $\lambda$-calculus

Theorem

$$(\lambda x. M)N \rightarrow M[x:=N] \text{ has the Z-property, for } \bullet \text{ full development contracting all redexes present:}$$

\[
\begin{align*}
  x^\bullet &= x \\
  (\lambda x. M)^\bullet &= \lambda x. M^\bullet \\
  (MN)^\bullet &= M'[x:=N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x. M'
  &= M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]

Proof.

By induction on $M$:

- (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
- (Self) $M \rightarrow M^\bullet$;
- (Rhs) $M^\bullet[x:=N^\bullet] \rightarrow M[x:=N]^\bullet$; and
Example: λ-calculus

Theorem

\((\lambda x. M) N \rightarrow M[x:=N]\) has the Z-property, for • full development contracting all redexes present:

\[
\begin{align*}
  x^\bullet &= x \\
  (\lambda x. M)^\bullet &= \lambda x. M^\bullet \\
  (MN)^\bullet &= M'[x:=N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x. M' \\
  &= M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]

Proof.

By induction on \(M\):

▶ (Substitution) \(M[y:=P][x:=N] = M[x:=N][y:=P][x:=N]\);

▶ (Self) \(M \rightarrow M^\bullet\);

▶ (Rhs) \(M^\bullet[x:=N^\bullet] \rightarrow M[x:=N]^\bullet\); and

▶ (Z) \(M \rightarrow N \Rightarrow N \rightarrow M^\bullet \rightarrow N^\bullet\).
Example: $\lambda$-calculus

**Theorem**

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for full development contracting all redexes present:

$x^\bullet = x$

$(\lambda x. M)^\bullet = \lambda x. M^\bullet$

$(MN)^\bullet = M'[x:=N^\bullet]$ if $M$ is an abstraction, $M^\bullet = \lambda x. M'$

$= M^\bullet N^\bullet$ otherwise

**Proof.**

By induction on $M$:

- (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
- (Self) $M \rightarrow M^\bullet$;
- (Rhs) $M^\bullet[x:=N^\bullet] \rightarrow M[x:=N]^\bullet$; and
- (Z) $M \rightarrow N \Rightarrow N \rightarrow M^\bullet \rightarrow N^\bullet$.

Same method works for all orthogonal first/higher-order TRSs
Example: λ-calculus

Theorem

\((\lambda x. M) N \rightarrow M[x:=N]\) has the Z-property, for • full super-development contracting all redexes present or upward created:

\[\begin{align*}
\dot{x} & \equiv x \\
\dot{(\lambda x. M)} & \equiv \lambda x.\dot{M} \\
\dot{(MN)} & \equiv M'[x:=\dot{N}]
\end{align*}\]

if \(M\) is a term, \(\dot{M} = \lambda x.\dot{M}'\)

\[\begin{align*}
\dot{M} & \equiv \dot{M} \cdot \dot{N}
\end{align*}\]

otherwise
Example: $\lambda$-calculus

Theorem

$$(\lambda x. M)N \rightarrow M[x := N]$$ has the Z-property, for full super-development contracting all redexes present or upward created:

$$x^\bullet = x$$

$$(\lambda x. M)^\bullet = \lambda x. M^\bullet$$

$$(MN)^\bullet = M'[x := N^\bullet] \quad \text{if } M \text{ is a term, } M^\bullet = \lambda x. M'$$

$$= M^\bullet N^\bullet \quad \text{otherwise}$$

Example

- $I^\bullet = I; (I = \lambda x.x)$
- $(I(II))^\bullet = I, (III)^\bullet = I$
- $((\lambda xy.x)zw)^\bullet = z$
- $((\lambda xy. lyx)zl)^\bullet = lz$
Example: $\lambda$-calculus

**Theorem**

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for • full super-development contracting all redexes present or upward created:

$x^\bullet = x$

$(\lambda x. M)^\bullet = \lambda x. M^\bullet$

$(MN)^\bullet = M'[x:=N^\bullet]$ if $M$ is a term, $M^\bullet = \lambda x. M'$

$= M^\bullet N^\bullet$ otherwise

**Proof.**

Same (‘an abstraction’⇨‘a term’) proof by induction on $M$:

- (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$
Example: \(\lambda\)-calculus

Theorem
\[(\lambda x. M) N \rightarrow M[x:=N] \text{ has the Z-property, for } \bullet \text{ full super-development contracting all redexes present or upward created:}
\]
\[
x^\bullet = x
\]
\[
(\lambda x. M)^\bullet = \lambda x. M^\bullet
\]
\[
(MN)^\bullet = M'[x:=N^\bullet] \text{ if } M \text{ is a term, } M^\bullet = \lambda x. M'
\]
\[
= M^\bullet N^\bullet \text{ otherwise}
\]

