On Categorical Interpretations of ILL and Lambek Calculus

Vlasta and Julia

1 Introduction

Category theory is a relatively new field of mathematics. It was discovered in the middle of the 19th centry. Samuel Eilenberg and Saunders Mac Lane provided a first definition of a category and William Lawvare expanded the applications of the filed. Nowadays category theory is even considered to be an alternative to set theory as a foundation for mathematics. In the 1960's, it was Joachim Lambek who first combined Gerhard Gentzen's methods with category theory and linguistics.

The aim of this paper is to present categorical interpretations of linguistically interesting substructural logics such as ILL and the Lambek calculus. We will provide the reader with the necessary background in category theory. No preliminary knowledge of that field is required. Following the seminal ideas of Lambek we will examine categorical interpretations thoroughly from a proof theoretical point of view. Ultimately, we will focus on the categorical interpretation of intuitionistic linear logic and the Lambek calculus following the lines of the paper of Richard Blute and Philip Scott [2] and Lambek's paper *"Categorial and Categorical Grammar"* [7], respectively. In our presentation we will particularly try to spell out some details that are left out in the mentioned literature.

In the next section we will introduce the needed definitions of category theory, followed by a section on the connection between category theory and general proof theory. In Section 4 we eloborate the categorial translation of ILL and the associative Lambek calculus L. In the conclusion we will give a brief summary of the presented material and also give a small outlook how the presented concepts can be extended.

2 Preliminaries in category theory

In this section we will introduce some basic notions of category theory such as the definitions of a category, functor, natural transformation and adjunction. We will also present the notion of a monoidal category, which will be of great interest for our considerations in Section 4 on the categorical interpretation of some substructural logics. For a more thorough introduction to category theory the reader is referred to Steve Awodey's book [1].

Definition 1. A *category* $\mathcal{C}$ consists of the following:

- A class of *objects*, denoted by $\mathrm{Ob}(\mathcal{C})$.
- A class of *arrows*, denoted by $\mathrm{Ar}(\mathcal{C})$.
- Mappings $\mathbf{dom}, \mathbf{cdom} : \mathrm{Ar}(C) \longrightarrow \mathrm{Ob}(C)$, that assign to the arrows their domain and codomain. We often write $f : A \longrightarrow B$ to indicate that $f \in \mathrm{Ar}(\mathcal{C})$ is an arrow, with $\mathbf{dom}(f) = A$ and $\mathbf{cdom}(f) = B$.
- For all objects A, B and C and arrows $f : A \longrightarrow B, g : B \longrightarrow C$ there is an arrow $g \circ f : A \longrightarrow C$. The arrow $g \circ f$ is called the composition of g and f.
- for any object A there is an identity arrow $1_A : A \longrightarrow A$.

For all objects A, B, C and D and arrows $f : A \longrightarrow B, g : B \longrightarrow C$ and $h : C \longrightarrow D$ the above must satisfy the following axioms:

- $f \circ 1_A = f = 1_B \circ f$ (Identity law)
- $h \circ (g \circ f) = (h \circ g) \circ f$ (Associativity)

The variety of categories is endless. In order to have a slight inside we will present two examples of categories and give a detailed explanation why they satisfy the imposed axioms. The most "famous" category is the category **Set**, the category of sets. The class of objects of this category is the class of all sets and the collection of arrows is the class of all possible functions on sets. The mappings **dom** and **cdom** are defined in the obvious way by sending a function to its actual domain and codomain, respectively. Composition is given by set theoretical function composition. For every set there is a canonical identity function that serves also as the identity function in the categorical sense. It is easy to check that given these definitions, the axioms of associativity and identity are satisfied. A bit more abstract kind of category is given by considering a fixed partial order as a category. Let $\mathcal{P} = (P, \leq)$ be a partial order on a set P. The partial order can be seen as a category as follows. The objects of the category are the elements of the set P and we require an arrow from the element p to q if and only if $p \leq q$ in $\mathcal{P}$. With this interpretation the identity axiom follows from the reflexivity axiom of the partial order, whereas function composition holds by transitivity of the partial order. Since there is at most one arrow between two elements associativity condition holds and also the unit law is satisfied.
In the next definition we will present the product category. Building the product of two categories will be for us mostly of technical interest.

