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Theoretical Philosophy
Universiteit Utrecht
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this month at LIX

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Z

Z for $\lambda$-calculi

Z or not
A rewrite relation $\rightarrow$ has the Z-property
A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects.
A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects such that for any step from $a$ to $b$. 
A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects such that for any step from $a$ to $b$ there is a reduction from $b$ to $a^\bullet$. 
A rewrite relation $\rightarrow$ has the Z-property if there is a map $\bullet$ from objects to objects such that for any step from $a$ to $b$ there is a reduction from $b$ to $a^\bullet$ and there is a reduction from $a^\bullet$ to $b^\bullet$. 
\( \exists \bullet : A \rightarrow A, \forall a, b \in A : a \rightarrow b \Rightarrow b \rightarrow a\bullet, a\bullet \rightarrow b\bullet \)
This talk: (short) history, interest, and (non-)examples
self-distributivity: \[ xyz \rightarrow xz(yz) \]

**Theorem**

*self-distributivity has the Z-property*
self-distributivity: $xyz \rightarrow xz(yz)$

**Theorem**

*Self-distributivity has the Z-property*

**Map**

\[
\begin{align*}
    x^* &= x \\
    (ts)^* &= t^*[x_1:=x_1s^*, x_2:=x_2s^*, \ldots]
\end{align*}
\]
self-distributivity: $xyz \rightarrow xz(yz)$

**Theorem**

*Self-distributivity has the Z-property*

Map

$$x^* = x$$

$$(ts)^* = t^*[x_1:=x_1^*, x_2:=x_2^*, \ldots]$$

**Example**

$$(xy)^* = xy$$

Proof. This works: Braids and Self-distributivity (Dehornoy 2000)
self-distributivity: $xyz \rightarrow xz(yz)$

**Theorem**

*self-distributivity has the Z-property*

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    (xy)^\bullet &= xy \\
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**Example**

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\begin{align*}
(xy) \bullet &= xy \\
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**Proof.**

This works: Braids and Self-distributivity (Dehornoy 2000)
Theorem

Every normalising and confluent rewrite relation has the Z-property
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Let $\bullet$ map every object to its normal form
(exists by normalisation, unique by confluence)
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Let $\bullet$ map every object to its normal form
(exists by normalisation, unique by confluence)

Proof.
If $a \rightarrow b$, then $b \rightarrow a^\bullet \rightarrow b^\bullet$ since $b$ reduces to its normal form $b^\bullet$
which is the same as the normal form $a^\bullet$ of $a$. $\square$
Theorem

Every normalising and confluent rewrite relation has the Z-property

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Corollary

$Z$-property for $\beta$-reduction in typed $\lambda$-calculi by using meta-theory
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Every normalising and confluent rewrite relation has the Z-property

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Corollary
$Z$-property for $\beta$-reduction in typed $\lambda$-calculi by using meta-theory

Here reverse: $Z$-property to establish meta-theory
Theorem

If a rewrite relation has the Z-property then it is confluent
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Proof.

\[ a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \cdots \rightarrow a_{n+1} \]
Theorem

*If a rewrite relation has the Z-property then it is confluent*

Proof.

\[
\begin{array}{c}
a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_{n+1} \\
\end{array}
\]
If a rewrite relation has the Z-property then it is confluent

Proof.
Theorem

*If a rewrite relation has the Z-property then it is confluent*

Proof.
Theorem

If a rewrite relation has the Z-property then it is confluent

Proof.
Theorem

If a rewrite relation has the $Z$-property then it is confluent

Proof.
Z ⇒ hyper-cofinal

Definition (●-strategy)

\( a \rightarrow_{\bullet} b \) if \( a \) is not a normal form and \( b = a^\bullet \)
Hyper-cofinality of $\rightarrow$:
for any reduction which eventually always contains $\rightarrow$-step
any co-initial reduction can be extended to reach the first
Theorem

\[ \Rightarrow \text{is hyper-cofinal} \]

Proof.
Theorem
\[ \bullet \rightarrow \text{is hyper-cofinal} \]

Proof.

Summary: \[ \bullet \rightarrow \text{confluent, (hyper-)normalising, bullet-fast,} \]
\[ \beta \text{ has } Z \]

**Theorem**

\[(\lambda x. M)N \rightarrow M[x:=N] \text{ has the } Z\text{-property for } \lambda\text{-calculus}\]
\( \beta \) has Z

**Theorem**

\[(\lambda x. M)N \rightarrow M[x:=N] \text{ has the } Z\text{-property for } \lambda\text{-calculus} \]

**Proof.**

**Full-development** map (contract all redexes present)

\[
\begin{align*}
x^\bullet &= x \\
(\lambda x. M)^\bullet &= \lambda x. M^\bullet \\
(MN)^\bullet &= M'[x:=N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x. M' \\
&= M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]

**Example**

\[
\begin{align*}
\rightarrow I^\bullet = I; \quad (I = \lambda x. x) \\
\rightarrow I(II)^\bullet = I, \quad III^\bullet = II; \\
\rightarrow (\lambda xy. x)zw^\bullet = (\lambda y.z)w; \\
\rightarrow ((\lambda xy. lyx)zl)^\bullet = (\lambda y.yz)l;
\end{align*}
\]
Theorem

\((\lambda x. M) N \rightarrow M[x:=N]\) has the Z-property for \(\lambda\)-calculus

Proof.