Proof.
Same (‘an abstraction’ \(\rightarrow\) ‘a term’) proof by induction on \(M\):

\(\blacktriangleright\) (Substitution) \(M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]\);

\(\blacktriangleright\) (Self) \(M \rightarrow M^\bullet\);
Example: $\lambda$-calculus

**Theorem**

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for $\bullet$ full super-development contracting all redexes present or upward created:

- $x^\bullet = x$
- $(\lambda x. M)^\bullet = \lambda x. M^\bullet$
- $(MN)^\bullet = M'[x:=N^\bullet]$ if $M$ is a term, $M^\bullet = \lambda x. M'$
- $= M^\bullet N^\bullet$ otherwise

**Proof.**

Same (‘an abstraction’$\mapsto$ ‘a term’) proof by induction on $M$:

- (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
- (Self) $M \rightarrow M^\bullet$;
- (Rhs) $M^\bullet[x:=N^\bullet] \rightarrow M[x:=N]^\bullet$; and
Example: $\lambda$-calculus

Theorem

$(\lambda x. M)N \rightarrow M[x:=N]$ has the Z-property, for full super-development contracting all redexes present or upward created:

$x^* = x$

$(\lambda x. M)^* = \lambda x. M^*$

$(MN)^* = M'[x:=N^*]$ if $M$ is a term, $M^* = \lambda x. M'$

$= M^* N^*$ otherwise

Proof.

Same (‘an abstraction’ $\mapsto$ ‘a term’) proof by induction on $M$:

1. (Substitution) $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
2. (Self) $M \rightarrow M^*$;
3. (Rhs) $M^*[x:=N^*] \rightarrow M[x:=N]^*$; and
4. (Z) $M \rightarrow N \Rightarrow N \rightarrow M^* \rightarrow N^*$. 

$\blacksquare$
Example: $\lambda$-calculus

Theorem

$(\lambda x. M) N \rightarrow M[x:=N]$ has the $Z$-property, for • full super-development contracting all redexes present or upward created:

\[
\begin{align*}
\bullet^* &= \bullet \\
(\lambda x. M)^* &= \lambda x. M^* \\
(MN)^* &= M'[x:=N^*] \quad \text{if } M \text{ is a term, } M^* = \lambda x. M' \\
&= M^* N^* \quad \text{otherwise}
\end{align*}
\]

Proof.

Same (‘an abstraction’ $\mapsto$ ‘a term’) proof by induction on $M$:

- (Substitution) \quad $M[y:=P][x:=N] = M[x:=N][y:=P[x:=N]]$;
- (Self) \quad $M \rightarrow M^*$;
- (Rhs) \quad $M^*[x:=N^*] \rightarrow M[x:=N]^*$; and
- (Z) \quad $M \rightarrow N \Rightarrow N \rightarrow M^* \rightarrow N^*$.

Moral: possibly more than one witnessing map for $Z$-property
Example: \( \lambda \)-calculus with explicit substitutions

**Theorem**

\( \lambda \sigma \) has the Z-property, for • the map composed of first \( \sigma \)-normalisation (\( \triangleright \)), then a Beta-full development (\( \rightarrow \rightarrow \))

Works for other explicit substitution/proof calculi as well.
Example: $\lambda$-calculus with explicit substitutions

**Theorem**

$\lambda \sigma$ has the Z-property, for • the map composed of first $\sigma$-normalisation ($\triangleright$), then a Beta-full development ($\rightarrow\rightarrow$)

**Proof.**

\[
\Gamma \quad t \rightarrow s \\
\triangleleft \triangleleft \quad \tilde{t} = \tilde{s} \\
\triangleleft \triangleleft \quad t^* = s^* \\
\]

\[
E \quad \Gamma \quad t \rightarrow s \\
\triangleleft \triangleleft \quad t' \rightarrow \tilde{s} \\
\triangleleft \triangleleft \quad E \quad \Delta \\
\]

Works for other explicit substitution/proof calculi as well.
Example: $\lambda$-calculus with explicit substitutions

Theorem

$\lambda\sigma$ has the Z-property, for $\bullet$ the map composed of first $\sigma$-normalisation ($\triangleright$), then a Beta-full development ($\rightarrow\rightarrow$)

Proof.

