Lecture XXII

The ω-rule.

Recall that Gödel's theorem gives us a universally quantified statement (x)A(x) all of whose instances are provable but which is not itself provable. Thus, while intuitively it might seem like (x)A(x) follows from A(**0**), A(**0'**), ..., in fact, while all of the latter are provable, the former is not provable. However, it would be provable if we added to our formal system the following rule, known as the *ω-rule*: from A(**0**), A(**0'**), ... to infer (x)A(x). In fact, this was Hilbert's suggestion when he first heard about Gödel's result.

The ω-rule can't actually be applied in practice, since it has infinitely many premises and so a proof using the ω-rule would be infinitely long. Moreover, even if we can prove each of the instances of (x)A(x), we may not be in a position to know that they are all provable. For example, consider Goldbach's conjecture. Supposing that it is in fact true, we can easily prove each of its instances; nonetheless, we are not now in a position to know that all of its instances are provable, since we are not now in a position to prove that the statement itself is true.

Nonetheless, we can consider formal systems which contain the ω-rule, even if we cannot actually use such systems. If we add the ω-rule to an ordinary first-order deductive system (Q, for example), then not only will there be no true but unprovable Π_1 statements: all true statements will be provable. To see this, suppose we start out with a system which proves all true sentences of Lim, and which is such that every sentence of the language of arithmetic is provably equivalent to a Σ_n or Π_n sentence, for some n. If we add the ω-rule to such a system, then we will be able to prove every true Σ_n or Π_n sentence, and therefore every true sentence whatsoever. We show this by induction on n. (For the sake of the proof, we define a formula to be both Σ_0 and Π_0 if it is a formula of Lim.) We know it holds for n = 0, because by hypothesis all true sentences of Lim are provable. Suppose it holds for n, and let A be a Σ_{n+1} formula. Then A is (∃x)B(x) for some Π_n formula B(x). If A is true, then B($\mathbf{0}^{(m)}$) is true for some m, so by the inductive hypothesis B($\mathbf{0}^{(m)}$) is provable in the system, so A is also provable. Now let A be a Π_{n+1} formula. Then A is (x)B(x) for some Σ_n formula B(x). If A is true, then B($\mathbf{0}^{(m)}$) is true for all m, so by the inductive hypothesis, B($\mathbf{0}^{(m)}$) is provable for all m. Now we apply the ω-rule: from the sentences B(**0**), B(**0'**), ..., we can infer the sentence (x)B(x), i.e. the sentence A, so A is provable.

So as long as we stay within the first-order language of arithmetic, we can get around the Gödel theorem by allowing our formal systems to include the ω-rule. However, if we consider richer languages (e.g. languages with quantifiers over sets of numbers, or with extra predicates), we will not necessarily be able to get around the Gödel result in this way. In fact, there are languages richer than the first-order language of arithmetic such that, even when we allow formal systems to contain an ω-rule, we get a Gödel-type result. This was first discovered by Rosser, but it was not until much later, when extensions of the arithmetical hierarchy were being studied in the 50's, that his ideas were taken up again.

The Analytical Hierarchy.

We have already seen how to enrich the language of arithmetic by adding extra predicates and function symbols. We can also treat these new symbols as *variables*, and even quantify over them. The resulting formulae will then have two types of variables: one type for numbers and one type for sets (or functions); if a formula has n number variables and k set variables, then it defines an n+k-place relation between numbers and sets, in which the first n places are occupied by numbers and the remaining places are occupied by sets. Similarly, if there are k function variables, then the formula defines an n+k-place relation between numbers and functions. (The formula f(x) = y, for example, defines the 3-place relation $\{<x, y, f>: x, y \in \mathbf{N}$ and $f: \mathbf{N} \to \mathbf{N}$ and $f(x) = y\}$.) When the variables are function variables, their values are always *total* functions.

We could get by with only unary predicates, reducing functions and other predicates to unary predicates via standard methods. We could also use only unary function symbols. That is, we could rewrite $f(x_1, ..., x_n)$ as $f([x_1, ..., x_n])$, and replace sets by their characteristic functions. In principle it doesn't matter what we do, but it will turn out to be convenient to require all the new variables to be unary function variables, so we shall do so. We use lower case Greek letters for function variables.

In the case of Σ^0_1 formulae, a version of the monotonicity and finiteness theorems hold. That is, if $A(x_1, ..., x_n, \alpha_1, ..., \alpha_k)$ is a Σ^0_1 formula, then $<m_1, ..., m_n, f_1, ..., f_k>$ satisfies it iff there are finite initial segments $s_1, ..., s_k$ of $f_1, ..., f_k$, such that $<m_1, ..., m_n, s_1, ..., s_k>$ satisfies it. (Unary functions on **N** can be seen as infinite sequences of numbers; an initial segment of a function f is then a sequence $<f(0), ..., f(x)>$ for some x.) Actually, this way of putting it isn't quite correct, because we require the values of the variables to be total functions, so we must restate it as follows. Let A* be the result of replacing $\alpha_i(x) = y$ by $(\exists z)(\text{Seql}(s_i, x) \wedge x < z \wedge [x, y] \in s_i)$ wherever it occurs in A. (If function variables are embedded in A, we iterate this process.) Then $<m_1, ..., m_n, f_1, ..., f_k>$ satisfies A iff for some finite initial segments $s_1, ..., s_k$ of $f_1, ..., f_k$, respectively, $<m_1, ..., m_n, \alpha_1, ..., \alpha_k>$ satisfies A*.

Now let us consider formulae which may contain quantifiers over functions; a relation between natural numbers and functions defined by such a formula is called *analytical*. In particular, a set of numbers defined by such a formula is called analytical.

A Σ^1_n formula is a formula that consists of an alternating string of function quantifiers of length n, beginning with an existential quantifier, followed by a single number quantifier of the opposite type from the last variable quantifier in the string, followed by a formula of Lim. The definition of "Π^1_n" is the same except that we require the first quantifier to be universal. Thus, for example, the formula (α)(∃β)(x) α(x) = β(x) is a Π^1_2 formula. A relation is Σ^1_n or Π^1_n if it is defined by a Σ^1_n or Π^1_n formula, respectively; a relation is Δ^1_n if it is both Σ^1_n and Π^1_n. The hierarchy of Σ^1_n and Π^1_n sets is called the *analytical hierarchy*. This hierarchy was first studied by Kleene, who invented its name.

In general, a Σ^m_n or Π^m_n formula is an alternating string of type-m quantifiers of length n followed by a formula containing only quantifiers of type < m. Quantifiers over numbers are type-0, quantifiers over functions on **N**, sets of numbers, etc., are of type 1, quantifiers over sets of sets of numbers are of type 2, etc.

The analytical relations are not to be confused with the *analytic* relations, i.e. the Σ^1_1 relations. When Kleene first studied the analytical hierarchy, a certain class of functions had already been studied and were called "analytic"; it was only discovered later that these functions are precisely the Σ^1_1 functions. To avoid conflicting notations, the term "analytical" was chosen for the more inclusive class. Nowadays, in order to avoid confusion, the term "Σ^1_1" is generally used instead of "analytic".

Normal Form Theorems.

An arithmetical formula is a formula that does not contain any quantifiers over functions (though it may contain free function variables). We would like to show that every formula is equivalent to some Σ^1_n or Π^1_n formula (for some n), and in particular that every arithmetical formula is equivalent to some Σ^1_1 formula and to some Π^1_1 formula. At this point it should be far from obvious that this is the case, since a formula can have several number quantifiers, and a Σ^1_n or Π^1_n formula is only allowed to have a single number quantifier, and that of the opposite type from the last function quantifier. In this section we shall show how to find a Σ^1_n or Π^1_n equivalent for any formula of the language of arithmetic.

Clearly, any formula can be put into prenex form. (We consider a formula to be in prenex form if it consists of a string of unbounded quantifiers followed by a formula of Lim.) However, the initial string of quantifiers that results may not alternate, and it may also include number quantifiers. So to put the formula in the desired form, we must move the number quantifiers to the end of the string, collapse them to a single quantifier of the opposite type from the last function quantifier, and make the string of function quantifiers alternate.

First, let us work on moving the number quantifiers to the end. To do this, it suffices to show that any formula of the form (Qx)(Q'α)A is equivalent to a formula (Q'α)(Qx)A*, where Q and Q' are quantifiers and A differs from A* only in the part that is in Lim: if we have this result, then we can apply it repeatedly to any prenex formula to produce an equivalent prenex formula with all the number quantifiers at the end. This is easy to show when Q = Q': (∃x)(∃α)A is always equivalent to (∃α)(∃x)A, and (x)(α)A is always equivalent to (α)(x)A. So the only difficult case is when Q ≠ Q'.

Consider a formula of the form (x)(∃α)A. This is true just in case for every number x there is a function α_x such that A(x, α_x) holds. Letting Φ(x) = α_x, this implies that there is a function Φ such that for all x, A(x, Φ(x)) holds; conversely, if such an Φ exists, then obviously (x)(∃α)A(x, α) holds. Φ is a higher-order function, and the quantifiers in our formulae only range over functions from **N** to **N**, so we cannot rewrite (x)(∃α)A as (∃Φ)(x)A(x, Φ(x)). However, there is a way to get around this. Suppose Φ maps numbers onto functions; then let γ be the function from **N** to **N** such that γ([x, y]) = (Φ(x))(y). Let A*(x, γ) be the result of replacing all occurrences of α(t) in A by γ([x, t]), for any term t; clearly, A and A* differ only in the part that is in Lim. It is easy to see that A*(x, γ) holds iff A(x, Φ(x)) holds. Therefore, (x)(∃α)A holds iff there is a Φ such that for all x, A(x, Φ(x)) holds, iff there is a γ such that for all x, A*(x, γ) holds, iff (γ)(∃x)A*(x, γ) holds. So we have the desired result in this case.

There is only one remaining case, namely the case of formulae of the form (∃x)(α)A(x, α). But (∃x)(α)A(x, α) is equivalent to ~(x)(∃α)~A(x, α), which, as we have just seen, is equivalent to ~(∃γ)(x)~A*(x, g), which is equivalent to (γ)(∃x)A*(x, γ). So we have proved the following

Theorem: Any formula is equivalent to a prenex formula in which all the unbounded number quantifiers occur at the end.

Notice that, in moving from (x)(∃α)A to (∃Φ)(x)A(x, Φ(x)), we have assumed the axiom of choice: if the axiom of choice fails, then even though for every x there is an α such that A(x, α) holds, there may be no single function which takes x to an appropriate α.

The initial string of function quantifiers may not yet alternate. However, using the pairing function, we can collapse adjacent quantifiers of the same type into a single quantifier, and by repeating this process, we can make the initial string alternate. That is, for any formula A(α, β), let A*(γ) be a formula that differs from A only in the Lim part, and such that A(α, β) is equivalent to A*([α, β]) for all α, β. (Such an A* is easy to find.) Then (∃α)(∃β)A(α, β) is equivalent to (∃γ)A*(γ), and (α)(β)A(α, β) is equivalent to (γ)A*(γ). (Here we are assuming that our pairing function is onto.) Thus, we have the following

Theorem: Any formula is equivalent to a prenex formula consisting of an alternating string of function quantifiers followed by a first-order formula.

