Logische Technieken, Tentamen
Utrecht 2-5-2002

1. (a.) Exhibit proofs to show that $\mathbf{K} \vdash (\Box\phi \vee \Box\psi) \rightarrow \Box(\phi \vee \psi)$ and $\mathbf{K4} \vdash (\Box\phi \wedge \Box\Box\psi) \rightarrow \Box\Box(\phi \wedge \psi)$.

   (b.) Show for each of the above implications that the direction can not be reversed, that is, we have no equivalences in the respective logics.

2. (a.) Provide a proof in **K4** of $\Diamond\Diamond A \wedge \Box(A \rightarrow B) \rightarrow \Diamond(A \wedge B)$.

   (b.) Let A be a theorem of **GL**, that is, $\mathbf{GL} \vdash A$. Show that A can not be equivalent to a consistency statement in **GL**. Thus, for no modal formula C we have that $\mathbf{GL} \vdash A \leftrightarrow \Diamond C$.

3. Prove in PA that any number is either odd or even, that is, $\forall x(\exists y\ 2y = x \vee \exists y\ 2y + 1 = x)$.

4. (a.) Show that $\mathbf{K4} \nvdash \Diamond p \rightarrow \Diamond(p \wedge \Box\neg p)$.

   (b.) Show by semantical means that $\Diamond p \rightarrow \Diamond(p \wedge \Box\neg p)$ is valid on (Boolos says *in*) every transitive and converse well-founded frame. May we conclude that $\mathbf{GL} \vdash \Diamond p \rightarrow \Diamond(p \wedge \Box\neg p)$?

   (c.) Provide (the sketch of) a proof in **GL** of $\Diamond p \rightarrow \Diamond(p \wedge \Box\neg p)$.

   (d.) Let α be some arithmetical sentence such that $\mathsf{PA} \nvdash \neg\alpha$. Infer that $\mathbb{N} \models \mathsf{Con}(\ulcorner \alpha \wedge \mathsf{Bew}(\ulcorner \neg\alpha \urcorner) \urcorner)$.

5. Let the Solovay sentences S_i be as defined on page 127 of the book.

   (a.) Does S_i assert that i is the limit of the Solovay function h or does it assert that i is *not* the limit of the Solovay function h.

   (b.) Let i be a top-node in our model, that is, there are no nodes accessible from i. Show that $\mathsf{PA} \vdash S_i \rightarrow \mathsf{Bew}(\ulcorner \bot \urcorner)$.