Works for other explicit substitution/proof calculi as well.
Example: weakly orthogonal term rewriting systems

Definition
Rewrite system is weakly orthogonal, if only trivial critical pairs.
Example: weakly orthogonal term rewriting systems

Definition
Rewrite system is weakly orthogonal, if only trivial critical pairs.

Example

- $\lambda$-calculus with $\beta$ and $\eta$ : $\lambda x. Mx \to M$, if $x \notin M$;
- predecessor/successor $S(P(x))) \to x$, $P(S(x)) \to x$;
- parallel-or.
Example: weakly orthogonal term rewriting systems

Theorem

Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for • full inside-out development
Example: weakly orthogonal term rewriting systems

Theorem
Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for • full inside-out development
Example: weakly orthogonal term rewriting systems

Theorem
Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for • full inside-out development

Proof.

\[ \begin{align*}
  c(x) & \rightarrow x \\
  f(f(x)) & \rightarrow f(x) \\
  g(f(f(f(x)))) & \rightarrow g(f(f(x)))
\end{align*} \]

Then \( g(f(c(f(c(f(x)))))) \rightarrow g(f(f(f(f(x))))) \) gives Z:
\[ g(f(c(f(c(f(x)))))) = g(f(f(x))) = g(f(f(f(f(x))))) \]  
\[ \square \]
Example: weakly orthogonal term rewriting systems

Theorem

*Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for full inside-out development*

Proof.

\[ c(x) \rightarrow x \]
\[ f(f(x)) \rightarrow f(x) \]
\[ g(f(f(f(x)))) \rightarrow g(f(f(x))) \]

Then \( g(f(f(c(f(f(x)))))) \rightarrow g(f(f(f(f(x))))) \) gives Z:
\[ g(f(f(c(f(f(x)))))) \circ = g(f(f(x))) = g(f(f(f(f(x))))) \circ \]

Outside-in not monotonic: not \( g(f(f(x))) \rightarrow g(f(f(f(x)))) \)!
Z vs. angle

- Dehornoy:
  Z-property of \( \rightarrow \) for \( \bullet \);

- Takahashi:
  angle (⟨⟩) property of \( \rightarrow \) for \( \bullet \):
  \[ \exists \circ \rightarrow, \rightarrow \subseteq \circ \rightarrow \subseteq \rightarrow \]

\[ \begin{align*}
  &a \\
  &\circ \\
  &\rightarrow \\
  &\bullet \\
  &a
\end{align*} \]
Z vs. angle

- Dehornoy:
  \( Z \)-property of \( \rightarrow \) for \( \bullet \);

- Takahashi:
  angle (\( \langle \rangle \)) property of \( \rightarrow \) for \( \bullet \): \( \exists \rightarrow, \rightarrow \subseteq \rightarrow \subseteq \rightarrow \)

\( \rightarrow \rightarrow \) steps are divisors of \( \rightarrow \rightarrow \)
Z ⇔ angle

Theorem

for any map \( \bullet \), \( Z \iff \langle \)

Proof. \qed
Theorem

for any map \( \textbullet \), \( Z \iff \langle \)

Proof.

(If)

\[ a \rightarrow b \]
Theorem

for any map $\bullet$, $Z \Leftrightarrow \langle$

Proof.

(Iff)
Z \iff \text{angle}

**Theorem**

*for any map \bullet, Z \iff \langle*

**Proof.**

(If)

\[ a \rightarrow b \]

\[ a^\bullet \rightarrow b^\bullet \]
Theorem

for any map •, Z ⇔ ⟨

Proof.

(Iff)
Theorem
for any map \( \bullet \), \( Z \Leftrightarrow \langle \)

Proof.
(only if) Def. \( a \rightarrow b \) if \( b \) between \( a \) and \( a^\bullet \), i.e. \( a \rightarrow b \rightarrow a^\bullet \):

\[ \begin{align*}
\text{Suppose } a \rightarrow b. & \\
\quad & \\
\quad & \\
\end{align*} \]

\[ \begin{align*}
\text{Suppose } a \rightarrow b. & \\
\quad & \\
\quad & \\
\quad & \\
\quad & \\
\text{by definition of } \rightarrow. & \\
\text{by Z} & \\
\text{by definition of } \rightarrow. & \\
\end{align*} \]
Non-examples
Some properties of $\bullet$s

- if $a \rightarrow b$ then $a^\bullet \rightarrow b^\bullet$;
Some properties of $\bullet$s