Full-development map (contract all redexes present)

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\]

(Self) \(M \rightarrow M^\bullet\);

(Rhs) \(M^\bullet[x:=N^\bullet] \rightarrow M[x:=N]^\bullet\); and

(Z) \(M \rightarrow N \Rightarrow N \rightarrow M^\bullet \rightarrow N^\bullet\).

each by induction and cases on \(M\).
\( \beta \) has Z

Theorem

\((\lambda x. M)N \rightarrow M[x:=N]\) has the Z-property for \(\lambda\)-calculus

Proof.

Full-superdevelopment map (redexes present or upward-created)

\[
\begin{align*}
\bar{x}^* &= x \\
(\lambda x. M)^* &= \lambda x. M^* \\
(MN)^* &= M'[x:=N^*] \quad \text{if } M \text{ is a term, } M^* = \lambda x. M' \\
&= M^* N^* \quad \text{otherwise}
\end{align*}
\]

Example

\[
\begin{align*}
\&\quad I^* = I; \quad (I = \lambda x.x) \\
\&\quad I(I) = I, \quad III^* = I; \\
\&\quad (\lambda xy.x)zw^* = z; \\
\&\quad ((\lambda xy.lyx)z)l^* = lz
\end{align*}
\]
\[ \beta \text{ has } Z \]

**Theorem**
\[(\lambda x. M)N \rightarrow M[x:=N] \text{ has the } Z\text{-property for } \lambda\text{-calculus}\]

**Proof.**

Full-superdevelopment map (redexes present or upward-created)
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\begin{align*}
x^\bullet &= x \\
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 &= M^\bullet N^\bullet \text{ otherwise}
\end{align*}
\]

Replace ‘is an abstraction’ by ‘is a term’ in development proof.  \[\square\]
\( \beta \) has Z

Theorem
\[(\lambda x. M)N \rightarrow M[x:=N] \text{ has the Z-property for } \lambda\text{-calculus}\]

Proof.

Full-superdevelopment map (redexes present or upward-created)

\[
\begin{align*}
  x^\bullet &= x \\
  (\lambda x. M)^\bullet &= \lambda x. M^\bullet \\
  (MN)^\bullet &= M'[\!x:=N^\bullet] \quad \text{if } M \text{ is a term, } M^\bullet = \lambda x. M' \\
  &= M^\bullet N^\bullet \quad \text{otherwise}
\end{align*}
\]

Replace ‘is an abstraction’ by ‘is a term’ in development proof.

Moral: possibly more than one witnessing map for Z-property
Comparison

- Dehornoy:
  Z-property of $\rightarrow$ for $\bullet$;

- Tait–Martin Löf:
  $\rightarrow \subseteq \diamondsuit \subseteq \twoheadrightarrow$ and diamond (◊) property of $\diamondsuit \rightarrow$;

- Takahashi:
  $\rightarrow \subseteq \triangleleft \subseteq \twoheadrightarrow$ and angle (⟨⟩) property of $\triangleleft \rightarrow$ for $\bullet$. 
Comparison

- Dehornoy:
  Z-property of $\rightarrow$ for $\bullet$;

- Tait–Martin Löf:
  $\rightarrow \subseteq \circlearrowleft \subseteq \circlearrowright$ and diamond (◊) property of $\circlearrowright$;

- Takahashi:
  $\rightarrow \subseteq \circlearrowleft \subseteq \circlearrowright$ and angle (⟨⟩) property of $\circlearrowright$ for $\bullet$.

Mnemonics: $\rightarrow \bullet$ is full $\circlearrowright$
Comparison

- **Dehornoy:**
  Z-property of $\rightarrow$ for $\bullet$;

- **Tait–Martin L¨of:**
  $\rightarrow \subseteq \leftrightarrow \subseteq \rightarrow$ and diamond ($\diamond$) property of $\leftrightarrow$;

- **Takahashi:**
  $\rightarrow \subseteq \leftrightarrow \subseteq \rightarrow$ and angle ($\angle$) property of $\leftrightarrow$ for $\bullet$.

How do $Z$, $\diamond$, $\angle$ relate?
Angle property
Theorem

for any map \( \bullet \), \( Z \iff \text{both } \implies \circ \implies \subseteq \implies \rangle \text{ and } \langle \)

Proof.
Theorem

for any map •, Z ⇔ both → ⊆ −→ ⊆ ↠ and ⟨

Proof.

(Iff)

\[ a \rightarrow b \]
Theorem

for any map \( \bullet \), \( Z \iff both \to \subseteq \implies \subseteq \implies \) and \( \langle \)

Proof.

(Iff)
Theorem
for any map \( \bullet \), \( Z \iff both \to \subseteq \circ \subseteq \to \) and \( \langle \)

Proof.

