LMNLP 2012. Take-home assignment 2

1

Background: Wadler, A Taste of Linear Logic.

The ! connective establishes the relation between the multiplicative and additive conjunctions:

$$\langle\, !(A \mathbin{\&} B)\,\rangle \vdash\, !A \otimes\, !B \qquad \langle\, !A \otimes\, !B\,\rangle \vdash\, !(A \mathbin{\&} B)$$

EXERCISE prove these equivalences.

Below a translation embedding intuitionistic logic into linear logic.

$$\begin{array}{rcl} A \to B & = & !A \multimap B \\ A \times B & = & A \mathbin{\&} B \\ A + B & = & !A \oplus\, !B \end{array}$$

EXERCISE Show that the intuitionistic rules for $\times$, $+$ can be derived from the corresponding rules of linear logic, together with the ! intro/elim rules.

2

The following context-free grammar recognizes well-bracketed strings over two pairs of parentheses, $a, \overline{a}$ and $b, \overline{b}$.

$$G : \quad \begin{array}{l} S \longrightarrow a\ S\ \overline{a}\ S \mid b\ S\ \overline{b}\ S \\ S \longrightarrow \epsilon \end{array}$$

In this exercise we consider non-empty strings recognized by G, i.e. $L(G) \setminus \{\epsilon\}$

1. Lexicalize G. Get rid of the epsilon rule. Turn the 'open bracket' symbols (a, b) into the lexical anchors of the new rules. Call the lexicalized grammar G'.
2. Give a derivation in G' for the string $ab\overline{b}\overline{a}a\overline{a}$.[1]
3. Compute the dependency structure induced by that derivation (see Kuhlmann, §6.1)

3

In this exercise, we consider the lexicalized grammar G' you obtained in §2.

1. Turn G' into a typelogical grammar for **NL**. Recall that in **NL**, linear order and phrase structure (bracketing) are respected.

[1] This example replaces the one we did together in class.

2. Provide a bracketing for the string $ab\overline{b}\overline{a}a\overline{a}$ which allows you to derive conclusion s from the lexical type-assignments to the elements of that string.

3. Give the proof net corresponding to that derivation and verify that it satisfies the **NL** criteria (acyclicity/connectedness, planarity, balance).

4. Compute the lambda term corresponding to the **L** proof net for this derivation (i.e. treating the lexical assumptions as a string, rather than a bracketed string). Use constants for the lexical items.

5. Assume a map $h(p) = (* \multimap *)$ for atomic types p and the homomorphic extension $h(A/B) = h(B\backslash A) = h(B) \multimap h(A)$, where $* \multimap *$ is understood as the string type. Provide appropriate lexical terms for the vocabulary items $a, \overline{a}, b, \overline{b}$ under their different type assignments. Substitute these lexical terms in the proof term for the derivation of $ab\overline{b}\overline{a}a\overline{a}$. Apply β-reduction to compute the string image of the proof term.

For the last step, Slide 33 of the Part 2 slides provides an example (but with a different map h). Slide 47 explains the modeling of strings as functions of type $(* \multimap *)$, with concatenation as function composition, and $\lambda i.i$ for the empty string.

4

In this exercise, we return to L_{2s}, the language of two *shuffled* parenthesis systems: words $w \in \{a, \overline{a}, b, \overline{b}\}^+$ such that

$|w|_a = |w|_{\bar{a}}$ and $|w|_b = |w|_{\bar{b}}$,

for every prefix w' of w, $|w|_a \geq |w|_{\bar{a}}$ and $|w|_b \geq |w|_{\bar{b}}$

For those of you who like counting, the number of shuffled strings of length $2n$ is the sequence http://oeis.org/A005568, the product of two successive Catalan numbers $C_n C_{n+1}$:

$$1, 2, 10, 70, 588, 5544, 56628, 613470, 6952660, 81662152, 987369656, 12228193432, \ldots$$

Below, for $n = 3$, the thirty strings (out of seventy) of L_{2s} that are not recognized by your typelogical grammar of §3 (or by G' of §2).