To get the desired result, we must show how to collapse the number quantifiers into a single quantifier. We shall do this by proving that any first-order formula is equivalent to both a Σ^1_1 and a Π^1_1 formula. Once we have done this, we can prove our main result as follows. Let A be any formula, and take any prenex equivalent with all the function quantifiers in front. Suppose the last function quantifier is existential, and let B be the first-order part of the formula. Then A is equivalent to a formula $(Q\alpha_1)...(\exists\alpha_n)B$. Now let $(\exists\alpha_{n+1})(x)C$ be a Σ^1_1 equivalent of B; A is equivalent to $(Q\alpha_1)...(\exists\alpha_n)$ $(\exists\alpha_{n+1})(x)C$. We can collapse the adjacent quantifiers $(\exists\alpha_n)$ and $(\exists\alpha_{n+1})$; thus A is equivalent to $(Q\alpha_1)...(\exists\alpha_n)(x)D$, with D in Lim, i.e. A is equivalent to a Σ^1_n formula. If the last function quantifier is universal we argue similarly, this time using a Π^1_1 equivalent of B.

Theorem: Every first-order formula is equivalent to both a Σ^1_1 and a Π^1_1 formula.
Proof: Let A be any first-order formula. We know already that we can take A to be either Σ^0_n or Π^0_n, for some n. By adding vacuous quantifiers if necessary, we can assume that A is Π^0_n for some n and that n is even. Thus, A is equivalent to a formula $(x_1)(\exists y_1)...(x_m)(\exists y_m)B$ with B in Lim. Now any formula $(x)(\exists y)C(x, y)$ is equivalent to $(\exists\alpha)(x)C(x, \alpha(x))$, as we can see using the same sort of argument we used before. (If $(\exists\alpha)(x)C(x, \alpha(x))$ holds, then obviously $(x)(\exists y)C(x, y)$ holds; conversely, if $(x)(\exists y)C(x, y)$ holds, then $(\exists\alpha)(x)C(x, \alpha(x))$ holds, letting $\alpha(x)$ = the least y such that C(x, y) holds.) Iterating this, and moving the number quantifiers to the end, we see that A is equivalent to $(\exists\alpha_1)...(\exists\alpha_m)(x_1)...(x_m)B'$ for B' in Lim. We can collapse the existential function quantifiers, and we can also collapse the universal number quantifiers using a bounding trick. The result is Σ^1_1, so A is equivalent to a Σ^1_1 formula.

To see that A is also equivalent to a Π^1_1 formula, notice that the foregoing argument shows that the formula ~A is equivalent to some Σ^1_1 formula $(\exists\alpha)(x)B$, and so A itself is equivalent to the Π^1_1 formula $(\alpha)(\exists x)$~B.

By the foregoing remarks, we finally have our main result.

Theorem: Every formula is equivalent to some Π^1_n or Σ^1_n formula, for some n. Moreover, if A is a formula consisting of an alternating string of quantifiers of length n, the first quantifier of which is existential (universal), followed by a first order formula, then A is equivalent to a Σ^1_n (Π^1_n) formula.

(The trick of replacing $(x)(\exists y)C(x, y)$ by $(\exists\alpha)(x)C(x, \alpha(x))$ is due to Skolem. Notice that, in contrast to the previous case, we have not assumed the axiom of choice, since we defined $\alpha(x)$ to be the least y such that C(x, y). We were able to do this because we know that our domain (viz. **N**) can be well-ordered. Skolem's trick can be applied to any domain that can be well-ordered; however, if the axiom of choice fails, then there will be domains that cannot be well-ordered.)

As with the arithmetical hierarchy, we can define the *level* of an analytical relation to be the least inclusive Σ^1_n, Π^1_n, or Δ^1_n of which it is an element. The above discussion gives us ways of estimating the level of a given analytical relation.

All arithmetical relations are Δ^1_1, as we have seen. Moreover, if A is a Σ^1_n formula, then $(\exists\alpha)A$ is equivalent to a Σ^1_n formula since we can collapse $(\exists\alpha)$ with A's initial quantifier; similarly, if A is a Π^1_n formula, then $(\alpha)A$ is equivalent to a Π^1_n formula. In short, the Σ^1_n and

Π^1_n relations are closed under existential and universal functional quantification, respectively. Similarly, if A is Σ^1_n, then so are both (x)A and (∃x)A, and the same is true if A is Π^1_n. This is because, as we have seen, we can always move number quantifiers inwards without affecting the variable quantifiers.

It is also not hard to see that if A and B are Σ^1_n (or Π^1_n), then so are A ∧ B and A ∨ B. We can show this by induction on n. Since $\Sigma^1_0 = \Pi^1_0$ = {arithmetical relations}, this clearly holds for n = 0. Suppose it holds for n. If A and B are Σ^1_{n+1}, then they are (∃α)C and (∃β)D for Π^1_n formula C and D. Then A ∧ B and A ∨ B are equivalent to (∃α)(∃β)(C ∧ D) and (∃α)(∃β)(C ∨ D), respectively, which are Σ^1_{n+1}, by the inductive hypothesis and collapsing the quantifiers (∃α) and (∃β). If A and B are Π^1_{n+1}, we argue similarly.

Thus, the situation is similar to that of the arithmetical hierarchy, except that function quantifiers and unbounded number quantifiers play the role here that bounded and unbounded number quantifiers play in the arithmetical case. Using a similar argument to the one we gave there, we can see that if a relation is enumeration reducible to some Σ^1_n (resp. Π^1_n) relations, then it is Σ^1_n (resp. Π^1_n). It follows immediately that anything r.e. in a Δ^1_n relation is itself Δ^1_n; *a fortiori*, anything recursive in a Δ^1_n relation is Δ^1_n.

Exercise

1. Recall that A and B are *recursively isomorphic* (A ≡ B) iff there is a 1-1 total recursive function φ whose range is **N**, and such that B = {φ(x): x ∈ A}. Show that for all A and B, A ≡ B iff A $\equiv_1$ B. The following sketches a method of proof. If A ≡ B, then A $\equiv_1$ B follows easily, so suppose A $\equiv_1$ B. Let φ and ψ be 1-1 recursive functions such that x ∈ A iff φ(x) ∈ B and x ∈ B iff ψ(x) ∈ A, all x. Define, a sequence $a_1, a_2, \ldots$ and a sequence $b_1, b_2, \ldots$, as follows. Suppose $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ have been defined (where possibly n = 0). If n is even, then let a_{n+1} be the least number distinct from $a_1, \ldots, a_n$, and let b_{n+1} be such that a_{n+1} ∈ A iff b_{n+1} ∈ B and b_{n+1} is distinct from all of $b_1, \ldots, b_n$. If n is odd, do the same thing in reverse (i.e. let b_{n+1} be the least number distinct from $b_1, \ldots, b_n$, etc.). Moreover, do this in such a way that the function χ such that $\chi(a_n) = b_n$ for all n ∈ **N** is recursive. Conclude that χ is a 1-1 total recursive function whose range is **N**, and such that for all x, x ∈ A iff χ(x) ∈ B, and therefore that A ≡ B. Hint: Informally, the problem reduces to finding an appropriate b_{n+1} *effectively* from $a_1, \ldots, a_n, a_{n+1}$ and $b_1, \ldots, b_n$ (or a_{n+1} from $b_1, \ldots, b_{n+1}$ and $a_1, \ldots, a_n$, if n is odd). If $\phi(a_n) \notin \{b_1, \ldots, b_n\}$, then we can put $b_{n+1} = \phi(a_n)$. However, we may have $\phi(a_n) = b_i$ for some i = 1, ..., n; show how to get around this.

A *recursive isomorphism type* is a ≡-equivalence class. Conclude that 1-degrees are therefore recursive isomorphism types, and that there is a 1-degree (which is also an m-degree and a recursive isomorphism type) which consists of the creative sets.

Comment: Dekker proposed that the notions studied by recursion theory should all be invariant under recursive isomorphism. While all the notions studied in this course are invariant under recursive isomorphism, there is at least one notion, that of a *retraceable* set, which is not so invariant and which has been studied by recursion theorists. (Offhand, I don't know whether this notion was proved to be not recursively invariant before Dekker's proposal or only afterwards.)

Lecture XXIII

Relative Σ's and Π's.

The absolute notions Σ^0_n, Π^0_n, Σ^1_n, and Π^1_n can be relativized, just as we relativized the notions of recursiveness and recursive enumerability earlier. Let us say that a set is Σ^0_n *in* the unary functions $\alpha_1, ..., \alpha_n$ if it is definable by a Σ^0_n formula of the language of arithmetic with extra function symbols for the functions $\alpha_1, ..., \alpha_n$, and similarly for the notions Π^0_n, Σ^1_n, and Π^1_n in $\alpha_1, ..., \alpha_n$. So in particular, a relation between numbers, is Σ^0_1 in β (Δ^0_1 in β) just in case it is r.e. in β (recursive in β).

Another way of looking at this is as follows. Consider an arbitrary formula A($x_1, ..., x_n, y_1, ..., y_m, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q$) of the language of arithmetic (possibly with function quantifiers), where the x's and y's are free number variables and the α's and β's are free function variables. The formula A defines an m+n+p+q-place relation, with m+n places for numbers and p+q places for functions. (Of course, any of m, n, p, and q may be 0.) Now suppose we regard the y's and β's as having *fixed* values (the numbers $k_1, ..., k_m$ and the functions $f_1, ..., f_q$, say). Relative to these fixed values, A defines an n+p-place relation. In the case of the fixed number values, we can get the same effect by considering the formula A* in which each variable y_i is replaced by the numeral $\mathbf{0}^{(k_i)}$; however, we cannot treat functions in the same way, since we do not have a term in the language for each function. (In fact, as long as we only have countably many terms in the language, we cannot have a term for each function, since there are uncountably many functions.) Ignoring the y's and k's, then, if the relation defined by A (with the β's treated as variables) is Σ^0_n, then the relation defined by A with the values of the β's fixed will be Σ^0_n in $f_1, ..., f_n$ (and similarly for Π^0_n, Σ^1_n, and Π^1_n).

Equivalently, an n+p-place relation R is Σ^0_n (or Π^0_n, etc.) in $\beta_1, ..., \beta_q$ iff there is an n+p+q-place Σ^0_n (or Π^0_n, etc.) relation R' such that $R = \{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: <x_1, ..., x_n, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q> \in R'\}$. Thus, we can characterize the relative notions directly in terms of the corresponding absolute notions.