Definition 2. Given two categories $\mathcal{C}, \mathcal{D}$ the *product category* $\mathcal{C} \times \mathcal{D}$ is defined as follows.

- $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) := \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, where the $\times$ on the right hand side is the cartesian product of the two classes of objects.
- $\mathrm{Ar}(\mathcal{C} \times \mathcal{D}) := \mathrm{Ar}(\mathcal{C}) \times \mathrm{Ar}(\mathcal{D})$. More precisely, for arrows $f : C_1 \to C_2 \in \mathcal{C}$ and $g : D_1 \to D_2 \in \mathcal{D}$ of $\mathcal{D}$ there is an arrow $f \times g : C_1 \times D_1 \to D_1 \times D_2$.
- For an object $C \times D$ of $\mathcal{C} \times \mathcal{D}$ the identity arrow is defined as $1_C \times 1_D$.
- Composition is defined coordinate-wise.

The reader can easily check, that given the above definitions the identity and associativity axiom are inherited from the categories $\mathcal{C}$ and $\mathcal{D}$. So $\mathcal{C} \times \mathcal{D}$ is indeed a category.

In category theory we are often more interested in functions than in the objects. There are not only arrows within a category, but also on "higher levels". Our next definition will be the one of a functor, that provides us an appropriate notion of a map between categories. On one more level above we will introduce the concept of natural transformations as a notion of maps between functors.

Definition 3. A *functor* $\mathcal{F} : \mathcal{C} \to D$ between categories $\mathcal{C}, \mathcal{D}$ consists of:

- A map $\mathcal{F}_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$, i.e. it assigns to every object of $\mathcal{C}$ an object of $\mathcal{D}$.
- A map $\mathcal{F}_{Ar} : \mathrm{Ar}(\mathcal{C}) \longrightarrow \mathrm{Ar}(\mathcal{D})$ s.t. for all arrows $f : A \longrightarrow B$ of the category $\mathcal{C}$ there is an arrow $\mathcal{F}_{\mathrm{Ar}}(f) : \mathcal{F}_{\mathrm{Ob}}(A) \longrightarrow F_{\mathrm{Ob}}(B)$. One can observe that this notation implies that a functor has to preserve the domain and the codomain of an arrow.

Furthermore, it has to preserve the identity arrow and composition, i.e.

- $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ and
- $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

The notion of a functor can help us in comparing categories. It can also serve to impose a particular structure or operation on the objects within a category. An example of the latter is the product-functor $\times : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$, that assigns a pair of sets to its cartesian product and two function f, g to the pair of functions (f, g) defined point-wise. The reader can easily check that for this definition the functor axioms are satisfied. Functors that have a product category in their domain will come along throughout the whole paper. This is why we are going to present one important property of product functors already at this point. Let $\mathcal{F} : \mathcal{C} \times C' \longrightarrow \mathcal{D}$ be a functor and C a fixed object of the category $\mathcal{C}$. Now we can define a map $\mathcal{F}(C,-) : \mathcal{C} \longrightarrow \mathcal{D}$ by sending an object C' of $\mathcal{C}'$ to $\mathcal{F}(C, C')$ and an arrow $g : C_1' \to C_2'$ to $\mathcal{F}(1_C, g)$. It can be shown that this map inherits the functor axioms from $\mathcal{F}$. So fixing the first component of a product functor yields to a functor. A similar result holds, if we fix the second component.

We continue with the promised definition of a natural transformation, a notion of a map between to functors.

Definition 4. A *natural transformation* $\theta : \mathcal{F} \longrightarrow \mathcal{G}$ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ is given by a family $(\theta_A : \mathcal{F}(A) \longrightarrow \mathcal{G}(A))_{A \in \mathrm{Ob}(\mathcal{C})}$, s.t. for all for all arrows $f : A \longrightarrow B \in \mathrm{Ar}(C)$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ {\scriptstyle\theta_A}\downarrow & & \downarrow{\scriptstyle\theta_B} \\ \mathcal{G}(A) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(B) \end{array}$$

A natural transformation is called *natural isomorphism*, if all components are isomorphisms[1].