$aab\bar{a}\bar{a}\bar{b}$ $aab\bar{a}\bar{b}\bar{a}$ $aa\bar{a}b\bar{a}\bar{b}$ $aba\bar{a}\bar{a}\bar{b}$ $aba\bar{b}\bar{a}\bar{a}$ $abb\bar{a}\bar{b}\bar{b}$ $abb\bar{b}\bar{a}\bar{b}$ $ab\bar{a}a\bar{a}\bar{b}$ $ab\bar{a}a\bar{b}\bar{a}$ $ab\bar{a}b\bar{b}\bar{b}$
$ab\bar{a}\bar{b}a\bar{a}$ $ab\bar{a}\bar{b}b\bar{b}$ $ab\bar{b}b\bar{a}\bar{b}$ $a\bar{a}ab\bar{a}\bar{b}$ $a\bar{a}ba\bar{b}\bar{a}$ $baa\bar{a}\bar{b}\bar{a}$ $baa\bar{b}\bar{a}\bar{a}$ $bab\bar{a}\bar{b}\bar{b}$ $bab\bar{b}\bar{b}\bar{a}$ $ba\bar{a}a\bar{b}\bar{a}$
$ba\bar{b}a\bar{a}\bar{a}$ $ba\bar{b}b\bar{a}\bar{b}$ $ba\bar{b}b\bar{b}\bar{a}$ $ba\bar{b}\bar{a}a\bar{a}$ $ba\bar{b}\bar{a}b\bar{b}$ $bba\bar{b}\bar{a}\bar{b}$ $bba\bar{b}\bar{b}\bar{a}$ $b\bar{b}ba\bar{b}\bar{a}$ $b\bar{b}ab\bar{a}\bar{b}$ $b\bar{b}ba\bar{b}\bar{a}$

1. Generalize the **NL** grammar of §3 to **NL**$\Diamond$, with the extraction postulates $(P1)$, $(P2)$ below.

2. Provide appropriate type assignments for 'displaced' occurrences of the vocabulary items, and appropriate terms given the interpretation map h of §3, extended with $h(\Diamond A) = h(\Box A) = h(A)$, i.e. treating the control features as having no effect on the interpretation.

3. Give a derivation in $\mathbf{NL}\Diamond$ for $aba\bar{b}\overline{aa}$, where you obtain that string as a structural deformation of a string that is derivable with your **NL** grammar of §3.

4. Compute the proof term for that derivation, and its string image.

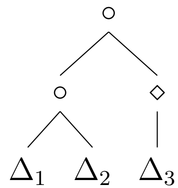
Below the postulates for extraction from right branches:

$$(A \otimes B) \otimes \Diamond C \vdash A \otimes (B \otimes \Diamond C) \quad (P1)$$
$$(A \otimes B) \otimes \Diamond C \vdash (A \otimes \Diamond C) \otimes B \quad (P2)$$

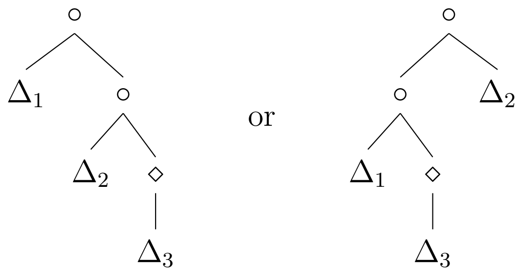
In rule format for use with the N.D. inference rules:

$$\frac{\Gamma[\Delta'] \vdash D}{\Gamma[\Delta] \vdash D} \; P1; P2$$

where Δ is



and Δ':



The following *derived* inference rule telescopes $(/I), (\Diamond E)$, a sequence (possibly empty) of $P1/P2$ restructurings, and finally $(\Box E)$ when the marked hypothesis has found the position where it can be used as a regular B. You can use (*xright*) to give your derivation in an abbreviated format.

$$\frac{\Gamma[\Delta \circ B] \vdash A}{\Gamma[\Delta] \vdash A/\Diamond\Box B}\ xright$$

Hint You can use the discussion of ‘Controlling structural resource management’ in §3.1 of the SEP article on Typelogical Grammar for inspiration. Assign an extra higher-order type to the closing brackets $\overline{a}, \overline{b}$, allowing them to appear in surface structure in a position to the right of the position where they would be canonically required, given the type assignments of the **NL** grammar you found for Assignment §3.