As with our other relative notions, we can reduce the general case to the case q = 1, this time using a pairing function on *functions*. There are several pairing functions that we could use. For example, we could take $[\beta_1, \beta_2]$ to be the function β such that $\beta(m) = [\beta_1(m), \beta_2(m)]$; alternatively, we could take it to be the function β such that $\beta(2n) = \beta_1(n)$ and $\beta(2n+1) = \beta_2(n)$ for all n. (The latter has the advantage of being an *onto* pairing function.) It is easy to verify that this successfully reduces the general case to the case of a single function.

We say that a relation R is $\mathbf{S}^0_n$ iff there is a function β such that R is Σ^0_n in β. We define $\mathbf{P}^0_n$, $\mathbf{S}^1_n$, and $\mathbf{P}^1_n$ similarly, and **D** is defined in the usual way. (So boldface letters are used for the notions with function parameters, lightface letters for the notions without function parameters.) This notion is not very interesting if R is a relation between numbers, since in that case R will *always* be $\mathbf{D}_1^0$ (since R is always Δ_1^0 in its characteristic function). However, this is not the case if some of R's places are occupied by functions (i.e. if p > 0).

Let's look at $\mathbf{S}_1^0$ in the case n = 0 and p = 1 (i.e. the case of $\mathbf{S}_1^0$ sets of functions). A set S is $\mathbf{S}_1^0$ iff there is a 2-place Σ_1^0 relation R and a function β such that S = {α: <α, β> ∈ R}. We can also characterize the $\mathbf{S}_1^0$ sets topologically. *Baire space* is the topological space whose points are total functions from **N** to **N** and whose open sets are those sets S such that for every function φ in S, there is a finite initial segment s of φ such that every function extending s is also in S. To verify that this is indeed a topological space, we must show that if two sets satisfy our characterization of the open sets then their intersection does as well, and that if F is a family of sets satisfying that characterization then ∪F also satisfies it. Alternatively, we can characterize Baire space as follows. For any finite sequence s, let O_s = {φ: φ is a total function which extends s}; then the sets of the form O_s form a basis for Baire space.

Theorem: The $\mathbf{S}_1^0$ sets are precisely the open sets of Baire space.
Proof: First, suppose S is $\mathbf{S}_1^0$. Then there is a 2-place Σ_1^0 relation R between functions and a particular function β such that S = {α: <α, β> ∈ R}. Suppose α ∈ S, i.e. <α, β> ∈ R. By the monotonicity and finiteness properties of Σ_1^0 relations, there is an initial segment s of α such that <γ, β> ∈ R for all γ extending s, and therefore γ ∈ S for all such γ. Since α was arbitrary, it follows that S is open.

Next, suppose S is open. Let F = {s: α ∈ S for all α extending s}. Then S = {α: α extends s for some s ∈ F}. Since F is a collection of finite sequences, we can let G = {n ∈ **N**: n codes some element of F}, and let γ be G's characteristic function. Then S is Σ_1^0 in γ, and therefore $\mathbf{S}_1^0$, since we can define S by the Σ_1^0 formula (∃s)(γ(s) = **0'** ∧ s ⊆ α). (Here s ⊆ α abbreviates the formula (n < s)(m < s)([n, m] ∈ s £ α(n) = m).)

Baire space is also homeomorphic to the irrational numbers under the usual topology. The onto pairing function mentioned earlier is a homeomorphism between Baire space and its direct product with itself; since Baire space is homeomorphic to the irrationals, this shows that the irrational plane is homeomorphic to the irrational line. Thus, the situation is very different from the case of the reals.

We can set up a similar topology on sets of natural numbers by identifying these sets with their characteristic functions; if we restrict Baire space to functions into {0, 1}, the result is a space which is homeomorphic to the Cantor set. (That is, the set of all reals in the interval [0, 1] whose base-3 expansions contain no 1's.) It is also identical to the space $\mathbf{2}^\omega$, where **2** is the space {0, 1} with the discrete topology.

Notice that since the $\mathbf{S}_1^0$ sets are precisely the open sets, the $\mathbf{D}_1^0$ sets are precisely the clopen sets (i.e. sets that are both closed and open). This is another difference between the reals and the rationals: whereas the only clopen subsets of **R** are **R** itself and Ø, clopen sets of irrationals exist in great abundance.

Another Normal Form Theorem.

Given a function α, let us define $\overline{\alpha}(n)$ to be some numerical code for the sequence <α(0), ..., α(n-1)>. It doesn't matter what particular code we choose; however, for definiteness, let us say that $\overline{\alpha}(n) = 2^{\alpha(0)+1} \cdot 3^{\alpha(1)+1} \cdot \ldots \cdot p_n^{\alpha(n-1)+1}$, where in general p_n is the nth prime. (This is essentially the coding scheme Gödel used.) As Quine has remarked, coding systems are not like matrimony, and we are free to switch back and forth between them as we please.

We now prove another normal form theorem, due to Kleene.

Theorem: If S is an n+p-place Σ^0_1 relation, then there is an n+p-place recursive relation R such that S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_n(z))\}$.

Proof: We shall prove the theorem for the case n = 0 and p = 1; the other cases are similar. Let S be a Σ^0_1 set of functions. For some relation L(α, y) definable in Lim, S = {α: (∃y)L(α, y)}. By monotonicity and finiteness, α ∈ S iff some initial segment of α is in S, so S = $\{\alpha: (\exists z)(\exists y)L(\overline{\alpha}(z), y)\}$. In fact, S = $\{\alpha: (\exists z)(\exists y < z)L(\overline{\alpha}(z), y)\}$: if $(\exists z)(\exists y < z)L(\overline{\alpha}(z), y)$ then certainly $(\exists z)(\exists y)L(\overline{\alpha}(z), y)$, and if $L(\overline{\alpha}(k), y)$, then let z > k, y; $L(\overline{\alpha}(z), y)$ by monotonicity, so $(\exists z)(\exists y < z)L(\overline{\alpha}(z), y)$. Let R'(z, s) ≡ (∃y < z)L(s, z): R' is a recursive relation, and S = $\{\alpha: (\exists z)R'(z, \overline{\alpha}(z))\}$. This is almost what we want. Let R(s) ≡ R'(lh(s), s), where lh(s) is the length of the sequence s; R is still recursive, and S = $\{\alpha: (\exists z)R(\overline{\alpha}(z))\}$.

This gives us a new normal form theorem for Π^1_1 relations.

Theorem: Every n+p-place Π^1_1 relation is $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\beta)(\exists z)R(x_1, ..., x_n, \overline{\alpha})_1(z), ..., \overline{\alpha}_p(z), \overline{\beta}(z))\}$ for some recursive relation R.

Proof: Let S be any n+p-place Π^1_1 relation. Then S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\beta)T(x_1, ..., x_n, \alpha_1, ..., \alpha_p, \beta)\}$ for some n+p+1-place Σ^0_1 relation T. By what we just proved, there is a recursive relation R such that $T(x_1, ..., x_n, \alpha_1, ..., \alpha_p, \beta)$ iff $(\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z), \overline{\beta}(z))$; it follows that S is $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\beta)(\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z), \overline{\beta})(z))\}$.

We can prove similar normal form theorems for the other Σ^1_n's and Π^1_n's. The main thing to note is that we have, so to speak, reduced the relation S, which may involve *functions*, to R, a recursive relation among *numbers*.

There is a related result about the various **S**'s and **P**'s.

Theorem: An n+p-place relation S is $\mathbf{S}^0_1$ iff for some β and some recursive R, S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z), \overline{\beta}(z))\}$, iff for some n+p-place relation R on **N** (not necessarily recursive), S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z))\}$.

Proof: The equivalence of the first two conditions is immediate. Suppose S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z), \overline{\alpha}(z))\}$, and let R' be the relation $\{< x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z)>: z \in$ **N** and $R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z), \overline{\beta}(z))\}$; then S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R'(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z))\}$. Conversely, suppose S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)R(x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z))\}$ for some relation R on **N**. Let β be the characteristic function of the set $\{[x_1, ..., x_n, y_1, ..., y_p]: R(x_1, ..., x_n, y_1, ..., y_p)\}$. Then S = $\{<x_1, ..., x_n, \alpha_1, ..., \alpha_p>: (\exists z)\ \beta([x_1, ..., x_n, \overline{\alpha}_1(z), ..., \overline{\alpha}_p(z)]) = 1\}$, so S is Σ^0_1 in β and is therefore $\mathbf{S}^0_1$.

Similar results hold for the other **S**'s and **P**'s.

The Hyperarithmetical Hierarchy.

Consider the hierarchy 0, 0', 0'', ..., $0^{(n)}$, ... of degrees. As we have seen, a set is arithmetical just in case it is recursive in one of these degrees. We also know that not all sets are arithmetical (e.g. the set of true sentences of the language of arithmetic), so there are sets which are not recursive in any of these degrees; therefore, there is a degree d which is not $\leq 0^{(n)}$ for any n. In fact, there are degrees d such that $0^{(n)} < d$ for all n: it is not too hard to see that the degree of the set of true sentences is such a degree. This suggests that we should be able to extend the hierarchy 0, 0', 0'', ..., $0^{(n)}$, ... into the transfinite in some way.

In particular, it suggests that there ought to be a natural next degree, which we can call $0^{(\omega)}$, beyond all of the degrees $0^{(n)}$. But what is $0^{(\omega)}$? A natural answer would be that $0^{(\omega)}$ is the least upper bound of the degrees 0, 0', 0'', ... However, by a result due to Spector, that collection of degrees does not have a least upper bound; so the most natural characterization of $0^{(\omega)}$ will not work.

However, the situation is not quite as bad as it first appears. While there is no least degree beyond 0, 0', 0'', ..., there *is* a least degree a such that a = d'' for some d > 0, 0', ... (This result is due to Enderton, Putnam and Sacks.) We can define $0^{(\omega)}$ to be the degree a. In fact, $0^{(\omega)}$ is the degree of the set of true sentences of the language of arithmetic.

We can use this idea to extend the hierarchy still further. In general, we say that a set is hyperarithmetical if it is recursive in $0^{(\alpha)}$ for some ordinal α for which $0^{(\alpha)}$ is defined. We can define the degrees $0^{(\omega+1)}$, $0^{(\omega+2)}$, etc. by $0^{(\omega+1)} = 0^{(\omega)\prime}$, $0^{(\omega+2)} = 0^{(\omega+1)\prime}$, etc.; in general, if $0^{(\alpha)}$ has been defined, we can define $0^{(\alpha+1)}$ to be $0^{(\alpha)\prime}$. We can define the next degree beyond all these, namely $0^{(\omega+\omega)}$, similarly to the way we defined $0^{(\omega)}$: there is a least degree a such that a = d'' for some d > $0^{(\omega)}$, $0^{(\omega+1)}$, ..., and we can define $0^{(\omega+\omega)}$ to be that degree a. In fact, we can use this technique to define $0^{(\alpha)}$ for quite an extensive class of ordinals (known as the *recursive ordinals*).