One of the most fundamental definitions in category theory is the definition of adjoint functors. Pairs of adjoint functors occur in various areas where category theory is applied. There are several equivalent definitions of this concept that are used in the literature. We will cite the one that is often referred as the "Hom-set definition", as it is the most convenient for our purposes.

Definition 5. Given categories $\mathcal{C}, \mathcal{D}$ and functors $\mathcal{F}$ and $\mathcal{G}$ satisfying:

$$\mathcal{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$

we say that $\mathcal{G}$ *is a right adjoint of* $\mathcal{F}$ (or equivalently $\mathcal{F}$ *is a left adjoint of* $\mathcal{G}$), if for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there is a bijection

$$\Phi : \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}C, D) \cong \mathrm{Hom}_{\mathcal{C}}(C, \mathcal{G}D),$$

that is natural in both, C and D. Here, $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}C, D)$ refers to the collection of arrows in $\mathcal{D}$ with domain $\mathcal{F}C$ and codomain D. Similarly for $\mathrm{Hom}_{\mathcal{C}}(C, \mathcal{G}D)$.

The last definition that we will introduce is the one of a monoidal category. This class of categories will be of great interest in our later attempt

[1]For a categorical definition of an isomorphism see [1].

in finding the appropriate category for certain substructural logics. On the other hand it is worth mentioning, that monoidal categories also occur in other branches of mathematics.

Definition 6. A *monoidal category* is a category $\mathcal{C}$ together with

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
- a distinguished object I,
- natural isomorphisms [2] α, r, l with the following components

$$\begin{aligned} &\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \\ &l_A : I \otimes A \longrightarrow A \\ &r_A : A \otimes I \longrightarrow A \end{aligned}$$

  satisfying the equations given by the commutative diagrams:

$$l_I = r_I : I \otimes I \longrightarrow I$$

$$\begin{array}{ccc} A \otimes (I \otimes C) & \xrightarrow{\alpha} & (A \otimes I) \otimes C \\ {\scriptstyle 1\otimes l_C}\downarrow & & \downarrow{\scriptstyle r_A\otimes 1} \\ A \otimes C & \xrightarrow[=]{} & A \otimes C \end{array}$$

$$\begin{array}{ccccc} A(B(CD)) & \xrightarrow{\alpha} & (AB)(CD) & \xrightarrow{\alpha} & ((AB)C)D \\ {\scriptstyle 1\otimes\alpha}\downarrow & & & & \uparrow{\scriptstyle \alpha\otimes 1} \\ A((BC)D) & & \xrightarrow{\alpha} & & (A(BC))D, \end{array}$$

  where in the last diagram we omitted the $\otimes$ connective for readability reasons.

The equations imposed by the last diagrams are often called *coherence conditions*. One can say that they assure that "all possible ways of defining a particular arrow lead to the same result". Mac Lane's coherence theorem states that if the three diagrams above commute, then all other diagrams build up in a similar manner commute as well. A proof of this theorem can be found in [9]. As the reader might infer, the name monoidal category refers to the algebraic structure of a monoid. Indeed, a monoidal category can be seen as a monoid by taking the class of objects as the underlying set of the monoid with the $\otimes$-tensor as a binary relation on it. The distinguished object I corresponds to the unit object of the monoid and the imposed natural isomorphisms make sure that the monoid axioms are satisfied.

[2]natural transformations are often specified by their components. In these cases the underlying functors are implicit.

3 Proof Theory and Category Theory

As mentioned, Lambek combined Gentzen's methods with category theory. He emphasized the idea that arrows in freely generated categories are equivalence classes of proofs, raising the question what these equations between proofs are. On the other hand, in the beginning of the 1970's, Dag Prawitz introduced a field of general proof theory by asking the philosophical question what a proof is. In this section we will try to explain what general proof theory is and how category theory may be used for investigating general proof theory.

3.1 General Proof Theory

General proof theory has its foundations in Gentzen's work [4]. By examining proof-structure it tries to provide a satisfactory definition of a proof. Prawitz specifies the following four topics in general proof theory in [10]:

(1) Defining the notion of a proof.

(2) Investigating the structure of different kinds of proofs (e.g. questions concerning normal forms).

(3) Representing proofs as derivations and investigating equivalence among them.

(4) Applying these insights to the other questions in logic.

