Exercises week 3.

1. Spell out the statement $\mathsf{Lcm}[m(i) : i < k](l)$ if $M(x, y)$ is the formula $y = x \cdot x$. (Is this a Σ-pterm?)

   Do the same for $\mathsf{Max}[m(i) : i < k](l)$ as defined on Page 31.

2. In the proof of the Chinese Remainder Theorem (Theorem (36)) we use induction on $n \leq k$. Formulate precisely this induction statement.

3. Write out $\mathsf{Beta}(a, b, i, r)$ in the language of PA.

4. Should the $\mathbf{i}$ in the definition of the $\mathsf{Beta}(a, b, i, r)$ predicate on Page 31 really be boldface?

5. Prove by term induction that every term has a Gödel number.

6. Prove by formula induction that every term has a Gödel number.

7. Give a strict Σ formula that is equivalent to $\mathsf{Pair}(x, y, z)$.

8. Calculate successively $\mathsf{pair}(0, 0)$, $\mathsf{pair}(0, 1)$ $\mathsf{pair}(1, 0)$, $\mathsf{pair}(0, 2)$ $\mathsf{pair}(1, 1)$, $\mathsf{pair}(2, 0)$ $\mathsf{pair}(0, 3)$, $\mathsf{pair}(1, 2)$ $\mathsf{pair}(2, 1)$ and $\mathsf{pair}(3, 0)$. Wite down these values in a diagram of $\mathbb{N} \times \mathbb{N}$.

9. Prove that $\forall x \forall y\ (\neg F(x, y) \leftrightarrow \exists z\ (F(x, z) \wedge \neg(y = z)))$ if F is a pterm.

10. Prove by the number of nested pterms (give a proper definition!) that Δ-formulas (Δ^0_1-formulas) are closed under substitution of Σ-pterms.

11. Recall the definition we gave of the arithmetical hierarchy. A formula φ is Δ^0_0, Σ^0_0 and Π^0_0 at the same time if φ is built up from atomic formulas by boolean operations and bounded quantification only. The formula sets Π^0_{n+1} and Σ^0_{n+1} are inductively defined as follows. If $\varphi \in \Sigma^0_n$, then $\forall \vec{x} \varphi \in \Pi^0_{n+1}$. If $\varphi \in \Pi^0_n$, then $\exists \vec{x} \varphi \in \Sigma^0_{n+1}$.

    - Show that every Σ^0_1-formula is also a Σ-formula according to Boolos' notion.
    - Show that if $\mathrm{PA} \vdash \forall x{<}y \exists z\ \varphi(x, z) \rightarrow \exists b \forall x{<}y \exists z{<}b \varphi(x, z)$, then indeed every strict Σ-formula in Boolos' sense is also a Σ^0_1-formula.
    - Does PA prove $\forall x{<}y \exists z\ \varphi(x, z) \rightarrow \exists b \forall x{<}y \exists z{<}b \varphi(x, z)$? This principle is called the Σ^0_1 *collection* principle.

12. What is the difference between $\pi(x, y)$, $\mathsf{pair}(x, y)$ and (x, y) on Page 35?

13. Calculate the code of the numeral of the code of the numeral of the number 0. Of how many symbols does its numeral consists?

14. Prove that every term of PA has an odd Gödel number or has a Gödel number of the form $\pi(i, j)$ with i odd. Do the same for formulas.

15. Give an inductive definition of the class of all subformulas of a formula φ. Show that $\mathsf{GN}(\varphi')<\mathsf{GN}(\varphi)$ whenever φ' is a proper subformula of φ. What is our induction?

16. Give a proof of Lemma (42) on Page 36.

17. Show that PA $\vdash \forall z \exists ! w\ \mathsf{Fst}(z, w)$.

18. Give a proof of Lemma (45) on Page 37.

19. Give a proof (by induction?) of Lemma (46) on Page 38.

20. In Lemma (50) of Chapter 2, nowhere it is demanded that $\mathsf{FinSeq}(s)$. Is this omission problematic?

21. Give a proof of Lemma (51) of Chapter 2.

22. Write down a Σ-formula $\mathsf{Sub}(t, i, x, y)$ that will yield the Σ pterm as claimed on the bottom of Page 43.

23. Do we have $\mathsf{FinSeq}(((0,0),4))$? Calculate $\mathsf{val}(((0,0),4),2)$. Also calculate $\mathsf{trunc}(((0,0),4),1,3)$.