The resulting extended hierarchy is called the *hyperarithmetical hierarchy*. The hyperarithmetical hierarchy was first studied by Martin Davis in his Ph.D. thesis at Princeton. It was also invented independently by Mostowski and by Kleene (who coined the expression "hyperarithmetical"). Most of the basic theorems about the hierarchy were proved by Kleene and Spector.

Another approach is to define the *set* $\varnothing^{(\alpha)}$, for suitable a, and then let $0^{(\alpha)}$ be the degree of $\varnothing^{(\alpha)}$. On this approach, we can let $\varnothing^{(\omega)}$ be $\{[m, n]: m \in \varnothing^{(n)}\}$; the degree of $\varnothing^{(\omega)}$ is then $0^{(\omega)}$ as we defined it before. Obviously, this can be carried further into the transfinite. For example, we could let $\varnothing^{(\alpha+1)} = \varnothing^{(\alpha)'}$ whenever $\varnothing^{(\alpha)}$ is defined, and we could define $\varnothing^{(\omega+\omega)}$, for example, to be the set $\{[m, n]: m \in \varnothing^{(\omega+n)}\}$. To define $\varnothing^{(\omega^2)}$, we can do essentially the same thing, except this time it's a bit trickier: let $\varnothing^{(\omega^2)} = \{[m, n]: m \in \varnothing^{(\omega.n)}\}$. We could continue in this manner for quite some time, thinking of new definitions of $\varnothing^{(\alpha)}$ for limit ordinals α as we need them, but we would like to give a uniform definition of $\varnothing^{(\alpha)}$ for all of the appropriate α. We do so as follows.

An ordinal is said to be *recursive* if it is the order type of some recursive well-ordering of **N**. For example, ω is recursive because it is the order type of <0, 1, 2, ...>, and $\omega+\omega$ is recursive because it is the order type of <0, 2, 4, ..., 1, 3, 5, ...>. The recursive ordinals go up quite far. Of course, not every ordinal is recursive, since every recursive ordinal is countable but not every ordinal is countable. In fact, not all countable ordinals are recursive: since there are only countable many recursive well-orderings, there are only countably many recursive ordinals, but there are uncountably many countable ordinals. Once we have fixed a recursive well-ordering R, individual natural numbers code the ordinals less than the order type of R: specifically, we let $|n|_R$ denote the order type of the set {m: m R n} ordered by R. (So m R n iff $|m|_R < |n|_R$.)

Let S be an arbitrary recursive set, and let R be an arbitrary recursive well-ordering. We define H_n as follows, for all n. If $|n|_R = 0$, then $H_n = S$. If $|n|_R = \alpha+1$ and $|m|_R = \alpha$, then $H_n = (H_m)'$. Finally, if $|n|_R$ is a limit ordinal, let $H_n = \{[x, y]: x \in H_y \text{ and } x R y\}$. A set is said to be *hyperarithmetical* if it is recursive in H_n, for some n and some choice of R and S. (This definition is quite close to the definitions of Kleene and Spector.)

Now it might seem as though H_n depends strongly on the choice of S and of R. However, this is not really the case. Suppose R and R' are recursive well-orderings of the same order type, and S, S' are any two recursive sets; then whenever $|m|_R = |n|_{R'}$, H_m and H_n' are of the same Turing degree (where H_n' is H_n defined in terms of R' and S' rather than R and S). (The proof of this is due to Spector. The proof, by the way, is a nice illustration of the use of the recursion theorem in the study of recursive ordinals.) Thus, we may define $0^{(\alpha)}$ to be the degree of H_n, where $\alpha = |n|_R$, for any recursive ordinal α.

If R is allowed to be arithmetical, or even hyperarithmetical, then the order type of R is still a recursive ordinal; that is, while R may not itself be recursive, there is a recursive well-ordering R' which is isomorphic to R. Moreover, if R and R' are allowed to be arithmetical, then H_m and H_n' are still of the same Turing degree, so the hierarchy is unaffected. If R is allowed to be hyperarithmetical, then the same sets get into the hierarchy, but the hierarchy may go up at a different rate.

The characterization of the hyperarithmetical sets that we have just given is *invariant*, in that while it involves S and R, which are extraneous to the hierarchy itself, the same hierarchy is given by any choice of S and R. A characterization in terms of double jumps (sketched at the beginning of this section), on the other hand, is *intrinsic* in the sense that such extraneous entities are not involved at all. This is certainly a virtue of the latter approach, although it relies on a rather more advanced result than the former approach, namely that for suitable sequences $a_1 < a_2 < ...$ of degrees there is a least d" such that d > a_1, a_2, ...

Another characterization of the hyperarithmetical sets is as follows. Consider those sequences <S_α: α a recursive ordinal> such that S_0 is recursive, $S_{\alpha+1} = S_\alpha'$ for all α, and when α is a limit ordinal, S_α is an upper bound of $\{S_\beta: \beta < \alpha\}$. (It doesn't matter which upper bound we choose.) Then a set will be hyperarithmetical just in case for *every* such sequence, it is recursive in some set in the sequence. There are many other equivalent characterizations of the hyperarithmetical sets.

Lecture XXIV

Hyperarithmetical and Δ^1_1 sets.

An important theorem about the hyperarithmetical sets, due to Kleene, is that they are all Δ^1_1. An even more important theorem, also due to Kleene (and whose proof is more difficult), is the converse. Thus, we have yet another characterization of the hyperarithmetical sets, this time in terms of the analytical hierarchy.

We shall prove the easier half of this theorem. In fact, we shall prove a somewhat stronger result. Let us say that a function ϕ is the *unique solution* of a formula $A(\alpha)$ if ϕ satisfies $A(\alpha)$ and is the only function that does so.

Theorem: If the characteristic function of a set S is the unique solution of an arithmetical formula, then S is Δ^1_1.
Proof: Let ϕ be the characteristic function of S, and let $A(\alpha)$ be an arithmetical formula of which ϕ is the unique solution. Then S is defined by the formula $(\exists\alpha)(A(\alpha) \wedge \alpha(x) = \mathbf{0}')$ and also by the formula $(\alpha)(A(\alpha) \supset \alpha(x) = \mathbf{0}')$. Since both $A(\alpha) \wedge \alpha(x) = \mathbf{0}'$ and $A(\alpha) \supset \alpha(x) = \mathbf{0}'$ are arithmetical formulae, the two formulae that define S are equivalent to Σ^1_1 and Π^1_1 formulae, respectively.

Notice that this argument goes through under the weaker assumption that $A(\alpha)$ is a Σ^1_1 formula.

Suppose S is hyperarithmetical; then there is a recursive well-ordering R of **N** such that S is recursive in H_n for some n, where $H_n = \emptyset$ when $|n|_R = 0$, $H_n = H_m'$ when $|n| = |m|_R+1$, and $H_n = \{[x, y]: x \in H_y \text{ and } y R n\}$ when $|n|_R$ is a limit ordinal. Let ψ be the characteristic function of $\{[m, n]: m \in H_n\}$. If ψ is the unique solution to some arithmetical formula, then that set is Δ^1_1. It follows easily (by the reasoning of the last section) that each H_n is Δ^1_1, so S is recursive in a Δ^1_1 set and is therefore itself Δ^1_1. Therefore, we need only find an arithmetical formula of which ψ is the unique solution.

Since R is r.e., there is an arithmetical formula B(x, y) that defines R, and let k be the R-least element of N. Define

$$\text{Zero}(n) =_{\text{df.}} n = \mathbf{0}^{(k)}$$
$$\text{Succ}(m, n) =_{\text{df.}} B(m, n) \wedge (y){\sim}(B(m, y) \wedge B(y, n))$$
$$\text{Limit}(n) =_{\text{df.}} {\sim}\text{Zero}(n) \wedge {\sim}(\exists m)\text{Succ}(m, n).$$

These formulae hold just in case $|n|_R = 0$, $|n|_R = |m|_R+1$, and $|n|_R$ is a limit ordinal, respectively. Next, define

$$\text{Jump}(m, n, \alpha) =_{df.} (z)(\alpha([z, n]) = \mathbf{0'} \equiv \quad (\exists e)(\exists m)(z = [e, m] \wedge (\exists s)[W(e, m, s) \wedge$$
$$(k)((\mathbf{0}^{(2k)} \in s \supset \alpha([k, m]) = \mathbf{0'}) \wedge$$
$$(\mathbf{0}^{(2k)}+\mathbf{0'} \in s \supset \alpha([k, m]) = \mathbf{0})))]))$$

Let ϕ be a function into {0, 1} and let M and N be the sets {x: $\phi([x, m]) = 1$} and {x: $\phi([x, n]) = 1$}, respectively. With a little work, we can see that Jump(m, n, ϕ) holds iff N = M'. We are now ready to define A(α): let A(α) be the formula

$$(x)[(\alpha(x) = \mathbf{0} \vee \alpha(x) = \mathbf{0'}) \wedge (\alpha(x) = \mathbf{0'} \supset (\exists y)(\exists n)\ x = [y, n]) \wedge$$
$$(y)(n)(x = [y, n] \supset$$
$$\{\text{Zero}(n) \supset \alpha(x) = \mathbf{0} \wedge$$
$$\text{Limit}(n) \supset (\alpha(x) = \mathbf{0'} \equiv (\alpha(y) = \mathbf{0'} \wedge B(K_2(y), n))) \wedge$$
$$(m)(\text{Succ}(m, n) \supset \text{Jump}(m, n, \alpha))\})].$$

Now let us verify that ψ is the unique solution of A(α). First, we show that ψ satisfies A(α). ψ is a function into {0, 1} which only takes the value 1 on arguments that code pairs, so the first line of the formula is satisfied. Let x = [y, n] be given. If $|n|_R = 0$, then $H_n = \emptyset$, so $y \notin H_n$ and $\psi([y, n]) = 0$, so the third line is satisfied. If $|n|_R$ is a limit ordinal, then $H_n = \{[z, w]: z \in H_w \text{ and } w\ R\ n\} = \{u: \psi(u) = 1 \text{ and } K_2(u)\ R\ n\}$, so the fourth line holds. Finally, if $|n|_R = |m|_R+1$, then $H_n = H_m'$, so the last line holds as well.

Conversely, suppose ϕ satisfies A(α). Then Range(ϕ) $\subseteq$ {0, 1}, and $\phi(x) = 0$ when x is a nonpair. Let $G_n = \{y: \phi([y, n]) = 1\}$ for all n; we will show by transfinite induction on $|n|_R$ that $G_n = H_n$, from which it follows that $\phi = \psi$. If $|n|_R = 0$, then $\phi([y, n]) = 0$ for all y, so $G_n = \emptyset = H_n$. If $|n|_R = |m|_R+1$, then Jump(m, n, ϕ) holds and $G_n = G_m'$; by the inductive hypothesis, $G_m = H_m$, so $G_n = H_m' = H_n$. Finally, if $|n|_R$ is a limit ordinal, then $G_n = \{[z, w]: \phi([z, w]) = 1 \text{ and } |w|_R < |n|_R\} = \{[z, w]: z \in G_w \text{ and } |w|_R < |n|_R\}$ = (by the inductive hypothesis) $\{[z, w]: z \in H_w \text{ and } |w|_R < |n|_R\} = H_n$. This completes the proof.