If one is trying to provide a definition of a proof then she also needs to be able to define when two proofs are identical. The notion of the identity of proofs is supposed to bring some light to the first basic question what a proof is. A proof should be an equivalence class of its representations. In this manner a precise mathematical answer is given to a philosophical question[3].

Of course, only proofs with the same assumptions and the same conclusions may be equivalent, but the whole field of general proof theory makes sense only if we assume that there can be more than one proof from the same set of premises to the same conclusion. Prawitz proposed an analysis of proof identity based on reduction to normal form in natural deduction. In the literature it is referred to this method as the *Normalization Conjecture* [3]. However, Lambek's work offers different a basis for proof equivalence which we will refer to as *Generality Conjecture*. Since this paper is focused on the connection between category theory and proof systems, we will later try to briefly explain this concept.

3.2 Proof Systems as Categories

The first stepping stone in connecting Gentzen's system with category theory was Lambek's definition of a deductive system. It opens up a general method how to interpret proof systems categorically. In such translations we first represent the logic in question as a deductive system. Then we impose additional equalities between the arrows in order to obtain a category.

Definition 7. A *deductive system* consists of:

- A class of *objects* (or *types*, or *formulas*) and
- a class of *arrows* (or *proofs*).
- The mappings source and target between them. They assign to an arrow a source and a target object, respectively. The notation $f : A \longrightarrow B$ implies, that f is an arrow with source A and target B.
- *Identity* arrow for each object A:

$$1_A : A \longrightarrow A$$

- *Composition* of arrows:

$$\frac{f : A \longrightarrow B \qquad g : B \longrightarrow C}{g \circ f : A \longrightarrow C}$$

In the translation from a given logic to a deductive system the formulas of the logic become the objects of the deductive system. A proof in a the logic corresponds to an arrow in the deductive system. More precisely, a proof of B with assumption A will be translated into an arrow with source A and target B. Imposing an identity arrow for every object, as it is stated in the above definition, means nothing else than having a proof of every formula from itself. Composition of arrows corresponds to the transitivity of the provability relation. Deductive systems can be enriched by adding more inference rules.

We are now going to explain the translation step from deductive systems to categories. It is noticeable that the definition of a deductive system is very similar to the definition of a category (compare Definition 1 from Section 2). In a category also the following equations need to be satisfied.

- $f \circ 1_A = f = 1_B \circ f$ (Identity Law)
- $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \longrightarrow B, g : B \longrightarrow C$ and $h : C \longrightarrow D$ (Associativity).

Indeed, we can turn a deductive system into a category by imposing the identity law and associativity condition between arrows. Now an arrow in the category corresponds to an equivalence class of proofs. Further inference rules are translated as natural transformations, where the interplay between introduction and elimination rules of logical connectives is given via adjunction. Altogether the translation yields the following picture:

Logic		**Category Theory**
Formulas	$\leftrightsquigarrow$	objects
Equivalence classes of proofs	$\leftrightsquigarrow$	arrows
Inference rules	$\leftrightsquigarrow$	Natural transformations

Ultimately, we will sketch the proof identity criterion in a categorical setting that was first proposed by Lambek. What we need to consider are generalizations of derivations that diversify variables without changing the rules of inference. As an example let us consider the two projections that correspond to two derivations of conjunction elimination:

$$\pi^1_{p,p} : p \wedge p \longrightarrow p \text{ and } \pi^2_{p,p} : p \wedge p \longrightarrow p.$$

The two arrows have a different generality. They generalize to $\pi^1_{p,q} : p \wedge q \longrightarrow p$ and $\pi^2_{p,q} : p \wedge q \longrightarrow q$ respectively. The difference in generality follows as these two arrows do not have the same codomain.[3]

Two derivations have the same generality when every generalization of one of them leads to a generalization of the other, so that the two generalizations have the same assumptions and conclusion. Now this provides a proof identity criterion, namely *Generality Conjecture.*

Definition 8. Generality Conjecture

Two derivations are equivalent if and only if they have the same generality.

There is no ultimate answer to the question what the equivalence between proofs is. In Subsection 4.2. we will see an example of two proofs in the Lambek calculus that look different on the first sight but are equivalent in the categorical translation.

