The Logic of Provability

Suggestions for exercises. Week 5.

1. (C)

   We have the following conditionals to be provable in the indicated logics.

   (a) $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
   (b) $\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$
   (c) $\mathbf{K4} \vdash \Box\varphi \to \Box\Box\Box\varphi$
   (d) $\mathbf{K4} \vdash (\Box\varphi \wedge \Box\Box\psi) \to \Box\Box(\varphi \wedge \psi)$

   Show for each of this implications that the direction can not be reversed, that is, we have no equivalences in the logics.

2. (C)

   Write down all the formulas (i.e. subsentences and negated subsentences) of $\Box p \to p$. Give all the maximal **K4**-consistent sets corresponding to $\Box p \to p$. Indicate which sets are related to each other in the sense of the **K4** relation defined on page 80 of the book and thus indicate what a **K4** countermodel of $\Box p \to p$ can look like.

3. (C)

   (a) Give an example of a **K4**-consistent formula which is not **S4**-consistent. (b) The same question for the logics **K** and **K4**.

4. (C)

   Show: $\Box(\Box p \to p) \to \Box p$ is true in all upwards well-founded[1], transitive Kripke models.

5. (C)

   Let M be a Kripke model and $x \in M$. Show: the set $\{\varphi : (M, x) \vDash \varphi\}$ is maximal consistent.

6. (C)

   Show that any maximal consistent set of formulas is closed under modus ponens.

[1] That is, there is no infinite chain $x_1 R x_2 R x_3 R \ldots$ of elements of the model.

7. (C) (A)

   Write down all the formulas (i.e. subsentences and negated subsentences) of $p \to \Box p$. Give all the maximal **K4**-consistent sets corresponding to $p \to \Box p$. Indicate which sets are related to each other in the sense of the **K4** relation defined on page 80 of the book and thus indicate what a **K4** countermodel of $p \to \Box p$ can look like. Relate this model to the one obtained in exercise 2.

8. (C) (A)

   Do the previous exercise for the sentence $\Box\Box p \to \Box p$.

9. (C) (A)

   Write down an explicit function $\mathrm{neg}(x)$ that assigns to a code of a function the code of the negation of that function. We are not interested what $\mathrm{neg}(x)$ does on values that are not codes of formulas. Also write down a function $\mathrm{impl}(x, y)$ that assigns to each pair of codes of formulas the corresponding conditional. That is, if x is the code of φ and y is the code of ψ, $\mathrm{impl}(x, y)$ should be the code of $\varphi \to \psi$.