The definition of A(α) is complicated, but the idea is simple. The sequence $\langle H_n: n \in \mathbf{N}\rangle$ is defined in terms of itself; specifically, each H_n is defined in terms of various H_m for $|m|_R < |n|_R$. So we can define the function ψ in terms of itself in a similar way; if we do things right, the result will be an arithmetical formula A(α) with ψ as its unique solution.

$\mathbf{S}^1_n$ and $\mathbf{P}^1_n$ sets of functions are called *projective*, and the $\mathbf{S}^1_n$-$\mathbf{P}^1_n$ hierarchy is called the *projective hierarchy*. The study of the projective hierarchy and related notions is called *descriptive set theory*. Projective sets were studied years before Kleene studied the analytical hierarchy, and Suslin proved an analog of Kleene's result that Δ^1_1 = hyperarithmetical. (Specifically, he showed that the Borel sets are precisely the $\mathbf{D}^1_1$ sets.) A unified result, of which the results of Suslin and Kleene are special cases, is called the *Suslin-Kleene theorem*.

Kleene was originally unaware of this earlier work on projective sets. People then noticed analogies between this work and that of Kleene; later on, it was seen that not only are the theories of projective sets and analytical sets analogous: in fact, they are really part of the same theory. Kleene originally called the analytical sets "analytic"; unfortunately, "analytic" was already a term of descriptive set theory for the $\mathbf{S}^1_1$ sets. To avoid confusion, Kleene's term was replaced by "analytical". Nowadays, to avoid confusion, most people say "$\mathbf{S}^1_1$" instead of "analytic".

<u>Borel Sets.</u>

The *Borel sets* are defined as follows: all open sets of Baire space are Borel; the complement of a Borel set is Borel; and if $<S_n: n \in \mathbf{N}>$ is any countable sequence of Borel sets, then $\cup_n S_n$ is also Borel. (It follows that $\cap_n S_n$ is Borel, since $\cap_n S_n = -\cup_n -S_n$.)

The Borel sets form a hierarchy, called the *Borel hierarchy*, defined as follows. The first level consists of the open sets and the closed sets, that is, the $\mathbf{S}^0_1$ sets and the $\mathbf{P}^0_1$ sets. The next level consists of countable unions of closed sets and countable intersections of open sets, or in other words, the $\mathbf{S}^0_2$ and $\mathbf{P}^0_2$ sets. (Countable unions of open sets are already open, and countable intersections of closed sets are already closed.) We can see that the $\mathbf{S}^0_2$ sets are precisely the countable unions of closed sets, as follows. We know already that the $\mathbf{P}^0_1$ sets are precisely the closed sets. On the one hand, suppose S is $\mathbf{S}^0_2$; then S is $\{\alpha: (\exists x)(y)R(x, y, \alpha, \beta)\}$ for some fixed β and some Π^0_1 relation R. For each n, let $S_n = \{\alpha: (y)R(n, y, \alpha, \beta)\}$; then $S = \cup_n S_n$, and each S_n is $\mathbf{P}^0_1$ and therefore closed, so S is a countable union of closed sets. Conversely, suppose $S = \cup_n S_n$, where each S_n is closed and therefore $\mathbf{P}^0_1$. For each n, $S_n = \{\alpha: (y)\ \overline{\alpha}(y) \in X_n\}$ for some set X_n of numbers, by our normal form theorem for $\mathbf{P}^0_1$. Let R be the relation $\{<x, n>: x \in X_n\}$; then $\alpha \in \cup_n S_n$ iff $(\exists n)(y)R(\overline{\alpha}(y), n)$, so $\cup_n S_n$ is $\mathbf{S}^0_2$. So the $\mathbf{S}^0_2$ sets are precisely the countable unions of closed sets, from which it follows that the $\mathbf{P}^0_2$ sets are precisely the countable intersections of open sets. In general, the $\mathbf{S}^0_n$ sets are the countable unions of $\mathbf{P}^0_{n-1}$ sets and the $\mathbf{P}^0_n$ sets are the countable intersections of $\mathbf{S}^0_{n-1}$ sets, by the same argument.

The various $\mathbf{S}^0_n$'s and $\mathbf{P}^0_n$'s do not exhaust the Borel hierarchy: we can find a countable collection of sets which contains sets from arbitrarily high finite levels of the hierarchy, and whose union does not occur in any of these finite levels. We therefore need another level beyond these finite levels. Let us call a set $\mathbf{S}^0_\omega$ if it is a countable union of sets, each of which is $\mathbf{S}^0_n$ for some n, and $\mathbf{P}^0_\omega$ if it is a countable intersection of sets, each of which is $\mathbf{P}^0_n$ for some n. In general, for countable infinite ordinals α we define a set to be $\mathbf{S}^0_\alpha$ if it is the union of a countable collection of sets, each of which is $\mathbf{P}^0_\beta$ for some $\beta < \alpha$, and $\mathbf{P}^0_\alpha$ if it is the intersection of a countable collection of sets, each of which is $\mathbf{S}^0_\beta$ for some $\beta < \alpha$. It turns out that new Borel sets appear at each level of this hierarchy. On the other hand, it is easy to see that every Borel set appears eventually in the hierarchy. For suppose not: then there is some countable family F of sets in the hierarchy such that $\cup F$ is not in the hierarchy. For each $S \in F$, let rank(S) = the least ordinal α such that $S \in \mathbf{P}^0_\alpha$. Then {rank(S): S ∈ F} is a countable collection of countable ordinals, and it therefore has a countable upper bound α. But then ∪S ∈ $\mathbf{S}^0_\alpha$.

Notice that there are two equivalent ways to characterize at least the finite levels of the Borel hierarchy. One is purely topological: the $\mathbf{S}^0_1$ sets are the open sets, the $\mathbf{P}^0_1$ sets are the closed sets, the $\mathbf{S}^0_2$ sets are countable unions of closed sets, the $\mathbf{P}^0_2$ sets are countable intersections of open sets, etc. This is the way the Borel hierarchy was originally conceived, before analogies with recursion theory were noticed. The other is in terms of definability: a set is $\mathbf{S}^0_1$ iff it is definable by a $\mathbf{S}^0_1$ formula with a single function parameter, etc. The **S**, **P** notation was borrowed from recursion theory; the original notation (still quite standard outside of logic) was more baroque. Countable unions of closed sets were called F_σ, countable intersections of open sets were called G_δ, countable unions of G_δ's were called $G_{\delta\sigma}$, etc.

It is fairly easy to show that all Borel sets are $\mathbf{D}^1_1$. To prove this, it suffices to show that all open sets are $\mathbf{D}^1_1$, and that $\mathbf{D}^1_1$ is closed under complements and countable unions. That $\mathbf{D}^1_1$ is closed under complements is immediate from its definition. Suppose S is an open set; then S is {α: R(α, β)} for some fixed β and some Σ^0_1 relation R; we know already that any Σ^0_1 relation is Δ^1_1, so S is $\mathbf{D}^1_1$. Finally, suppose {S_n: n ∈ **N**} is a countable family of $\mathbf{D}^1_1$ sets. In particular, each S_n is $\mathbf{P}^1_1$. Each S_n is {α: (β)(∃x)$R_n(\overline{\alpha}(x), \overline{\beta}(x))$} for some relation R_n on **N**. Let R be the relation {<y, z, n>: R_n(y, z)}; $\cup_n S_n$ = {α: (∃n)(β)(∃x)$R(\overline{\alpha}(x), \overline{\beta}(x)$, n))}. But we know already that the $\mathbf{P}^1_1$ relations are closed under number quantification, so $\cup_n S_n$ is $\mathbf{P}^1_1$. The proof that $\cup_n S_n$ is $\mathbf{S}^1_1$ is similar.

Borel sets are analogous in a number of ways to the hyperarithmetical sets. In particular, we can imitate the Borel hierarchy in the case of sets of numbers. It would not do to have the family of sets be closed under countable unions, since then as long as every singleton is included, every set whatsoever will be included. However, if we replace unions with recursive unions, we can get around this difficulty. Specifically, we can set up a system of notations for sets of numbers as follows. Let [0, m] code the set {m}; if n codes a set S, let [1, n] code the set -S; finally, if every element of W_e is already the code of some set, let [2, e] code the set ∪{S: S is coded by some element of W_e}. We might call the sets that receive codes under this scheme the *effective Borel sets*, and the hierarchy that they form the *effective Borel hierarchy*. It turns out that the effective Borel sets are precisely the hyperarithmetical sets.

Π^1_1 Sets and Gödel's Theorem.

It turns out that there are close analogies between the Π^1_1 sets and the recursively enumerable sets (and also between the Δ^1_1 sets and the recursive sets). For example, consider the following extension of the notion of a computation procedure. We can consider, if only as a mathematical abstraction, machines which are capable of performing infinitely many operations in a finite amount of time. (For example, such a machine might take one second to perform the first operation, half a second to perform the second one, and so on.) Such a machine will always be able to decide a Π^0_1 set, for such a set is of the form {x: (y)R(x, y)} for some recursive relation R, and so the machine can run through all the y's, checking in each case whether R(x, y) holds, and then concluding that x is in the set or that it isn't. Using similar reasoning, we can see that any arithmetical set can be decided by such a machine. In fact, if the notion is made precise, it will turn out that the Δ^1_1 sets are precisely those sets that can be decided by such a machine, and that the Π^1_1 sets are those that can be semi-computed by one.

Another way in which the Π^1_1 sets are analogous to r.e. sets concerns representability in formal systems. Specifically, if we consider formal systems with the ω-rule, then it will turn out that all the sets weakly representable in such systems are Π^1_1, and conversely that any Π^1_1 is weakly representable in such a system.

We could also characterize the Π^1_1 sets via definability in a language: analogously to the language RE, we could set up a language with conjunction, disjunction, unbounded number quantifiers, and universal function quantifiers, in which precisely the Π^1_1 sets would be definable.

As in the arithmetical hierarchy, we have the following theorem.

Enumeration Theorem: For all n > 0 and all n and p, there is an m+1+p-place Π^1_n relation that enumerates the m+p-place Π^1_n relations, and similarly for Σ^1_n.

Proof: In what follows, we use $\vec{x}$ to abbreviate $x_1, ..., x_m$, and $\vec{\beta}$ to abbreviate $\beta_1, ..., \beta_p$. Let S be any m+p-place Π^1_1 relation. S is $\{<\vec{x}, \vec{\beta}>: (\alpha)(\exists z)R(\overline{\alpha}(z), \vec{x}, \overline{\beta}_1(z), ..., \overline{\beta}_p(z))\}$ for some recursive relation R. Since R is r.e., R = W_e for some e, S = $\{<\vec{x}, \vec{\beta}>: (\alpha)(\exists z)W(e, \overline{\alpha}(z), \vec{x}, \overline{\beta}_1(z), ..., \overline{\beta}_p(z))\}$. So the relation $\{<e, \vec{x}, \vec{\beta}>: (\alpha)(\exists z)W(e, \overline{\alpha}(z), \vec{x}, \overline{\beta}_1(z), ..., \overline{\beta}_p(z))\}$ enumerates the m+p-place Π^1_1 relations. Moreover, that relation is itself Π^1_1, since it comes from an arithmetical relation by universal function quantification.