4 Categorical Interpretations of Lingusitcally Interesting Substructural Logics

Linear logic was introduced by Jean-Yves Girard in 1987 [5]. While classical logic emphasizes the notion of *truth* and intuitionisitc logic the notion of

[3] This subtle difference is not captured by normalization procedure, that we mentioned as the proof identity criterion proposed by Prawitz.

proof, linear logic sees formulas as *resources*. On the other hand, Lambek proposed the famous noncommutative logic 1958 in his paper *The Mathematics of Sentence Structure* [8]. This logic can be in particular used to model the combinatory possibilities of the syntax of natural languages. His calculus has become one of the fundamental formalisms of computational linguistics. In this chapter we will present adequate categories for two different substructual logics. Namely for intuitionistic linear logics and for the Lambek calculus.

Let us first briefly remind of the proof theoretic way for obtaining these logics. Restricting different structural rules provides different logics. Structural rules in question were considered by Gentzen in 1934 [4]. They are *weakining*, *contraction* and *permutation* and they can act on the left and on the right side of a sequent.

	Left	*Right*
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$
Contraction	$\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$	$\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash \Delta, A}$
Permutation	$\dfrac{\Gamma_1, A, \Gamma_2, B, \Gamma_3 \vdash \Delta}{\Gamma_1, B, \Gamma_2, A, \Gamma_3 \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta_1, A, \Delta_2, B, \Delta_3}{\Gamma \vdash \Delta_1, B, \Delta_2, A, \Delta_3}$

In the family of linear logics, contraction and weakening might get forbidden as structural rules. Permutation rule is responsible for commutativity. By abolishing the permutation rule one gets non-commutative systems such as the Lambek calculus.

Since intuitionistic logic accepts contraction and weakening on the left side of sequents, as the right side consists only of one formula, we can investigate intuitionistic linear logic by adding further constrains on intuitionistic logic. Intuitionistic linear logic (ILL) is a fragment of Girard's linear logic, i.e. intuitionistic logic without weakening and contraction.

4.1 Intuitionistic Linear Logic

In the first step towards a categorical interpretation of ILL we will formulate intuistionistic linear logic with operation $\otimes$ and $\multimap$ as a deductive system.

Definition 9. *ILL as a deductive system*:

- The class of objects contains a special type I and is closed under the binary operations $\otimes$ and $\multimap$.

- The class of arrows contains the following axioms and is closed under the listed rules of inference:

$$1_A : A \longrightarrow A \text{ (Identity)}$$
$$r_A : A \otimes I \longrightarrow A \text{ and } r_A^{-1} : A \longrightarrow A \otimes I \text{ (Unit-laws)}$$
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \text{ (Associativity)}$$
$$s_{A,B} : A \otimes B \longrightarrow B \otimes A \text{ (Commutativity)}$$

$$\frac{f : A \longrightarrow B \qquad g : B \longrightarrow C}{g \circ f : A \longrightarrow C}$$

$$\multimap\ \mathit{int}\ \frac{f : C \otimes A \longrightarrow B}{f^* : C \longrightarrow (A \multimap B)} \qquad \multimap\ \mathit{elim}\ \frac{g : C \longrightarrow (A \multimap B)}{*g : C \otimes A \longrightarrow B}$$

In order to get familiar with proof techniques in a deduction system we will consider some examples of such proofs. Note that every proof is given by the construction of a particular arrow and thus the proof that corresponds to an arrow can be reconstructed just by reading its name.
It is immediate to see that next to the imposed Right-unit-laws we can infer Left-unit-laws by using the commutativity arrow:

$$\frac{s_{I,A} : I \otimes A \to A \otimes I \qquad r_A : A \otimes I \to I}{r_A \circ s_{I,A} : I \otimes A \to I} \qquad \frac{r_A^{-1} : I \to A \otimes I \qquad s_{A,I} : A \otimes I \to I \otimes A}{s_{A,I} \circ r_A^{-1} : I \to I \otimes A}.$$

An example of a bit more involved derivation is given below. The result, namely the inference rule

$$\otimes \mathit{rule}\ \frac{f : A \to B \qquad g : C \to D}{(s_{D,B} \circ^* (s^*_{B,D} \circ f)) \circ (^*(s^*_{D,A} \circ g) \circ s_{A,C}) : A \otimes C \to B \otimes D},$$

has a clear categorical translation. For readability reasons we will omit the names of the arrows in the proof.

