Categorial and Categorical Grammars

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Categorial and Categorical; Introduction

- Categorial refers to the types.
- Categorical refers to the category theory.

We will first briefly introduce the following notions:

- *deductive system, category, functor and natural transformation*

and discuss:

- *syntactic system*,
- *semantic system.*

The Notion of Category I

A *deductive system* consists of:

- Two classes, the class of *arrows* (or *proofs*) and the class of *objects* (or *types*, or *formulas*) and two mappings between them: source$(f) = A$ and target$(f) = B$ (which is the same as $f = A \to B$). [this is just a graph]
- *Identity* arrow for each type A: $1_A = A \to A$
- Composition of arrows:

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

The Notion of Category II

A *category* is a deductive system with the following equations between arrows:

1. $f \circ 1_A = f = 1_B \circ f$,
2. $(h \circ g) \circ f = h \circ (g \circ f)$, for all $f : A \to B, g : B \to C$ and $h : C \to D$.

A *functor*

$$\mathcal{F} : \mathbb{C} \to \mathbb{D}$$

between categories $\mathbb{C}$ and $\mathbb{D}$ is a mapping of objects to objects and arrows to arrows, such that the following holds:

1. $\mathcal{F}(f : A \to B) = \mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$,
2. $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$,
3. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

$\mathcal{F}$ preserves domains and codomains, identity arrows, and composition.
Example: Cat.

Natural Transformation

For categories $\mathbb{C}, \mathbb{D}$ and functors

$$\mathcal{F}, \mathcal{G} : \mathbb{C} \to \mathbb{D}$$

a *natural transformation* $\vartheta : \mathcal{F} \to \ G$ is a family of arrows in $\mathbb{D}$

$$(\vartheta_C : \mathcal{F}C \to \mathcal{G}C)_{C \epsilon C_o}$$

such that, for any $f : C \to C'$ in $\mathbb{C}$, it holds that $\vartheta_{C'} \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \vartheta_C$.

$$\begin{array}{ccc} \mathcal{F}C & \xrightarrow{\vartheta_C} & \mathcal{G}C \\ {\scriptstyle \mathcal{F}f}\downarrow & & \downarrow{\scriptstyle \mathcal{G}f} \\ \mathcal{F}C' & \xrightarrow[\vartheta_{C'}]{} & \mathcal{G}C' \end{array}$$

For all $A, B, C \subseteq M$, where M is a semigroup (or monoid) the following holds:

1. $A \cdot B \subseteq C$ iff $A \subseteq C/B$
2. $A \cdot B \subseteq C$ iff $B \subseteq A \setminus C$
3. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
4. $I \cdot A = A = A \cdot I$, where $I = \{1\}$. If the semigroup is a monoid with unity element 1.

Words of a language can generate a free multiplicative system, semigroup or monoid. Then, sets of strings of words will be called *syntactic types*.

Syntactic Calculus as a Deductive System

The *syntactic calculus* is a deductive system where the class of types contains a special type I and is closed under the three binary operations $\cdot, /$ and $\backslash$ with the following axioms and rules of inference:

$$\alpha_{A,B,C} : (A \cdot B) \cdot C \to A \cdot (B \cdot C),$$

$$\alpha^{-1}_{A,B,C} : A \cdot (B \cdot C) \to (A \cdot B) \cdot C,$$

$$\rho_A : A \cdot I \to A, \qquad \lambda_A : I \cdot A \to A,$$

$$\rho_A^{-1} : A \to A \cdot I, \qquad \lambda_A^{-1} : A \to I \cdot A,$$

$$\frac{f : A \cdot B \to C}{f^* : A \to C/B} \qquad \frac{f : A \cdot B \to C}{{}^*f : B \to A\backslash C}$$

Syntactic Calculus as a Deductive System; Example of a Proof

$$\frac{g : B \to A \backslash C}{{}^{+}g : A \cdot B \to C} \qquad \frac{g : A \to C/B}{g^{+} : A \cdot B \to C}$$

Where the following abbreviations are used:

f^* for $\beta_{A,B,C}(f)$, $\quad {}^{*}f$ for $\gamma_{A,B,C}(f)$,
g^+ for $\beta^{-1}_{A,B,C}(g)$, $\quad {}^{+}g$ for $\gamma^{-1}_{A,B,C}(g)$.