(Iff)

\[
\begin{array}{c}
a \\
\downarrow \quad \langle \\
\downarrow \\
a^* \\
\end{array}
\quad \begin{array}{c}
\rightarrow \quad \rightarrow \\
\rightarrow \subseteq \\
\rightarrow \subseteq \\
\rightarrow \to \\
b^* \\
\end{array}
\]
Theorem
for any map \( \bullet \), \( Z \Leftrightarrow \) both \( \rightarrow \subseteq \leftarrow \subseteq \rightarrow \) and \( \langle \) 

Proof.
(Iff)
Theorem
for any map \( \bullet \), \( Z \iff \text{both} \to \subseteq \iff \subseteq \to \) and \( \langle \)

Proof.
(only if) Def. \( a \iff b \) if \( b \) between \( a \) and \( a^\bullet \), i.e. \( a \to b \to a^\bullet \):

\[
\begin{align*}
\text{Suppose } a \iff b. \\
\quad & a \to b \implies b \to a^\bullet \implies \to \subseteq \iff. \\
\quad & a \iff b \implies a \to b \implies \iff \subseteq \to. \\
\end{align*}
\]

\[\square\]
Theorem

\( \lambda \sigma \) has \( Z \) property
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Proof.
Map: first \( \sigma \)-normalise (\( \uparrow \)) then Beta-full development (\( \rightarrow \rightarrow \))
Theorem
\( \lambda \sigma \) has \( Z \) property

Proof.
Map: first \( \sigma \)-normalise (\( \triangleright \)) then \( Beta \)-full development (\( \rightarrow \rightarrow \))

\( \Delta \): angle property of \( \rightarrow \rightarrow \)

\( E \): \( Beta \) commutes with \( \sigma \)-normalisation

\( \Gamma \): \( \sigma \) is terminating and confluent
\( \lambda \beta \eta \) has \( Z \) property

**Theorem**

*Weakly orthogonal rewrite system* \( \Rightarrow \) *\( Z \) property*

**Proof.**

Map:
Contract maximal set of non-overlapping redexes *inside-out*

**Example**

\[
\begin{align*}
c(x) &\rightarrow x \\
f(f(x)) &\rightarrow f(x) \\
g(f(f(f(x)))) &\rightarrow g(f(f(x)))
\end{align*}
\]
\[\lambda \beta \eta \text{ has } Z \text{ property}\]

**Theorem**

*Weakly orthogonal rewrite system \( \Rightarrow \) Z property*

**Proof.**

Map:
Contract maximal set of non-overlapping redexes *inside-out*

**Example**

\[
\begin{align*}
c(x) & \rightarrow x \\
n(f(f(x))) & \rightarrow f(x) \\
g(f(f(f(x)))) & \rightarrow g(f(f(x))) \\
g(f(f(c(f(f(x)))))) & = g(f(f(x))) = g(f(f(f(f(f(x))))))
\end{align*}
\]
\(\lambda \beta \eta\) has Z property

**Theorem**

*Weakly orthogonal rewrite system \(\Rightarrow\) Z property*

**Proof.**

Map:

Contract maximal set of non-overlapping redexes inside-out

**Example**

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\begin{align*}
c(x) & \rightarrow x \\
f(f(x)) & \rightarrow f(x) \\
g(f(f(f(x)))) & \rightarrow g(f(f(x)))
\end{align*}
\]

Outside-in (Takahashi) does not give Z (in general)! 
\[g(f(f(c(f(f(x)))))) \rightarrow g(f(f(f(f(x)))))\] holds...
\( \lambda \beta \eta \) has Z property

**Theorem**

*Weakly orthogonal rewrite system \( \Rightarrow \) Z property*

**Proof.**
Map:
Contract maximal set of non-overlapping redexes **inside-out**

**Example**

\[
\begin{align*}
  c(x) & \rightarrow x \\
  f(f(x)) & \rightarrow f(x) \\
  g(f(f(f(x)))) & \rightarrow g(f(f(x)))
\end{align*}
\]

**Outside-in** *(Takahashi) does not give Z (in general)!*

\[ \ldots \text{not} \ g(f(f(x))) \rightarrow g(f(f(f(x))))! \]
Some more consequences of Z

- if $a \to b$ then $a^\bullet \to b^\bullet$ (monotonicity)
- $\to$ has Z-property iff $\to^=\equiv$ has (IZ-property)
- If $\bullet_1, \bullet_2$ have the Z-property for $\to$, so does their composition $\bullet_1 \circ \bullet_2$. Moreover, $a^{\bullet i} \to (a^{\bullet 2})^{\bullet 1}$

May be used to get ideas about systems which do not have Z
Easy to turn into a finite term rewriting system
Conclusions

- Surprising outsider (Dehornoy) input: simple yet not known
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- Conjecture: $\beta$ with restricted $\eta$-expansion does not have $Z$
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- Surprising outsider (Dehornoy) input: simple yet not known
- Conjecture: $\beta$ with restricted $\eta$-expansion does not have Z
- Problem: characterize systems having Z-property