$$\dfrac{\dfrac{A \otimes C \to C \otimes A \qquad \dfrac{\dfrac{C \to D \qquad \dfrac{D \otimes A \to A \otimes D}{D \to A \multimap A \otimes D}}{C \to A \multimap A \otimes D}}{C \otimes A \to A \otimes D}}{A \otimes C \to A \otimes D} \qquad \dfrac{\dfrac{\dfrac{A \to B \qquad \dfrac{B \otimes D \to D \otimes B}{B \to D \multimap D \otimes B}}{A \to D \multimap D \otimes B}}{A \otimes D \to D \otimes B} \qquad D \otimes B \to B \otimes D}{A \otimes D \to B \otimes D}}{A \otimes C \to B \otimes D}$$

We are now ready to present a categorical interpretation of ILL. Having a close look at the deductive system given above the reader may remember the monoidal categories that we introduced in Definition 6 of Section 2. Indeed, these will be roughly the ones that offer us an adequate interpretation. The $\otimes$-functor will serve as an interpretation of the logical connective $\otimes$. However, with no further restrictions on monoidal categories the structural rule of commutativity is not available. Furthermore, an interpretation of the $\multimap$-connective is missing. In order to obtain these we will consider a subclass of monoidal categories, namely symmetric monoidal closed categories.

Definition 10. Let $\mathcal{C}$ be a monoidal category. $\mathcal{C}$ is called *symmetric*, if there is a natural isomorphism with components $s_{A,B} : A \otimes B \longrightarrow B \otimes A$, that satisfy the following diagrams:

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{s_{A,B}} & B \otimes A \\
& \searrow^{1_{A \otimes B}} & \downarrow s_{A,B} \\
& & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
B \otimes I & \xrightarrow{s_{B,I}} & I \otimes B \\
& \searrow^{r_B} & \downarrow l_B \\
& & B
\end{array}
$$

$$
\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{1 \otimes s} & A \otimes (C \otimes B) & \xrightarrow{a} & (A \otimes C) \otimes B \\
\downarrow a & & & & \downarrow s \otimes 1 \\
(A \otimes B) \otimes C & \xrightarrow{s} & C \otimes (A \otimes B) & \xrightarrow{a} & (C \otimes A) \otimes B
\end{array}
$$

Similar as in the definition of monoidal categories, the imposed commutative diagrams ensure coherence for the inferences.

Definition 11. A *symmetric monoidal closed category* is a symmetric monoidal category $\mathcal{C}$ s.t. for all $A \in \mathcal{C}$ the functor $- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint $A \multimap - : \mathcal{C} \longrightarrow \mathcal{C}$.

We will now provide a detailed explanation why symmetric monoidal closed categories serve as a good interpretation of ILL. Let $\mathcal{C}$ be such a category. First we need to give adequate translations of the formulas into the objects of the category. Then we need to check whether all the axioms and rules of inference that we imposed in Definition 9 -where we presented ILL as a deductive system- are satisfied. As our category $\mathcal{C}$ is monoidal it contains a distinguished object I that serves as the special object I of the deductive system. Now for any two objects A and B of $\mathcal{C}$ we need to find objects $A \otimes B$ and $A \multimap B$ in $\mathcal{C}$. The first is given by applying the $\otimes$-functor to the object (A, B) of $\mathcal{C} \times \mathcal{C}$. The latter is the image of B under the functor $A \multimap -$. This shows that the class of objects is as the translation requires. We continue with interpretations of the axioms. The identity arrow exists for all $A \in \mathcal{C}$ simply because $\mathcal{C}$ is a category. Arrows for the associativity condition and the unit laws are given because the functor $\otimes$ provides a monoidal structure on $\mathcal{C}$. The components of the natural isomorphism $s_{A,B}$ serve as proofs of the substructural rule of commutativity. It remains to infer the introduction and elimination rule for the $\multimap$ connective:

$$\multimap\ int\ \frac{f : C \otimes A \longrightarrow B}{f^* : C \longrightarrow (A \multimap B)} \qquad \multimap\ elim\ \frac{g : C \longrightarrow (A \multimap B)}{*g : C \otimes A \longrightarrow B}$$