A

Here is the unlexicalized grammar G.

$$G: \qquad \begin{array}{l} S \longrightarrow a\ S\ \overline{a}\ S \mid b\ S\ \overline{b}\ S \\ S \longrightarrow \epsilon \end{array} \tag{1}$$

In order to lexicalize it, we have to do two things: (i) get rid of the epsilon rule; (ii) have a single lexical anchor for each rule. For (i), we multiply the rules for the four possibilities of having the first/second rhs S empty. For (ii), we pick the opening brackets a, b as anchors, and introduce new non-terminals for the closing brackets $\overline{a}, \overline{b}$: A, B. The resulting lexicalized grammar is G'.

$$G': \qquad \begin{array}{lcl} S & \longrightarrow & a\ A \mid a\ A\ S \mid a\ S\ A \mid a\ S\ A\ S \\ S & \longrightarrow & b\ B \mid b\ B\ S \mid b\ S\ B \mid b\ S\ B\ S \\ A & \longrightarrow & \overline{a} \\ B & \longrightarrow & \overline{b} \end{array} \tag{2}$$

B

1. Recasting G' as a **NL** categorial grammar is straightforward. Below the lexicon with type assignments in one-to-one correspondence with the rules of G'. Notice that the

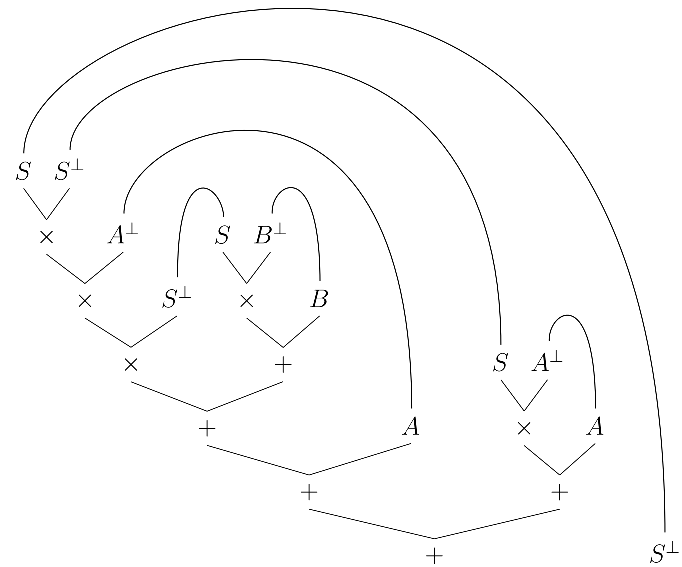
lexicon is a *relation* associating vocabulary items with one or more types.

$$\begin{array}{lcl} a & :: & S/A \mid (S/S)/A \mid (S/A)/S \mid ((S/S)/A)/S \\ b & :: & S/B \mid (S/S)/B \mid (S/B)/S \mid ((S/S)/B)/S \\ \overline{a} & :: & A \\ \overline{b} & :: & B \end{array} \tag{3}$$

2. The natural deduction below has the lexical type assignments at the leaf nodes; for the judgements $\Gamma \vdash A$ at the internal nodes, type information is dropped on the lhs of the turnstyle. The final step of the derivation has the required bracketing to show that $ab\overline{b}\overline{a}a\overline{a}$ is a well-formed expression of type S.