Just as we derived the general enumeration theorem for the arithmetical hierarchy from the special case of Σ^0_1, we can derive the present theorem from the case of Π^1_1. For example, consider the case of m+p-place Π^1_n relations with n odd. Any such relation is $\{<\vec{x}, \vec{\beta}>: (\alpha_1)(\exists\alpha_2)...(\exists\alpha_{n-1})S(\vec{x}, \vec{\beta}, \vec{\alpha})\}$ for some Π^1_1 relation S (where naturally $\vec{\alpha}$ abbreviates $\alpha_1, ..., \alpha_{n-1}$). But then by the enumeration theorem for Π^1_1 relations, this is $\{<\vec{x}, \vec{\beta}>: (\alpha_1)(\exists\alpha_2)...(\exists\alpha_{n-1})R(e, \vec{x}, \vec{\beta}, \vec{\alpha})\}$ for some e, where R is a Π^1_1 enumeration of the m+p+(n-1)-place Π\O(1,1) relations. So the relation {<e, \O(x,→), \O(β,→)>: $(\alpha_1)(\exists\alpha_2)...(\exists\alpha_{n-1})R(e, \vec{x}, \vec{\beta}, \vec{\alpha})\}$ is a Π^1_n enumeration of the m+p-place Π^1_n relations. The other three cases are treated similarly.

(A similar theorem, called the *parameterization theorem*, holds for $\mathbf{S}^1_n$ and $\mathbf{P}^1_n$ relations;

in that case, relations have functions rather than numbers as indices.)

We can use the enumeration theorem to prove the following.

Hierarchy Theorem: For all n, $\Sigma^1_n \neq \Pi^1_n$.
Proof: Let R be a Π^1_n enumeration of the Π^1_n sets of numbers, and let D = {x: R(x, x)}. Then D is clearly Π^1_n, so -D is Σ^1_n. But -D is not Π^1_n, for if it were, we would have -D = {x: R(e, x)} for some e, and so e ∈ -D iff R(e, e) iff e ∈ D. So -D ∈ $\Sigma^1_n - \Pi^1_n$.

So in particular, there is a set D ∈ $\Sigma^1_1 - \Pi^1_1$. This D is analogous to K; we may as well call it K^Π.

Most of our earlier discussion of Gödel's theorem can be duplicated in the present case. (Of course, if a system has the ω-rule, or in general has Π^1_1 inference rules, it may decide every arithmetical statement. However, this is not to say that it decides every second-order statement.) Just as we showed that any system with an r.e. set of axioms and r.e. rules has an r.e. set of theorems, we want to show that the set of theorems generated by a finite set of Π^1_1 rules is Π^1_1.

First, let us associate with each rule of inference with the relation {<x, α>: x follows by the rule from premises in the set with characteristic function α}, and say that a rule is Π^1_1 if the corresponding relation is. Thus, the ω-rule is to be identified with the relation {<(x)A(x), α>: α is the characteristic function of some set that contains A($\mathbf{0}^{(n)}$) for all n}. Let χ be a recursive function such that for all formulae A(x), if m is the Gödel number of (x)A(x), then χ(m, n) = the Gödel number of A($\mathbf{0}^{(n)}$); then the ω-rule is Π^1_1, since the corresponding relation is defined by the formula (y) α(y) ≤ **0'** ∧ (n) α(χ(x, n)) = **0'**. If S is a set of sentences, then we can get the effect of taking all of the sentences in S as axioms by having the single rule *from any set of premises to infer any sentence in S.* This rule corresponds to the relation defined by (y) α(y) ≤ **0'** ∧ x ∈ S, which is Π^1_1 if S is. Finally, if R_1, ..., R_n are P^1_1 rules, then the relation R = {<x, α>: R_1(x, α) ∨ ... ∨ R_n(x, α)} is a Π^1_1 relation, and a sentence is a theorem of the formal system consisting of the rules R_1, ..., R_n just in case it is a theorem of the single rule R. Thus, if we can show that the set of theorems of a single Π^1_1 rule is itself Π^1_1, it will follow that the set of theorems of a system with a Π^1_1 set of axioms, a finite number of Π^1_1 rules, and the ω-rule is Π^1_1.

Given a rule R, let ψ be the following operator on sets: ψ(S) = {x: R(x, S's characteristic function)}. Let φ be the corresponding operator on functions: if α is the characteristic function of a set S, then φ(α) = the characteristic function of ψ(S) = the characteristic function of {x: R(x, α)}. If R is a rule of inference in any reasonable sense, then ψ will be monotonic, since ψ(S) = the set of sentences that follow via R from sentences in S: if S ⊆ S' and A follows from some sentences in S, then A also obviously follows from some sentences in S' as well. The set of theorems of R is the least fixed point of ψ. Recall that the least fixed point of ψ is the set ∩{S: ψ(S) ⊆ S} = {x: (S)(ψ(S) ⊆ S ⊃ x ∈ S)}. In terms of the operator φ, this set is {x: (α)((y)[(φ(α))(y) = 1 ⊃ α(y) = 1] ∧ α is a characteristic function ⊃ α(x) = 1)}. Since (ϕ(α))(y) = 1 iff R(x, α), this set is defined by the formula $(\alpha)([(y)(R(x, \alpha) \supset \alpha(y) = \mathbf{0'}) \wedge (y)\, \alpha(y) \leq \mathbf{0'}] \supset \alpha(x) = \mathbf{0'})$. We must check that this formula is indeed Π^1_1. Since R is Π^1_1 and R(x, α) occurs in the antecedent of a conditional, the formula $(y)(R(x, \alpha) \supset \alpha(y) = \mathbf{0'})$ is Σ^1_1. However, that formula itself occurs in the antecedent of a conditional, so the formula $[(y)(R(x, \alpha) \supset \alpha(y) = \mathbf{0'}) \wedge (y)\, \alpha(y) \leq \mathbf{0'}] \supset \alpha(x) = \mathbf{0'}$ is Π^1_1. Finally, when (α) is added, the formula remains Π^1_1. We therefore have the following

Theorem: If a formal system has Π^1_1 set of axioms and a finite number of Π^1_1 rules (possibly including the ω-rule), then the set of theorems of the system is itself Π^1_1.

The definition of "weakly represents" for such formal systems is the same as for ordinary formal systems. Let S be a set of numbers which is weakly representable in some such system. Then S = {n: A($\mathbf{0}^{(n)}$) is a theorem} for some formula A(x). Let χ be a recursive function such that χ(n) = the Gödel number of A($\mathbf{0}^{(n)}$); then χ reduces S 1-1 to the set of theorems of the system, and so S is Π^1_1. So any set weakly representable in such a system is Π^1_1.

Conversely, we can find formal systems in the second-order language of arithmetic which weakly represent all the Π^1_1 sets, just as all the r.e. sets are weakly representable in Q. In particular, if Γ is such a system, then the set of theorems of the system, being Π^1_1, is weakly representable in the system itself. We can use this fact to construct a sentence that says '"Gödel heterological" is Gödel heterological', and prove that the sentence is true but unprovable if the system is consistent.

If Γ is a system all of whose theorems are true, then we can show directly that Γ is incomplete, by showing that the set of theorems of Γ is not the set of true sentences. For if it were, then the set of true sentences of the language would be Π^1_1 and therefore definable in the language itself. But then by the usual argument satisfaction would also be definable, which is impossible because the language has negation.

If Γ is Π^1_1-complete (i.e. if every true Π^1_1 sentence is provable) and consistent, then we can get a closer analog of Gödel's theorem. Let S be any Π^1_1 set of numbers that is not Σ^1_1; K^{Π} would do, for example. Then there is a Σ^1_1 formula A(x) that defines -S. Just as we did in the original Gödel theorem, we can prove that there are statements of the form A($\mathbf{0}^{(n)}$) that are true but unprovable in the system.

Arithmetical Truth is Δ^1_1.

We have proved that all hyperarithmetical sets are Δ^1_1; since we know already that not all hyperarithmetical sets are arithmetical, it follows that there are Δ^1_1 sets that are not arithmetical. There is also a direct proof of this, due to Tarski. We know that the set of true arithmetical sentences is not arithmetical; we can use Tarski's famous definition of truth to show that this set is Δ^1_1.

We showed that for a set to be Δ^1_1 it is sufficient that its characteristic function be the unique solution of some arithmetical formula (or even Σ^1_1 formula) A(α). Recall the usual inductive definition of truth:

$\mathbf{0}^{(m)} = \mathbf{0}^{(n)}$ is true iff m = n;
A($\mathbf{0}^{(m)}$, $\mathbf{0}^{(n)}$, $\mathbf{0}^{(p)}$) is true iff m + n = p;
M($\mathbf{0}^{(m)}$, $\mathbf{0}^{(n)}$, $\mathbf{0}^{(p)}$) is true iff m·n = p;
~A is true iff A is not true;
(A ⊃ B) is true iff either A is not true or B is true;
(x)A(x) is true iff for all n, A($\mathbf{0}^{(n)}$) is true.

We can obtain A(α) by writing out this definition in the language of arithmetic, replacing "x is true" by "α(x) = 1". From our previous work, we have arithmetical formulae Sent(x), At(x) and TrAt(x) which define the set of sentences of the language of arithmetic, the set of atomic sentences, and the set of true atomic sentences, respectively. We can therefore write A(α) as follows:

$$
\begin{aligned}
&(x)[\alpha(x) \leq \mathbf{0'} \wedge (\alpha(x) = \mathbf{0'} \supset \mathrm{Sent}(x)) \wedge \\
&\quad (\mathrm{At}(x) \supset (\alpha(x) = \mathbf{0'} \equiv \mathrm{TrAt}(x))) \wedge \\
&\quad (y)(z)(i)\{(\mathrm{Neg}(x, y) \supset \alpha(x)+\alpha(y) = \mathbf{0'}) \wedge \\
&\qquad (\mathrm{Cond}(x, y, z) \supset [\alpha(x) = \mathbf{0'} \equiv (\alpha(y) = \mathbf{0} \vee \alpha(z) = \mathbf{0'})]) \wedge \\
&\qquad (\mathrm{UQ}(x, y, i) \supset [\alpha(x) = \mathbf{0'} \equiv (n)(w)(\mathrm{Subst}_2(y, w, i, n) \supset \alpha(w) = \mathbf{0'})])\}]
\end{aligned}
$$

Where Neg(x, y) holds iff x is the negation of y, Cond(x, y, z) holds iff x is the conditional (y ⊃ z), and UQ(x, y) holds iff x is the result of attaching the universal quantifier (x_i) to x. We leave it to the reader to verify that this works.