For this purpose we have to examine the Definition 5 of adjoint functors. In order to obtain the first rule we keep object A of the category $\mathcal{C}$ fixed. Now $A \multimap - : \mathcal{C} \longrightarrow \mathcal{C}$ being a right adjoint of the functor $- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}$ requires that for B, C in $\mathcal{C}$ there is a bijection

$$\Phi : \mathrm{Hom}_{\mathcal{C}}(C \otimes A, B) \cong \mathrm{Hom}_{\mathcal{C}}(C, (A \multimap B)).$$

In particular, for an arrow $f : C \otimes A \longrightarrow B$ there is an arrow $f^* : C \longrightarrow (A \multimap B)$, which shows that the $\multimap$-introduction rule holds in our category. For the $\multimap$-elimination rule just consider the bijection Φ in the other direction.

4.2 Lambek calculus

We will now give a similar characterization for the Lambek calculus. The presented solution was first stated by Lambek in his seminal paper [7].

The most substantial difference to the system that we studies above is the absence of the commutativity rule. Similarly as in the previous chapter we will formulate the associative Lambek calculus L as a deductive system and then introduce the corresponding categorical notion. Furthermore, we will present an example of two proofs in the Lambek calculus that are equivalent in the categorical translation.

Definition 12. The *Lambek calculus* is a deductive system such that :

- The class of types contains a special type I and is closed under the binary operations $\cdot$, / and \.
- The class of arrows contains for all types A, B, C the following arrows and is closed under the listed rules of inferences

$$1_A : A \longrightarrow A$$
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$
$$\alpha^{-1}_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$$
$$r_A : A \otimes I \longrightarrow A \text{ and } r_A^{-1} : A \longrightarrow A \otimes I$$
$$l_A : I \otimes A \longrightarrow A \text{ and } l_A^{-1} : A \longrightarrow I \otimes A$$

$$\frac{f : A \longrightarrow B \qquad g : B \longrightarrow C}{g \circ f : A \longrightarrow C}$$

$$\frac{f : A \otimes B \longrightarrow C}{f^* : A \longrightarrow C/B} \quad \frac{f : A \otimes B \longrightarrow C}{{}^*f : B \longrightarrow A\backslash C}$$

$$\frac{g : A \longrightarrow C/B}{g^+ A \otimes B \longrightarrow C} \quad \frac{g : B \longrightarrow A\backslash C}{{}^+g : A \otimes B \longrightarrow C}$$

As for the case of ILL we take the monoidal categories from Section 2 as a starting point for our interpretation. As previously the $\otimes$-functor serves as the interpretation of $\otimes$-connective. However, the symmetric monoidal closed categories would not serve as a good interpretation, because the symmetry is not consistent with the absence of the permutation rule in the Lambek calculus. A solution, that was first proposed by Lambek in [7], lies in the notion of a biclosed monoidal category.

Definition 13. A *biclosed monoidal category* is a monoidal category $\mathcal{C}$, where for all $A \in \mathcal{C}$ the functor $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $-/A : \mathcal{C} \to \mathcal{C}$. In addition, for all $A \in \mathcal{C}$ also the functor $A \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A\backslash - : \mathcal{C} \to \mathcal{C}$.

The functor $-/A : \mathcal{C} \to \mathcal{C}$ serves as an interpretation for the /-connective and the functor $A\backslash - : \mathcal{C} \longrightarrow \mathcal{C}$ as an interpretation of the \-connective, respectively. It is easy to see that the inference rules of the deductive system can be obtained by the bijections that is given by the adjunctions. We will not spell out the details of the translation as they are very easily inferred from the description that we gave in the previous chapter.