$$\dfrac{\dfrac{\dfrac{\dfrac{a}{((S/S)/A)/S} \quad \dfrac{\dfrac{b}{S/B} \quad \dfrac{\overline{b}}{B}}{b \cdot \overline{b} \vdash S}\,[/E]}{a \cdot (b \cdot \overline{b}) \vdash (S/S)/A}\,[/E] \quad \dfrac{\overline{a}}{A}}{(a \cdot (b \cdot \overline{b})) \cdot \overline{a} \vdash S/S}\,[/E] \quad \dfrac{\dfrac{a}{S/A} \quad \dfrac{\overline{a}}{A}}{a \cdot \overline{a} \vdash S}\,[/E]}{((a \cdot (b \cdot \overline{b})) \cdot \overline{a}) \cdot (a \cdot \overline{a}) \vdash S}\,[/E] \tag{4}$$

3. The proof net satisfies all criteria for **NL**: acyclicity/connectedness, planarity, balance. Notice that the cotensor structure at the bottom of the net corresponds to the bracketing of the endsequent in the natural deduction proof: $((a + (b + \overline{b})) + \overline{a}) + (a + \overline{a})$.

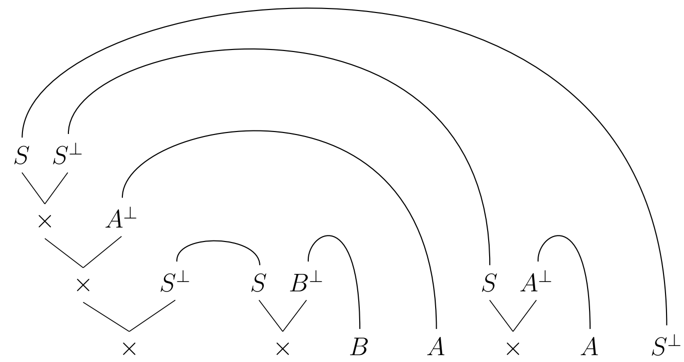


(5)

4. To compute the proof term with the net traversal algorithm, we drop the lower cotensor structure, keeping a flat sequence of assumptions. Labeling these from left to right with free variables ('constants') $x_0, \ldots, x_5$, we compute the following pure application

term:

$$((x_0\ (x_1\ x_2))\ x_3)\ (x_4\ x_5)$$



(6)

5. We now turn to the ACG picture of the relation between the proof term of a derivation at the ‘abstract syntax’ level and its string image obtained by a compositional mapping. We build up the proof term this time from abstract constants—you find them in the left column of the table below, together with their type for the abstract source calculus. Notice that type assignment is now a *function* associating an abstract constant with its single type.

$$
\begin{array}{ll|ll}
\text{A}_0 & A & \sigma & \lambda i.(\bar{\mathtt{a}}\ i) \\
\text{A}_1 & S/A & \sigma \multimap \sigma & \lambda x \lambda i.(\mathtt{a}\ (x\ i)) \\
\text{A}_2 & (S/S)/A & \sigma \multimap \sigma \multimap \sigma & \lambda x \lambda y \lambda i.(\mathtt{a}\ (x\ (y\ i))) \\
\text{A}_3 & (S/A)/S & \sigma \multimap \sigma \multimap \sigma & \lambda x \lambda y \lambda i.(\mathtt{a}\ (x\ (y\ i))) \\
\text{A}_4 & ((S/S)/A)/S & \sigma \multimap \sigma \multimap \sigma \multimap \sigma & \lambda x \lambda y \lambda z \lambda i.(\mathtt{a}\ (x\ (y\ (z\ i))))
\end{array} \tag{7}
$$

A compositional mapping h sends the source types and derivations/proof terms to target types/derivations for a string interpretation. We abbreviate the string type $* \multimap *$ as σ. In the third and fourth columns of the table you find the image of the abstract syntax types and of the abstract constan under the mapping h. The interpretation of the constants is built from target constants $\mathtt{a}$ and $\bar{\mathtt{a}}$ of type σ. (For the second pair of brackets, replace $\text{A}, A, \mathtt{a}$ by $\text{B}, B, \mathtt{b}$.)