Once we know that all Δ^1_1 sets are hyperarithmetical, it will turn out that the set of truths of the language of arithmetic is also hyperarithmetical. We can also give a direct proof that this set is hyperarithmetical; in fact, it turns out to be recursively isomorphic to the set H_n, where $|n|_R = \omega$, that is, it appears at the first level of the hyperarithmetical hierarchy that is beyond the arithmetical sets.

Lecture XXV

The Baire Category Theorem.

A subset S of Baire space is said to be *dense* if for any finite sequence s, there is an $\alpha \in S$ that extends s. (This definition coincides with the general definition of "dense" for topological spaces.)

Theorem: The intersection of a countable family of dense open sets is nonempty.
Proof: Let $O_1, O_2, \ldots$ be dense open sets. We shall construct a function $\alpha \in \cap_n O_n$ as follows. Let s_0 be the empty sequence. If s_n has been defined, let $\alpha \in O_{n+1}$ be such that α extends s_n; this is possible because O_{n+1} is dense. Since O_{n+1} is open, there is an initial segment t of α such that every function extending t is in O_{n+1}. Let s_{n+1} be some finite sequence that properly extends both s_n and t.

We have thus defined a sequence $s_0, s_1, \ldots$ of finite sequences such that $i > j$ implies that s_i properly extends s_j, and such that any function extending s_n (for $n > 0$) is an element of O_n. Let $\alpha = \cup_n s_n$; α is a total function. Moreover, since α extends each s_n, $\alpha \in O_n$ for all n, i.e. $\alpha \in \cap_n O_n$.

This is a special case of a more general theorem, known as the *Baire Category Theorem*. (The proof of the general theorem is essentially the same as the present proof.) Notice that for the theorem to go through, it suffices that each O_n *contain* some dense open set, since if for all n O_n' is a dense open subset of O_n, then we can apply the theorem to find $\alpha \in \cap_n O_n'$, whence $\alpha \in \cap_n O_n$. (Any set containing a dense set is itself dense, so if O_n contains a dense open set at all, the interior or O_n (i.e. the union of all the open sets contained in O_n) will be a dense open set. Thus, we can take O_n' to be the interior of O_n.) Notice also that O_1 need not be dense, but merely nonempty and open, since then we can let s_1 be any sequence all of whose total extensions are in O_1.)

The Baire Category Theorem turns out to have many applications in logic, and if there is a single most important principle in logic, it is probably this theorem. It is usually applied in the following way. Suppose we want to show that there is a function that satisfies a certain condition C. If we can break C down into a countable family of conditions, then we can find such a function if we can find a single function that satisfies all of those conditions simultaneously. If we can arrange things so that each of these conditions is dense and open (or contains a dense open condition), then the theorem guarantees that such a function exists.

Cohen's famous proof of the independence of the continuum hypothesis can be seen as an application of the category theorem. The theorem can also be seen as a generalization of

Cantor's diagonal argument. In particular, we can use it to show that there are uncountably many total functions on **N**. To see this, let F be any countable family of such functions, and for each $\alpha \in F$, let $O_\alpha = \{\beta: \beta \neq \alpha\}$. Each O_α is open, since two functions are different iff they disagree on some initial segment, and each O_α is dense, since any finite sequence can be extended to a function different from α. It follows that there is a function β such that $\beta \in O_\alpha$ for each $\alpha \in F$, i.e. such that $\beta \notin F$. (This application of the category theorem really boils down to Cantor's own proof, since in the latter a function outside F is constructed stage by stage in just the same way that the function α is constructed in the former.)

Incomparable Degrees.

Let us now consider an application of this theorem. For all we have said so far, the Turing degrees might be linearly ordered. It turns out that they are far from being linearly ordered; in this section, we shall construct a pair of incomparable degrees, i.e. degrees a and b such that neither $a \leq b$ nor $b \leq a$.

Call a pair of functions *recursively incomparable* if neither is recursive in the other. To find a pair of incomparable degrees, it suffices to find a pair of recursively incomparable functions, for then those functions will be of incomparable degrees. Recall that a function α is recursive in β just in case α is definable in the language RE with an extra function symbol for β. Let us define W_e^β to be the relation $\{<k, p>: (\exists s)(s$ is an initial segment of β and $W(e, s, k, p))\}$, and let us identify functions with their graphs. Then α is recursive in β just in case $\alpha = W_e^\beta$ for some e, and α is nonrecursive in β iff $\alpha \neq W_e^\beta$ for all e. Thus, α and β will be recursively incomparable if they satisfy all of the conditions $\alpha \neq W_e^\beta$ and $\beta \neq W_e^\alpha$ simultaneously; to find such α and β, we need only show that those conditions contain dense open conditions.

Theorem: There are incomparable Turing degrees.
Proof: For any e, let $A_e = \{[\alpha, \beta]: \alpha \neq W_e^\beta\}$ and $B_e = \{[\alpha, \beta]: \beta \neq W_e^\alpha\}$. If each of the A_e's and B_e's has a dense open subset, then we can apply the Baire category theorem to obtain α and β such that $[\alpha, \beta]$ is in A_e and B_e for each e, from which it follows that α and β are recursively incomparable. We show that A_e has a dense open subset; the proof that B_e does is the same.

Let $A_e' = \{\gamma \in A_e$: $(\exists s)(s$ is an initial segment of γ, and any function extending s is in $A_e)\}$. A_e' is open, for let $\gamma \in A_e'$ and let $s \subseteq \gamma$ be such that any function extending s is in A_e; then any function extending s is also in A_e'. (In fact, A_e' is the interior of A_e.) We need only show that A_e' is dense.

Let s be any finite sequence, and let s_1 and s_2 be the even and odd parts of s, respectively (that is, if $s = <x_0, ..., x_n>$, with $n = 2m$, then $s_1 = <x_0, x_2, ..., x_{2m}>$ and $s_2 = <x_1, x_3, ..., x_{2m-1}>$, and similarly if $n = 2m+1$); we need to show that some function extending s is in A_e'. It suffices to find an s' extending s (or extended by s) such that any function extending s' is in A_e. Notice that if $\gamma = [\alpha, \beta]$, then γ extends s iff α extends s_1 and β extends s_2.

Case 1: $W_e^\beta \subseteq s_1$ for all β extending s_2. In that case, $W_e^\beta \neq \alpha$ whenever α and β extend s_1 and s_2, because any such α is total and therefore properly extends s_1, so we can let s = s'.

Case 2: $W_e^\beta \not\subseteq s_1$ for some β extending s_2. Fix β, and let $<k, p> \in W_e^\beta - s_1$. Let s_2' be an initial segment of β such that W(e, s_2', k, p); then $<k, p> \in W_e^{\beta'}$ for all β' extending s_2'. We can find an extension s_1' of s_1 such that $<k, p> \notin \alpha'$ for all α' extending s_1: either s_1' has a kth element that is different from p, in which we can let $s_1' = s_1$, or s_1' has no kth element, in which we can let s_1' be an extension of s_1 whose kth element is different from p. Let s' be an extension of s such that the even and odd parts of s' extend s_1' and s_2'. Then whenever $[\alpha, \beta]$ extends s', $[\alpha, \beta] \in A_e$, as required.

We can also give a direct proof that does not appeal directly to the category theorem.

Second Proof: We construct finite sequences $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ such that $\alpha = \cup_n s_n$ and $\beta = \cup_n t_n$ are total functions; we then show that α and β are recursively incomparable.

Let $s_0 = t_0 = \emptyset$. Suppose s_n and t_n have been defined. If n = 2m, we proceed as follows. If $W_m^\beta \subseteq s_n$ for all extensions β of t_n, then let s_{n+1} and t_{n+1} be any finite sequences that extend s_n and t_n. Otherwise, find an extension t_n' of t_n and a pair <k, p> such that W(e, t_n', k, p), and let s_n' be an extension of s_n such that $\alpha(k) \neq p$ for all extensions α of s_n', as in the first proof. Let $s_{n+1} = s_n'$ and $t_{n+1} = t_n'$. If n = 2m+1, then do exactly the same, except reversing the roles of s and t.

Now let $\alpha = \cup_n s_n$ and $\beta = \cup_n t_n$. If α is recursive in β, then $\alpha = W_m^\beta$ for some m; but α and β extend s_{2m} and t_{2m}, and it is clear from the construction of the s's and t's that $\alpha \neq W_m^\beta$ for any such α and β. So α is not recursive in β, and by same argument β is not recursive in α, i.e. α and β are recursively incomparable.

The construction of the s's and t's in this proof is not effective, since if it were, α and β would be recursive and therefore recursively comparable. In particular, we cannot effectively decide whether $W_m^\beta \subseteq s_n$ for all extensions β of t_n, since that would involve surveying all the infinitely many extensions of t_n. However, if we had an oracle which gave us the answer to this question, we could use it to effectively construct α and β, so α and β would be recursive in the oracle. We can therefore modify the proof to place an upper bound on the Turing degrees of α and β.

Theorem: There are incomparable Turing degrees below 0'.
Proof (sketch): It suffices to show that the functions α and β constructed in the above

proof are recursive in 0'. Consider the relation R = {<s, t, m>: $W_m^\beta \subseteq$ s for all β extending t}. The relation -R is r.e., since <s, t, m> ∈ -R just in case (∃t' extending t)(∃k)(∃p)(W(m, t', k, p) ∧ <k, p> ∉ s). It follows that both R and -R are recursive in 0'. Let φ be a partial function which uniformizes the r.e. relation {<s, t, [t', k, p]>: t' extends t ∧ W(m, t', k, p) ∧ <k, p> ∉ s}.

We can then construct s_n and t_n effectively in terms of R and φ. Specifically, we set s_0 = t_0 = the code of the empty sequence. If n = 2m, we set s_{n+1} = s_n^<0> (i.e. the concatenation of s_n with the unit sequence <0>) and t_{n+1} = t_n^<0> if R(s_n, t_n, m) holds. If R(s_n, t_n, m) doesn't hold, let [t', k, p] = φ(s_n, t_n). Then t' extends t_n, and for any β extending t', <k, p> ∈ W_m^β - s_n. We then let t_{n+1} = t' and let s_{n+1} be some extension of s_n such that the kth element of s_{n+1} exists and is different from p. (s_{n+1} can obviously be found effectively.) If n = 2m+1, we do the same, but with the roles of s and t reversed.

So we see that the maps n → s_n and n → t_n are recursive in 0'. α and β are therefore also recursive in 0', since α(n) = the nth member of the sequence $s_{2(n+1)}$ and β(n) = the nth member of the sequence $t_{2(n+1)}$.

This theorem was originally proved by Kleene and Post. Notice that it does not show that there are any incomparable r.e. degrees, since a degree can be below 0' without containing any r.e. sets. In fact, the proof that incomparable r.e. degrees exist is a souped-up version of the proof we just gave.