We complete this Section by giving some remarks on further categorical structure and on the equality of proofs that results from the categorical interpretation.It can be shown that that in biclosed monoidal categories we can define functors

$$/ : \mathcal{C} \times \mathcal{C}^{op} \longrightarrow \mathcal{C}$$
$$\backslash : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C},$$

where $\mathcal{C}^{op}$[4] is the opposite category of $\mathcal{C}$. These functors are called internal homomorphisms. Their construction can be found in [6]. The advantage of the latter is that all the connectives are treated uniformly as binary operations on the category $\mathcal{C}$. In this presentation we also see immediately that we have inferences such as

$$\frac{f : A \longrightarrow B \qquad g : C \longrightarrow D}{f/g : A/D \longrightarrow B/C} \quad \text{and} \quad \frac{f : A \longrightarrow B \qquad g : C \longrightarrow D}{f\backslash g : B\backslash C \longrightarrow A\backslash D}.$$

[4]For a category $\mathcal{C}$ the category $\mathcal{C}^{op}$ is the opposite category. It contains exactly the same objects as $\mathcal{C}$ and there is an arrow $f^* : B \longrightarrow A$ in $\mathcal{C}^{op}$ if and only if there is an arrow $f : A \longrightarrow B$ in $\mathcal{C}$.

They arise by applying the functors / and \ to the given arrows. Of course, such inferences can also be made in the Lambek calculus.

Before finishing this chapter we will present some observations concerning the equality of proofs that we already considered in Section 3. As mentioned in Section 3 a categorical interpretation of deductive systems gives rise to a natural notion of the equality of proofs. So let us consider an example. Here are two proofs of the same formula:

$$\dfrac{\dfrac{f : A \longrightarrow B \qquad g : X \longrightarrow Y}{f/g : A/Y \longrightarrow B/X} \qquad \dfrac{f' : B \longrightarrow C \qquad g' : Z \longrightarrow X}{f'/g' : B/X \longrightarrow C/Z}}{f'/g' \circ f/g : A/Y \longrightarrow C/Z}$$

and

$$\dfrac{\dfrac{f : A \longrightarrow B \qquad f' : B \longrightarrow C}{f' \circ f : A \longrightarrow C} \qquad \dfrac{g : X \longrightarrow Y \qquad g' : Z \longrightarrow X}{g' \circ g : Z \longrightarrow Y}}{f' \circ f/g' \circ g : A/Y \longrightarrow C/Z}$$

The given proofs look different. Nevertheless, categorically they are equivalent. The reason for that lies in the functoriality of the map $/ : \mathcal{C} \times \mathcal{C}^{op} \longrightarrow \mathcal{C}$, that we stated above. Beeing a functor particularly implies that the map preserves composition of arrows. For arrows f, f', g, g' of $\mathcal{C}$ as above this is expressed by the equation

$$f' \circ f/g' \circ g = f'/g' \circ f/g,$$

where on the left side of the equation we first composed (f', g') and (f, g) in $\mathcal{C} \times \mathcal{C}^{op}$ and then applied the /-functor to the solution $(f' \circ f, g' \circ g)$. Whereas on the right hand side we applied the /-functor separately to the pairs (f', g') and (f, g) and then took the composite. Of course, there are many other examples of proofs that become are via the categorical interpretation. In order to detect them, one needs to observe carefully the equations forced by axioms, coherence and naturality conditions that we imposed throughout the paper.

5 Conclusion

We introduced the field category theory and showed how proof systems can be interpreted categorically. For this purpose we explained Lambek's notion of a deductive system and its connection to categories. An important benefits of this interpretation is the *Generality Conjecture* which we briefly introduced. It is a criterion for proof identity and therefore an important contribution for defining a proof. Nevertheless expressing a precise proof identity criterion still represents a challenge for logicians.

Furthermore,we focused on providing categorical translation for two specific substructural logics that are relevant in linguistic applications. We saw that symmetric monoidal categories provide an adequate categorical translation of intuitionistic linear logic. Moreover, biclosed monoidal categories play the same role in the case of the Lambek calculus. In our presentation we tried to contribute by explaining subtle details of these correspondences. Furthermore, we gave some illustrative examples.

What is left out is whether the correspondences can be extended, if we add more connectives and logical constants, such as $\perp$ or ! in linear logics, or impose more substructural rules. In the paper [2] Blute and Scott elaborate a great variety of solutions for such extensions. The notion of *-autonomous categories, that are symmetric monoidal categories with extra structure play a central role for these concepts. Another kind of modification lies in considering an even more restricted system such as NL, the non-associative Lambek calculus. Finding a categorical translation for the latter would imply dropping monoidal categories as the starting point for our translation, because their "conventional" definition includes arrows for associativity.

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