Below the abstract proof term for derivation (4), and its image under the mapping h.

$$
\begin{array}{rcl}
t & = & ((\text{A}_4\ (\text{B}_1\ \text{B}_0))\ \text{A}_0)\ (\text{A}_1\ \text{A}_0) \\
h(t) & = & \lambda i.(\mathtt{a}\ (\mathtt{b}\ (\bar{\mathtt{b}}\ (\bar{\mathtt{a}}\ (\mathtt{a}\ (\bar{\mathtt{a}}\ i))))))
\end{array} \tag{8}
$$

C

1. and 2. We extend the lexicon with higher-order entries for *displaced* closing brackets. In the formula tree for a judgement $\Gamma \vdash S$, A_x, B_x will appear *to the right* and *above* the position where the hypotheses $\Diamond\Box A, \Diamond\Box B$ are canonically needed, i.e. consumed as regular A, B. For the string image under h, the correct translations for A_x, B_x are the

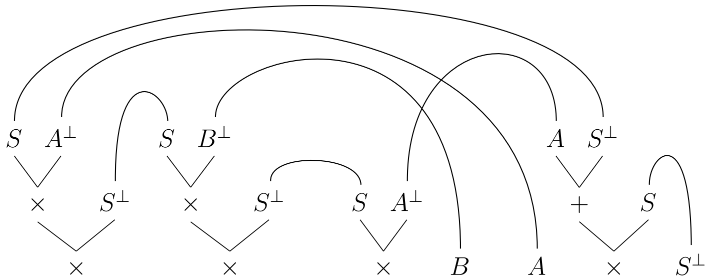
(†) variants. The parameter q here is a function from strings to strings. It is provided with the *empty string* (the term $\lambda i.i$) for the $\Diamond\Box-$ hypothesis, and then concatenated with the string constants for the displaced closing brackets. (We'll comment on the (‡) variant in a moment.)

$$\begin{array}{ll|lll} \mathrm{A}_x & (S/\Diamond\Box A)\backslash S & (\sigma \multimap \sigma) \multimap \sigma & \lambda q\lambda i.((q\ \lambda j.j)\ (\bar{\mathtt{a}}\ i)) & (\dagger) \\ & & & \lambda q.(q\ \bar{\mathtt{a}}) & (\ddagger) \\ \mathrm{B}_x & (S/\Diamond\Box B)\backslash S & (\sigma \multimap \sigma) \multimap \sigma & \lambda q\lambda i.((q\ \lambda j.j)\ (\bar{\mathtt{b}}\ i)) & (\dagger) \\ & & & \lambda q.(q\ \bar{\mathtt{b}}) & (\ddagger) \end{array} \tag{9}$$

3. The derivation in (10) obtains the string $aba\bar{b}\overline{aa}$ as a deformation of the context-free pattern $aba\bar{a}\bar{b}\bar{a}$. The displacement constant A_x is used for the final closing bracket element. The $\Diamond\Box A$ hypothesis finds its place as a right sister of A_1 by means of the $P1/P2$ postulates.

