Opgaves Wiskunde voor CKI

18 December 2004

1. Let $[a, b]$ denote $\{x \in \mathbb{R} : a \le x \le b\}$, $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$ and similarly for the other types of intervals. Describe bijections between the following sets:
   (a) $(-\pi, \pi)$ and $\mathbb{R}$;
   (b) $\mathbb{R}$ and $(-\infty, 0)$;
   (c) $[0, 1]$ and $[0, 1)$;
   (d) $[0, 1]$ and $[0, 1] \setminus \{\frac{1}{2}\}$.
2. Show that the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.
3. Show that $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$, are equivalent.
4. Using Cantor's diagonal method show that there is no bijection between A and $\mathcal{P}(A)$, for any set A.
5. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of countable sets A_i. Show that $\bigcup_{i=1}^{\infty} A_i$ is countable. (Recall that $x \in \bigcup_{i=1}^{\infty} A_i$ iff $x \in A_i$, for some $i \in \mathbb{N}$.)
6. Show that if A is a countable set, then the set of all finite sequences of elements of A is countable. (A finite sequence of length n is a function $f : \{1, \ldots, n\} \to A$.) Hint: use the statement of the previous problem.
7. A number $x \in \mathbb{R} \setminus \mathbb{Q}$ is called a *quadratic irrationality*, if it is a root of an equation of the form
   $$ax^2 + bx + c = 0,$$
   with $a, b, c \in \mathbb{Q}$. Show that the set of all quadratic irrationalities is countable.