CATEGORIAL AND CATEGORICAL GRAMMARS

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- Categorial refers to the types.
- Categorical refers to the category theory.

We will first briefly introduce the following notions:

deductive system, category, functor and natural transformation

and discuss:

- syntactic system,
- semantic system.

A deductive system consists of:

- Two classes, the class of arrows (or proofs) and the class of objects (or types, or formulas) and two mappings between them: source(f) = A and target(f) = B (which is the same as f = A → B). [this is just a graph]
- *Identity* arrow for each type A: $1_A = A \rightarrow A$
- Composition of arrows:

$$\frac{f: A \to B \quad g: B \to C}{g \circ f: A \to C}$$

A *category* is a deductive system with the following equations between arrows:

•
$$f \circ 1_A = f = 1_B \circ f$$
,
• $(h \circ g) \circ f = h \circ (g \circ f)$, for all $f : A \to B, g : B \to C$ and $h : C \to D$.

A functor

$$\mathcal{F}:\mathbb{C}\to\mathbb{D}$$

between categories $\mathbb C$ and $\mathbb D$ is a mapping of objects to objects and arrows to arrows, such that the following holds:

• $\mathcal{F}(f : A \to B) = \mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$, • $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$, • $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

 ${\mathcal F}$ preserves domains and codomains, identity arrows, and composition. Example: Cat.

NATURAL TRANSFORMATION

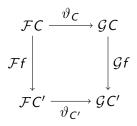
For categories \mathbb{C},\mathbb{D} and functors

$$\mathcal{F}, \mathcal{G}: \mathbb{C} \to \mathbb{D}$$

a natural transformation $\vartheta: \mathcal{F} \to G$ is a family of arrows in \mathbb{D}

$$(\vartheta_{\mathcal{C}}:\mathcal{FC}\to\mathcal{GC})_{\mathcal{C}\in\mathcal{C}_{o}}$$

such that, for any $f: C \to C'$ in \mathbb{C} , it holds that $\vartheta_{C'} \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \vartheta_C$.



For all $A, B, C \subseteq M$, where M is a semigroup (or monoid) the following holds:

- $A \cdot B \subseteq C \text{ iff } A \subseteq C/B$
- $A \cdot B \subseteq C \text{ iff } B \subseteq A \setminus C$
- $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
- $I \cdot A = A = A \cdot I$, where $I = \{1\}$. If the semigroup is a monoid with unity element 1.

Words of a language can generate a free multiplicative system, semigroup or monoid. Then, sets of strings of words will be called *syntactic types*.

The *syntactic calculus* is a deductive system where the class of types contains a special type I and is closed under the three binary operations \cdot , / and \setminus with the following axioms and rules of inference:

$$\begin{aligned} &\alpha_{A,B,C} : (A \cdot B) \cdot C \to A \cdot (B \cdot C) ,\\ &\alpha_{A,B,C}^{-1} : A \cdot (B \cdot C) \to (A \cdot B) \cdot C ,\\ &\rho_A : A \cdot I \to A ,\\ &\rho_A^{-1} : A \to A \cdot I , \end{aligned} \qquad \qquad \lambda_A : I \cdot A \to A ,\\ &\lambda_A^{-1} : A \to I \cdot A ,\end{aligned}$$

$$\frac{f: A \cdot B \to C}{f^*: A \to C/B} \qquad \frac{f: A \cdot B \to C}{*f: B \to A \setminus C}$$

Syntactic Calculus as a Deductive System; Example of a Proof

$$\frac{g: B \to A \setminus C}{f^+ g: A \cdot B \to C} \qquad \qquad \frac{g: A \to C/B}{g^+: A \cdot B \to C}$$

Where the following abbreviations are used:

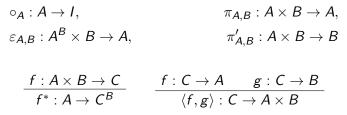
$$\begin{array}{ll} f^* \text{ for } \beta_{A,B,C}(f), & {}^*f \text{ for } \gamma_{A,B,C}(f), \\ g^+ \text{ for } \beta_{A,B,C}^{-1}(g), & {}^+g \text{ for } \gamma_{A,B,C}^{-1}(g). \end{array}$$

Example of a proof:

$$\frac{1_{A/B}: A/B \to A/B}{1^+_{A/B}: (A/B) \cdot B \to A}$$
$$*(1^+_{A/B}): B \to (A/B) \setminus A$$

- A categorial grammar of a language may be viewed as consisting of the syntactic calculus freely generated from a finite set {*S*, *N*...} of basic types together with a dictionary which assigns to each word of the language a finite set of types composed from the basic types and *I* by the three binary operations.
- Categorial grammar assigns type S to a string A₁A₂...A_n of words iff the dictionary assigns type B_i to A_i(i = 1, ..., n) and B₁B₂...Bn → S is a theorem in the freely generated syntactic calculus.

A *semantic calculus* is a deductive system with a specified type *I* and the class of types closed under two binary operations:



This is an intuitionisitic propositional calculus.

Our next goal:

Find a categorical interpretation of

- ${\scriptstyle \bullet}$ the semantic calculus \rightarrow catesian closed categories
- ${\scriptstyle \bullet}\,$ the syntactic calculus \rightarrow biclosed monoidal categories
- an interpretation functor between them

Recall:A *category* is a deductive system with the following equations between arrows:

A *biclosed monoidal category* is a syntactic calculus which satisfies the equations of a category as well as the following

$$\bullet \quad \bullet \, \cdot : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

$$\bullet \ \backslash : \mathbb{C} \times \mathbb{C}^{OP} \longrightarrow \mathbb{C}$$

•
$$/ : \mathbb{C}^{OP} \times \mathbb{C} \longrightarrow \mathbb{C}$$

are bifunctors

• $\rho, \lambda, \alpha, \beta, \gamma$ are natural isomorphisms

In fact, every cartesian closed category can be regarded as a biclosed monoidal category by an appropriate interpretation. So for example

$$A \cdot B = A \times B$$
$$A/B = A^B = B \setminus A$$

Then we already have f.e.

$$\frac{f: A \times B \to C}{f^*: A \to C^B}$$

Recall: A categorial grammar has two components

- the free syntactic calculus generated by $\mathcal{B} = \{S, N, \dots\}$
- a dictionary assigning a finite number of compound types to each word of the language
- So we define the category SYNTAX by:
 - form the free biclosed monoidal category on \mathcal{B} , call it $\mathcal{F}(\mathcal{B})$
 - Adjoin to *F*(*B*) the set of arrows *X* = {*John_N*, *works_{N\S}*,...} determined by the dictionary, i.e.
 John_N : *I* → *N*, *works_{N\S}* : *I* → *N\S*.

So we get

$$SYNTAX = \mathcal{F}(\mathcal{B})[\mathcal{X}]$$

- The category SEMANTICS can be any cartesian closed category that has
 - ${\scriptstyle \bullet}$ a distinguished object ${\cal J}$ of individuals
 - an object Ω of truth values

By an interpretation we mean a functor

$\Phi: \mathsf{SYNTAX} \to \mathsf{SEMANTICS}$

That preserves also the particular structure of SYNTAX. So for example : $\Phi(I) = I$ and $\Phi(A \cdot B) = \Phi(A) \times \Phi(B)$, etc. Furthermore, we require that $\Phi(S) = \Omega$ and $\Phi(N) = \mathcal{J}$.

$\Phi(John_N) = \text{John where } John : I \to \mathcal{J} \text{ in SEMANTICS}$ $\Phi(works_{N \setminus S} =) = \lambda x \in N(x \text{ works})$