Example of a proof:

$$\cfrac{\cfrac{1_{A/B} : A/B \to A/B}{1^{+}_{A/B} : (A/B) \cdot B \to A}}{*(1^{+}_{A/B}) : B \to (A/B) \backslash A}$$

Motivation

- A *categorial grammar* of a language may be viewed as consisting of the syntactic calculus freely generated from a finite set $\{S, N...\}$ of basic types together with a dictionary which assigns to each word of the language a finite set of types composed from the basic types and I by the three binary operations.
- Categorial grammar assigns type S to a string $A_1A_2...A_n$ of words iff the dictionary assigns type B_i to $A_i(i = 1, ..., n)$ and $B_1B_2...Bn \rightarrow S$ is a theorem in the freely generated syntactic calculus.

Semantic Calculus as a Deductive System

A *semantic calculus* is a deductive system with a specified type I and the class of types closed under two binary operations:

$$\circ_A : A \to I, \qquad \pi_{A,B} : A \times B \to A,$$

$$\varepsilon_{A,B} : A^B \times B \to A, \qquad \pi'_{A,B} : A \times B \to B$$

$$\frac{f : A \times B \to C}{f^* : A \to C^B} \qquad \frac{f : C \to A \qquad g : C \to B}{\langle f, g \rangle : C \to A \times B}$$

This is an intuitionisitic propositional calculus.

Our next goal:
Find a categorical interpretation of

- the semantic calculus $\rightarrow$ catesian closed categories
- the syntactic calculus $\rightarrow$ biclosed monoidal categories
- an interpretation functor between them

Cartesian Closed Categories

Recall:A *category* is a deductive system with the following equations between arrows:

1. $f \circ 1_A = f = 1_B \circ f$,
2. $(h \circ g) \circ f = h \circ (g \circ f)$, for all $f : A \to B, g : B \to C$ and $h : C \to D$.

A *cartesian closed category* is a semantic calculus that satisfies the equations of a category and also:

$f = \circ_A$ for all $f : A \to I$

$\pi_{A,B}\langle f, g\rangle = f$ for all $f : C \to A$, $g : C \to B$

$\pi'_{A,B}\langle f, g\rangle = g$

$\langle \pi_{A,B} h, \pi'_{A,B} h\rangle = h$ for all $h : C \to A \times B$

$\epsilon_{A,B}\langle h * \pi_{C,B}, \pi'_{C,B}\rangle = h$ for all $h : C \times B \to A$,

$\left(\epsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle\right)^* = k$ for all $k : C \to A^B$

Biclosed Monoidal Category

A *biclosed monoidal category* is a syntactic calculus which satisfies the equations of a category as well as the following

-
  - $\cdot : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$
  - $\backslash : \mathbb{C} \times \mathbb{C}^{OP} \longrightarrow \mathbb{C}$
  - $/ : \mathbb{C}^{OP} \times \mathbb{C} \longrightarrow \mathbb{C}$

  are bifunctors
- $\rho, \lambda, \alpha, \beta, \gamma$ are natural isomorphisms

In fact, every cartesian closed category can be regarded as a biclosed monoidal category by an appropriate interpretation.
So for example

$$A \cdot B = A \times B$$

$$A/B = A^B = B\backslash A$$

Then we already have f.e.

$$\frac{f : A \times B \to C}{f^* : A \to C^B}$$

The Category SYNTAX

Recall: A categorial grammar has two components

- the free syntactic calculus generated by $\mathcal{B} = \{S, N, \dots\}$
- a dictionary assigning a finite number of compound types to each word of the language

So we define the category SYNTAX by:

- form the free biclosed monoidal category on $\mathcal{B}$, call it $\mathcal{F}(\mathcal{B})$
- Adjoin to $\mathcal{F}(\mathcal{B})$ the set of arrows $\mathcal{X} = \{John_N, works_{N\backslash S}, \dots\}$ determined by the dictionary, i.e. $John_N : I \to N, works_{N\backslash S} : I \to N\backslash S$.

So we get

$$SYNTAX = \mathcal{F}(\mathcal{B})[\mathcal{X}]$$

The Category SEMANTICS

- The category SEMANTICS can be any cartesian closed category that has
  - a distinguished object $\mathcal{J}$ of individuals
  - an object Ω of truth values

By an interpretation we mean a functor

$$\Phi : \text{SYNTAX} \to \text{SEMANTICS}$$

That preserves also the particular structure of SYNTAX.
So for example :
$\Phi(I) = I$ and $\Phi(A \cdot B) = \Phi(A) \times \Phi(B)$, etc.
Furthermore, we require that
$\Phi(S) = \Omega$ and $\Phi(N) = \mathcal{J}$.

Examples

$$\Phi(John_N) = \text{John where John} : I \rightarrow \mathcal{J} \text{ in SEMANTICS}$$

$$\Phi\left(works_{N\backslash S} =\right) = \lambda x \in N(x \text{ works})$$