Opgaves Wiskunde voor CKI, uitwerking

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1. (a) Consider $f(x) := \tan(x)$. Then $f : (-\pi, \pi) \to \mathbb{R}$ bijectively.

   (b) Consider $f(x) = -2^x$. Then $f : \mathbb{R} \to (-\infty, 0)$ bijectively.

   (c) In order to bijectively map $[0,1]$ to $[0,1)$ consider a sequence of numbers $\frac{1}{2}, \frac{1}{4}, \ldots$ Define:

   $$f(x) := \begin{cases} \frac{1}{2^{n+1}}, & \text{if } x = \frac{1}{2^n}, \text{ for some } n \in \mathbb{N} \cup \{0\}; \\ \text{x}, & \text{otherwise.} \end{cases}$$

   It is easy to verify that this function is, indeed, a bijection.

   (d) First, using the previous problem bijectively map $[0, \frac{1}{2})$ to $[0, \frac{1}{2}]$. Second, identically map $(\frac{1}{2}, 1]$ to $(\frac{1}{2}, 1]$. Then combine the two mappings into one from $[0,1] \setminus \{\frac{1}{2}\}$ to $[0,1]$.

2. If the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ were countable, then so would be its union with a countable set $\mathbb{Q}$. But we know that $\mathbb{R}$ is uncountable by Cantor's theorem.

3. *Show that* $\mathbb{R}$ *and* $\mathcal{P}(\mathbb{N})$*, the set of all subsets of* $\mathbb{N}$*, are equivalent.*

   We only outline the idea of a proof. There is a bijection between subsets of $\mathbb{N}$ and functions $f : \mathbb{N} \to \{0,1\}$ (i.e., infinite binary sequences). Indeed, we map a subset $A \subseteq \mathbb{N}$ to its characteristic function

   $$f(n) := \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$$

   Conversely, having a sequence f we associate with it the set $A := \{n \in \mathbb{N} : f(n) = 1\}$. These two mappings are mutually inverse, hence establish a bijection.

Any binary sequence $a_1a_2a_3\ldots$ corresponds to a real number in the interval $[0,1)$ whose binary representation is $0,a_1a_2a_3\ldots$ This mapping is surjective but not injective, because of the sequences ending with $111111\ldots$ There are several ways to overcome this difficulty. We establish a bijection between all sequences and 'good' sequences (not ending with $1111\ldots$).

'Bad' sequences have the form $\alpha1111\ldots$, where α is a finite sequence not ending with a 1. We map such a sequence to $\alpha10101010\ldots$. In turn, $\alpha10101010\ldots$ should be mapped somewhere, e.g. to the sequence $\alpha100100100100\ldots$. In general, period of the form $1\underbrace{0\ldots0}_{n\text{ times}}$ will be replaced by $1\underbrace{0\ldots0}_{n+1\text{ times}}$. All other sequences will not be changed. One can easily verify that this procedure bijectively maps all sequences to good sequences.

This shows that $\mathcal{P}(\mathbb{N})$ is equivalent to $[0,1)$.

Using the method analogous to the one from Problem 1, it is not difficult to establish a bijection between $[0,1)$ and the whole of $\mathbb{R}$.

4. *Using Cantor's diagonal method show that there is no bijection between A and $\mathcal{P}(A)$, for any set A.*

   Proof: Assume that $f : A \to \mathcal{P}(A)$ is a bijection and g is its inverse. Define $B := \{x \in A : x \notin f(x)\}$. Let $b := g(B)$, then $f(b) = B$. We have by the definition of B (as in Russel's Paradox)

$$b \in B \iff b \notin f(b) \iff b \notin B.$$

   A contradiction.

   Analyze this proof and see in that this argument boils down to the usual proof of Cantor's theorem if one considers a representation of $\mathbb{R}$ as the set of all infinite binary sequences.

5. *Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of countable sets A_i. Show that $\bigcup_{i=1}^{\infty} A_i$ is countable.*

   Hint: the argument is almost the same as in the proof of the fact that countable sets are closed under cartesian product.

6. *Show that if A is a countable set, then the set of all finite sequences of elements of A is countable. (A finite sequence of length n is a function $f : \{1, \dots, n\} \to A$.)*

   One can represent the set of all finite sequences as

   $$A \cup A^2 \cup A^3 \cup \cdots$$

   We know that if A is countable then so are A^2, A^3, etc. (formally, one proves it by induction on n). So, the statement reduces to the previous one.

7. *A number $x \in \mathbb{R} \setminus \mathbb{Q}$ is called a* quadratic irrationality*, if it is a root of an equation of the form*

   $$ax^2 + bx + c = 0,$$

   with $a, b, c \in \mathbb{Q}$. Show that the set of all quadratic irrationalities is countable.

   There is a surjective function from $\mathbb{Q}^3$ to the union of $\mathbb{Q}$ and the set of quadratic irrationalities. Map $\langle a, b, c \rangle \in \mathbb{Q}$ to the minimal root of the corresponding equation $ax^2 + bx + c = 0$, if there is such a root, and to 0, otherwise. But $\mathbb{Q}^3$ is countable.