We can also get a refinement of these results:

Theorem: For any nonrecursive degree a, there is a degree incomparable with a.
Proof: Let α be a total function of degree a; we need to find a function β recursively incomparable with α. For all e, let A_e = {β: α ≠ W_e^β} and B_e = {β: β ≠ W_e^α}; it suffices to show that each A_e and each B_e has a dense open subset. B_e is dense and open already, as is easily seen. Let A_e' be the interior of A_e as before, i.e. A_e' = {β: (∃s ⊆ β)(β')(if β' extends s then α ≠ $W_e^{\beta'}$}. We need only show that A_e' is dense.

Let s be any finite sequence; we need to show that A_e' contains some function extending s, i.e. that there is a sequence s' extending s such that for all β extending s', α ≠ W_e^β. Suppose this is not the case; then for all s' extending s there is a β extending s' such that α = W_e^β. In that case, α = {<k, p>: (∃s')(s ⊆ s' ∧ W(e, s', k, p))}. (Suppose <k, p> ∈ α; since there is a β extending s such that α = W_e^β, there is an s' extending s such that W(e, s', k, p). On the other hand, if s' extends s and W(e, s', k, p), then <k, p> ∈ W_e^β for all β extending s'; since W_e^β = α for *some* such β, it follows that <k, p> ∈ α.) But in that case, α is an r.e. relation and is therefore a recursive function, contradicting our assumption that a is a nonrecursive degree.

We can refine this a bit further and show that for any nonrecursive degree a, there is a degree b below a' that is incomparable with a. The proof of this is like the proof that there are incomparable degrees below 0'. (Notice that this result does not directly imply that there are incomparable degrees below 0', because we cannot take a = 0.)

The Separation Theorem for $\mathbf{S}^1_1$ Sets.

In this section, we show that every $\mathbf{D}^1_1$ set is Borel. In fact, we shall prove something stronger. Call a pair (S_1, S_2) *Borel separable* if there is a Borel set which contains S_1 and is disjoint from S_2. We shall prove a theorem due to Lusin, namely that any disjoint pair of $\mathbf{S}^1_1$ sets is Borel separable. If S is $\mathbf{D}^1_1$, then (S, -S) is a disjoint pair of $\mathbf{S}^1_1$ sets, so there is a Borel set B which separates S from -S; but then S = B, and therefore S is Borel.

Notice that a set S is Borel inseparable from a set T iff there is no Borel set B with $S \subseteq B \subseteq -T$. We begin by proving the following.

Lemma: If $S = \cup_n S_n$ and S is Borel inseparable from T, then there is an n such that S_n is Borel inseparable from T.
Proof: Suppose S_n is Borel separable from T for each n. For each n, let B_n be a Borel set such that $S_n \subseteq B_n \subseteq -T$. Then $\cup_n S_n \subseteq \cup_n B_n \subseteq -T$. But then $S = \cup_n S_n$ is Borel separable from T, since $\cup_n B_n$ is Borel.

Corollary: If the sets of two countable unions are pairwise Borel separable, then the two unions are Borel separable.

Theorem: Any two disjoint $\mathbf{S}^1_1$ sets are Borel separable.
Proof: Let S_1 and S_2 be any two $\mathbf{S}^1_1$ sets, and assume that S_1 and S_2 are Borel inseparable. We show that $S_1 \cap S_2 \neq 0$. Since S_1 and S_2 are Σ^1_1, there are relations R_1 and R_2 on **N** such that

$$S_1 = \{\beta_1: (\exists\alpha_1)(x)R_1(\bar{\alpha}_1(x), \bar{\beta}_1(x))\},$$
$$S_2 = \{\beta_2: (\exists\alpha_2)(x)R_2(\bar{\alpha}_2(x), \bar{\beta}_2(x))\}.$$

We construct four infinite sequences $<a_1^{(n)}>$, $<b_1^{(n)}>$, $<a_2^{(n)}>$, $<b_2^{(n)}>$, where $a_1^{(n)}$, etc. are sequences of length n. First, set $a_1^{(0)} = b_1^{(0)} = a_2^{(0)} = b_2^{(0)}$ = the empty sequence. If s and t are finite sequences, let

$$S_1{}^{s,\,t} = \{\beta_1: (\exists\alpha_1)(s \subseteq \alpha_1 \wedge t \subseteq \beta_1 \wedge (x)R_1(\bar{\alpha}_1(x), \bar{\beta}_1(x)))\}$$

and define $S_2{}^{s,\,t}$ similarly. In particular, let $S_1^{(n)} = S_1{}^{a_1{}^{(n)},\, b_1{}^{(n)}}$ and $S_2^{(n)} = S_2{}^{a_2{}^{(n)},\, b_2{}^{(n)}}$. Then

$$S_1^{(n)} = \{\beta_1: (\exists\alpha_1)(\bar{\alpha}_1(n) = a_1^{(n)} \wedge \bar{\beta}_1(n) = b_1^{(n)} \wedge (x)R_1(\bar{\alpha}_1(x), \bar{\beta}_1(x)))\}$$

and similarly for $S_2^{(n)}$. Now suppose $a_1^{(n)}$, $b_1^{(n)}$, etc. have been defined and $S_1^{(n)}$ is Borel inseparable from $S_2^{(n)}$, and define $a_1^{(n+1)}$, etc. as follows. Notice that $S_1^{(n)} = \cup\{S_1^{s, t}$: s and t are sequences of length n+1 that extend the sequences $a_1^{(n)}$ and $b_1^{(n)}$, respectively}, so by our lemma, we can find such s and t so that $S_1^{s, t}$ is Borel inseparable from $S_2^{(n)}$. Let $a_1^{(n+1)} = s$ and $b_1^{(n+1)} = t$ for some such s and t. So $S_1^{(n+1)}$ is Borel inseparable from $S_2^{(n)}$. Similarly, we can find sequences s and t of length n+1 which extend $a_2^{(n)}$ and $b_2^{(n)}$ and such that $S_2^{s, t}$ is Borel inseparable from $S_1^{(n)}$; let $a_2^{(n)}$= s and $b_2^{(n)}$= t for some such s and t. $S_1^{(n+1)}$ and $S_2^{(n+1)}$ are therefore Borel inseparable.

Thus, the $S_1^{(n)}$'s and $S_2^{(n)}$'s are progressively narrower subsets of S_1 and S_2 that are Borel inseparable if S_1 and S_2 themselves are. Moreover, $a_1^{(n)}$ properly extends $a_1^{(m)}$ when n > m, and similarly for b_1, a_2, and b_2; so we can define $\alpha_1 = \cup_n a_1^{(n)}$, $\beta_1 = \cup_n b_1^{(n)}$, and similarly for α_2 and β_2.

Observe that $\beta_1 = \beta_2$. For suppose not; then $\beta_1(n) \neq \beta_2(n)$ for some n. Let $p = \beta_1(n)$, and let $O = \{\beta: \beta(n) = p\}$. O is open, as is easily seen, and is therefore Borel. However, $S_1^{(n+1)} \subseteq O$ and $S_2^{(n+1)} \subseteq -O$, so $S_1^{(n+1)}$ and $S_2^{(n+1)}$ are Borel separable, contradiction.

We now show that S_1 intersects S_2 by showing that $\beta_1 \in S_1 \cap S_2$. To prove this, it suffices to show that $(x)R_1(\overline{\alpha}_1(x), \overline{\beta}_1(x))$ and $(x)R_2(\overline{\alpha}_2(x), \overline{\beta}_2(x))$. We prove the former; the proof of the latter is the same. Suppose $\sim(x)R_1(\overline{\alpha}_1(x), \overline{\beta}_1(x))$. Then for some n, $\sim R_1(\overline{\alpha})_1(n), \overline{\beta}_1(n))$. By our definition of $a_1^{(n)}$ and $b_1^{(n)}$, this just means that $\sim R_1(a_1^{(n)}, b_1^{(n)})$. It follows that $S_1^{(n)} = \varnothing$. But then $S_1^{(n)}$ is a Borel set that separates itself from $S_2^{(n)}$, which is impossible. This concludes the proof.

We have already seen that all Borel sets are $\mathbf{D}_1^1$; we have therefore established Suslin's theorem: $\mathbf{D}_1^1$ = Borel. This is an unusually simple proof for such a sophisticated result. Notice that it is similar in flavor to the proof of the existence of incomparable degrees; in particular, the function β_1 is constructed via a stage-by-stage process.

Exercises

1. Consider the language of arithmetic with one extra predicate P(x).

(a) Consider a system in this languaßge whose axioms are an r.e. set of the language of arithmetic containing Q. Add the axioms $P(\mathbf{0}^{(n)})$ for each $n \in S$, where S is any set. No axioms except these contain the extra predicate P. Assume that the resulting system is ω-consistent. On these assumptions, characterize the sets weakly representable in the system and prove the characterization.

(b) Make the same assumptions as in (a) except that now we have $P(\mathbf{0}^{(n)})$ for each $n \in S$ and $\sim P(\mathbf{0}^{(n)})$ for each $n \notin S$. Characterize the weakly representable sets and prove the answer.

(c) Under the assumptions of (b), characterize the strongly representable (binumerable) sets, and prove the answer.

2. Consider a system in the second order language of arithmetic (i.e., the language of arithmetic supplemented with variables and quantifiers for 1-place number theoretic functions), with a Π^1_1 set of axioms containing at least the axioms of Q, and with the ω-rule added to the usual logical rules.

(a) Under the assumption that all the axioms are true, define a statement analogous to '"Gödel heterological" is Gödel heterological' and show that it is true but undecidable.

(b) Show that every true Π^1_1 sentence is provable. Use this to show that if all of the axioms are true, then the sets weakly representable in the system are precisely the Π^1_1 sets. (Hint: Prove the contrapositive (i.e., that if a Π^1_1 sentence is not provable, then it is not true) by a method similar to the proof of the $\mathbf{S}^1_1$ separation theorem and its corollary the Suslin characterization of the $\mathbf{D}^1_1$ sets. You may assume in your proof that the system contains any reasonable axioms, over and above those in the language of arithmetic, to handle function quantifiers; in particular, relevant axioms might include (m)((n)(α(m)=n⊃A(α)))⊃A(α).)

(c) Let A(x) be a Σ^1_1 formula with one free number variable. Using (b), show that if the system is consistent and the set defined is not Π^1_1, then some sentence of the form $A(\mathbf{0}^{(n)})$ is true but unprovable.

(d) Show that if the system is consistent (not necessarily true), then the statement '"Gödel heterological" is Gödel heterological' of part (a) must be true but unprovable.

3. Let R be any binary relation on the natural numbers. Suppose that for any partial recursive function φ there is a total recursive function ψ such that R(ψ(x),φ(x)) whenever φ(x) is defined. Prove that, under this hypothesis, for any total recursive function χ there is a number m such that R(m,χ(m)). Show that immediate consequences of this principle for suitable choices of R are: the self-reference lemma, that every maximal enumeration has the fixed-point property, and that there are two disjoint r.e. sets without recursive separation.