$$
\begin{gathered}
\dfrac{\mathrm{A}_1}{s/a} \qquad \dfrac{[\text{\textvisiblespace} \vdash \Box a]^2}{\langle \text{\textvisiblespace} \rangle \vdash a}\,[\Box E] \\[1ex]
\dfrac{s/a \qquad \langle \text{\textvisiblespace} \rangle \vdash a}{\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle \vdash s}\,[/E] \\[1ex]
\dfrac{\dfrac{\mathrm{B}_3}{(s/b)/s} \qquad \mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle \vdash s}{\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle) \vdash s/b}\,[/E] \\[1ex]
\dfrac{\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle) \vdash s/b \qquad \dfrac{\mathrm{B}_0}{b}}{(\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0 \vdash s}\,[/E] \\[1ex]
\dfrac{\dfrac{\mathrm{A}_3}{(s/a)/s} \qquad (\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0 \vdash s}{\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0) \vdash s/a}\,[/E] \\[1ex]
\dfrac{\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0) \vdash s/a \qquad \dfrac{\mathrm{A}_0}{a}}{(\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s}\,[/E] \\[1ex]
\dfrac{(\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot (\mathrm{A}_1 \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s}{(\mathrm{A}_3 \cdot (((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \langle \text{\textvisiblespace} \rangle) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s}\,[P1] \\[1ex]
\dfrac{(\mathrm{A}_3 \cdot (((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \langle \text{\textvisiblespace} \rangle) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s}{(\mathrm{A}_3 \cdot (((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0) \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{A}_0 \vdash s}\,[P2] \\[1ex]
\dfrac{(\mathrm{A}_3 \cdot (((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0) \cdot \langle \text{\textvisiblespace} \rangle)) \cdot \mathrm{A}_0 \vdash s}{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \langle \text{\textvisiblespace} \rangle) \cdot \mathrm{A}_0 \vdash s}\,[P1] \\[1ex]
\dfrac{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \langle \text{\textvisiblespace} \rangle) \cdot \mathrm{A}_0 \vdash s}{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0) \cdot \langle \text{\textvisiblespace} \rangle \vdash s}\,[P2] \\[1ex]
\dfrac{[\text{\textvisiblespace} \vdash \Diamond \Box a]^1 \qquad ((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0) \cdot \langle \text{\textvisiblespace} \rangle \vdash s}{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0) \cdot \text{\textvisiblespace} \vdash s}\,[\Diamond E]^2 \\[1ex]
\dfrac{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0) \cdot \text{\textvisiblespace} \vdash s}{(\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s/\Diamond \Box a}\,[/I]^1 \\[1ex]
\dfrac{(\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0 \vdash s/\Diamond \Box a \qquad \dfrac{\mathrm{A}_x}{(s/\Diamond \Box a)\backslash s}}{((\mathrm{A}_3 \cdot ((\mathrm{B}_3 \cdot \mathrm{A}_1) \cdot \mathrm{B}_0)) \cdot \mathrm{A}_0) \cdot \mathrm{A}_x \vdash s}\,[\backslash E]
\end{gathered}
\tag{10}
$$

Drawing an **LP** proof net (i.e. relaxing the planarity constraint) gives a more uncluttered picture of the dependencies. We ignore the structural control operators $\Diamond, \Box$ in this net. (This will change when we reach Part 3 of the course.)



(11)

4. The proof term t for derivation (10), or for the proof net (11), is given in (12) below, together with the image under h, for the (†) and (‡) versions of the translation of the constants. The (†) translation correctly yields the desired string image $aba\overline{b}\overline{a}\overline{a}$. The (‡) translation substitutes the closing $\overline{a}$ bracket for the λ-bound variable for the $\Diamond\Box A$ hypothesis, and thus produces the canonical context-free pattern underlying the string $aba\overline{b}\overline{a}\overline{a}$ . . . !

$$\begin{array}{rcl} t & = & (\mathrm{A}_x\ \lambda y_2.((\mathrm{A}_3\ ((\mathrm{B}_3\ (\mathrm{A}_1\ y_2))\ \mathrm{B}_0))\ \mathrm{A}_0)) \\ (\dagger)\quad h(t) & = & \lambda i.(\mathtt{a}\ (\mathtt{b}\ (\mathtt{a}\ (\overline{\mathtt{b}}\ (\overline{\mathtt{a}}\ (\overline{\mathtt{a}}\ i)))))) \\ (\ddagger)\quad h(t) & = & \lambda i.(\mathtt{a}\ (\mathtt{b}\ (\mathtt{a}\ (\overline{\mathtt{a}}\ (\overline{\mathtt{b}}\ (\overline{\mathtt{a}}\ i)))))) \end{array} \qquad (12)$$

Discussion

Here are some questions we have to address:

Q1. Does the **NL**$\Diamond$ grammar of (3)+(9) recognize all and only parenthesis shuffle patterns from L_{2s}? Positive answer for the 'only' part is easy. What about the 'all' part? Looking at the sample of 30 length-6 patterns, 24 of them are obtained with the aid of a single displacement type. For the others, more that one displacement type is needed.

Q2. If the language of the **NL**$\Diamond$ grammar of (3)+(9) is a proper subset of L_{2s}, then how can we characterize this subset exactly? Is well-nestedness the discriminating feature?

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