- if $a \rightarrow b$ then $a^\bullet \rightarrow b^\bullet$;  
- $\rightarrow$ has Z-property iff $\rightarrow^=\$ has IZ-property;
Some properties of ●s

- if $a \rightarrow b$ then $a\overset{●}{\rightarrow} b\overset{●}{\rightarrow}$;
- $\rightarrow$ has Z-property iff $\overset{●}{\rightarrow}$ has IZ-property;
- $\overset{●}{1} \circ \overset{●}{2}$ has Z, if $\overset{●}{i}$ do.
Some properties of $\bullet$s

- if $a \rightarrow b$ then $a^\bullet \rightarrow b^\bullet$;
- $\rightarrow$ has Z-property iff $\rightarrow^{=} = \text{has IZ-property}$;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- slower order: $\bullet_1 \leq \bullet_2$, if $\forall a$, $a^{\bullet_1} \rightarrow a^{\bullet_2}$;
Some properties of ⋄s

- if $a \to b$ then $a^\bullet \to b^\bullet$;
- $\to$ has Z-property iff $\to^=\equiv$ has IZ-property;
- $\bullet_1 \circ \bullet_2$ has Z, if $\bullet_i$ do.
- slower order: $\bullet_1 \leq \bullet_2$, if $\forall a, a^{\bullet_1} \to a^{\bullet_2}$;
- $\bullet_i \leq \bullet_1 \circ \bullet_2$;
Some properties of $\triangleright$s

- if $a \rightarrow b$ then $a \triangleright \rightarrow b \triangleright$;
- $\rightarrow$ has $Z$-property iff $\xrightarrow{=}$ has $IZ$-property;
- $\triangleright_1 \circ \triangleright_2$ has $Z$, if $\triangleright_i$ do.
- *slower* order: $\triangleright_1 \leq \triangleright_2$, if $\forall a, a \triangleright_1 \rightarrow a \triangleright_2$;
- $\triangleright_i \leq \triangleright_1 \circ \triangleright_2$;
- no slowest/initially slow/fastest/finally fast;

For normalising/finite systems: go to ‘normal’ form fast.
Some properties of ▶

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Used to get ideas about (confluent) systems which do not have Z
$\mathbb{Z}$ does not have $\mathbb{Z}$
$\mathbb{Z}$ does not have $\mathbb{Z}$

for given integer, no upperbound on steps from it
\( \mathbb{Z} \) does not have \( \mathbb{Z} \)

not finitely branching, no finite TRS

for given integer, no upperbound on steps from it
\hat{Z} \text{ does not imply } Z
\( \hat{Z} \) does not imply \( Z \)

\[
\text{finitely branching, finite TRS}
\]

\[
\begin{align*}
n(x) &\rightarrow p(x) \quad n(1) \rightarrow 0 \quad 0 \rightarrow p(1) \\
n(s(x)) &\rightarrow n(x) \\
p(x) &\rightarrow p(s(x))
\end{align*}
\]
\[ \hat{Z} \text{ does not imply } Z \]

finitely branching, finite TRS

not monotonic (e.g. for \(-3\))

\[
\begin{align*}
n(x) & \rightarrow p(x) \\
n(1) & \rightarrow 0 \\
0 & \rightarrow p(1) \\
n(s(x)) & \rightarrow n(x) \\
p(x) & \rightarrow p(s(x))
\end{align*}
\]
$\mathbb{Z}^p$ does have $\mathbb{Z}$

\[\rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow\]
\(\mathbb{Z}^p\) does have \(Z\)

finitely branching, finite TRS, no transitivity

\[
\begin{array}{ccccccc}
\rightarrow & -2 & \rightarrow & -1 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow \\
\end{array}
\]
$\mathbb{Z}^+$ does have $\mathbb{Z}$

finitely branching, finite TRS, no transitivity

\[ \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \]

$\mathbb{Z}$ trivial ($i^* = i + 1$)
Z does have Z

finitely branching, finite TRS, no transitivity

\[ \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \]

Z trivial \((i^\bullet = i + 1)\)

Examples show:
- confluent $\not\Rightarrow$ Z
- transitivity might be harmful
Conclusions

- Surprise: $Z \iff \text{angle};$
Conclusions

- Surprise: $Z \Leftrightarrow \text{angle}$;
- Claim: gives simplest confluence proofs;
- Conjecture: $\beta$ with restricted $\eta$-expansion does not have $Z$;
- Problem: characterise systems having $Z$-property;
- Puzzle: is $Z$ a modular property of TRSs?;
- Further work: Garside categories $\Leftrightarrow$ residual systems.
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