Additional view on binary relations
Course: Wiskunde voor AI
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Question: Is it possible for symmetry and anti-symmetry to co-exist in one binary relation ***R***?

It will suffice to produce one situation that shows this is possible.

In a model of binary relations we take the following definitions:

1- Symmetry: $\forall x,y\ (xRy \rightarrow yRx)$
2- Anti-Symmetry: $\forall x,y\ ((xRy$ **and** $yRx) \rightarrow (x = y))$

One example will be: ***R*** = {<x,y>: (x,y $\in$ {x}) and (x = y)}

Intuitively this relation states that a pair of 2 elements (x and y) from the set containing only the element x has a relation x = y.
This means the pair of <x,y> will actually always be the pair <x,x> because there is no other element than x.

- We can see that symmetry holds: symmetry states if x has a relation with y then y has this relation with x. Using the relation defined it states: if x equals y then y equals x. Precisely in this model it means: if x equals x then x equals x (which is always true ofcourse!).
- Now to see if Anti-symmetry holds: Anti-Symmetry states that if x has a relation with y and y has a relation with x then x and y are the same element. We already saw that x has a relation with y and y with x, or in using the relation defined: x equals x and x equals x. Anti-Symmetry states that if this is the case, the element x and x should be the same (which is also always true!).

Now we have shown that symmetry and Anti-Symmetry can co-exist in one model!

We can generalize this problem to a model for mathematics and/or logics (modal logics to be more precise).
In this model we represent elements from the set as worlds or nodes and relations as arrows between worlds. Often this way it will be much more easy to see.

To generalize we will ask the following question:

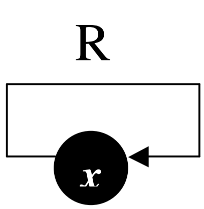
Question: Is it possible for symmetry and anti-symmetry to co-exist in one (modal logic) model?

In this model the following will hold:

1- Symmetry: $\forall x,y\ (xRy \rightarrow yRx)$
2- Anti-Symmetry: $\forall x,y\ ((xRy$ **and** $yRx) \rightarrow (x = y))$
3- $\boldsymbol{R} = \{<x,y>: (x,y \in \{x\})$ and $(x = y)\}$

Again we construct the model taking the elements of the set as worlds (or nodes) with the relation ***R*** as an arrow between worlds.

Model:



In this model we have one world x with a reflexive relation (a relation to itself) called ***R***.
This relation is still the same relation as above.
In this model it should be clear that symmetry and Anti-Symmetry holds. In short we can state the following.
- symmetry: if x has a relation with x then x has a relation with x.
- Anti-Symmetry: if x has a relation with x and x has a relation with x then x and x are the same.

You can check this with the proof on the previous page!

Two little extra additions by Lev D. Beklemishev and me:

1- Check that in a model with only world x **and no relation** (so the empty relation holds) symmetry and Anti-Symmetry simultaneously hold.
Here you can see that it makes a difference which definition for anti-symmetry you use.
I used the definition $\forall$x,y ((xRy **and** yRx) $\leftrightarrow$ (x = y)), the definition which is written with a bi-implication, so goes both ways. Lev used a little different definition, namely $\forall$x,y ((xRy **and** yRx) $\rightarrow$ (x = y)), so only one way. The other way around is usually assumed (and valid) but it makes a little difference with regard to the empty relation. Using the bi-implication we can derive that the empty relation is NOT simultaneously symmetric and anti-symmetric, using the normal implication we can derive that the empty relation is simultaneously symmetric and anti-symmetric.
Check this for yourself!

2- We formulate the following general fact: The only binary relation that is simultaneously symmetric and anti-symmetric is the equality relation.

Recall:

1- Symmetry: $\forall$x,y (xRy $\rightarrow$ yRx)
2- Anti-Symmetry: $\forall$x,y ((xRy **and** yRx) $\leftrightarrow$ (x = y))
3- Reflexivity: $\forall$x (xRx)

Proof: by Anti-Symmetry (as written above) we automatically have reflexivity: xRx, for any x, because x=y implies xRy.

The other way around: if xRy then yRx by symmetry, then x=y by anti-symmetry.

So, we have proved x=y if and only if xRy.