24. Give the code of the sequence of ten zeros. (We do not ask for a proof that this is indeed the code!)

25. Give the code of the sequence of three one's. (We do not ask for a proof that this is indeed the code!)

26. Give the code of the sequence of length one only containing the element n.

27. In Lemma (53) we demand in the antecedent that s is a finite sequence. Show (by means of a counterexample) that this requirement is necessary for the Lemma to hold.

28. Let t be a term. We define $\exists\, x{<}t\ \varphi_0$ to be $\exists\, x{<}y(x = t \wedge\ \varphi_0)$. If φ_0 is a Δ_0-formula, show that $\exists\, x{<}t\ \varphi_0$ can be rewritten as a Π_1 formula and as a Σ_1 formula that are equivalent over PA. Can we do the same for $\exists\, x{<}x+1\ \varphi_0$?

29. Prove that $\forall x\ (\mathsf{Formula}(x) \to \mathsf{Formula}((\ulcorner\to\urcorner, x, \ulcorner\bot\urcorner)))$ is provable in PA.

30. Give a Δ_0-formula $\mathsf{Imp(x)}$ that says that x is the code of a formula of the form $A \to (B \to A)$.

31. Show that once Lemma (56) of Chapter 2 is proved, we see that $\mathsf{Term}(t)$ is equivalent to a Π_1-formula.

32. Provide the code of $\bot \to \bot$. If we were to code $\neg(v_0 = v_1)$, how should we treat the brackets?

33. What is the least common multiple of 3 and 5? Find x smaller than this least common multiple such that $x \equiv 2(3)$ and $x \equiv 3(5)$.

34. Prove (for example by induction) that $\Sigma_{k=0}^{n} k = \frac{n(n+1)}{2}$.

35. Shoenfield's coding is given by $\pi(x,y) = (x+y)(x+y)+x+1$. Determine the least positive natural number that is not in the range of this function. (For example by just calculating the "first values".) (Determine the lengths of consecutive gaps in the range of this pairing function.)

36. When using real numbers we can write down a function for the second component of a pair, so, if $z = (x,y)$ then $y = \frac{z}{2} - (\lfloor \sqrt{\frac{z}{2}} \rfloor^2 + 1)$. Show this and give a similar function for x.

37. Does the number 356645864186 code a formula? If so, which formula?

38. Find $x < \mathrm{lcm}(3,4,7)$ such that $x \equiv 2(3)$, $x \equiv 1(4)$ $x \equiv 5(7)$.

39. We have defined $\exists x < yA(x)$ to be short for $\exists x(x < y \wedge A(x))$ and dually $\forall x < yA(x)$ to be short for $\forall x(x < y \rightarrow A(x))$. Show that $\exists x < yA(x) \leftrightarrow \neg\forall x < y\neg A(x)$ is provable in PA. Do the same for the dual statement.

40. Show that $\mathrm{PA} \vdash v_0 = \overline{16} \rightarrow \overline{\mathrm{GN}(v_0)} = \overline{17}$.

41. Show by elementary means that

$$\mathrm{PA} \vdash \mathsf{Bew}(\ulcorner A \urcorner) \wedge \mathsf{Bew}(\ulcorner B \urcorner) \rightarrow \mathsf{Bew}(\ulcorner A \wedge B \urcorner).$$

Also show that

$$\mathsf{Bew}(\ulcorner A \wedge B \urcorner) \rightarrow \mathsf{Bew}(\ulcorner A \urcorner)$$

and

$$\mathsf{Bew}(\ulcorner A \wedge B \urcorner) \rightarrow \mathsf{Bew}(\ulcorner B \urcorner).$$

Conclude that

$$\mathsf{Bew}(\ulcorner A \wedge B \urcorner) \leftrightarrow \mathsf{Bew}(\ulcorner A \urcorner) \wedge (\ulcorner B \urcorner).$$

42. Prove that

$$\mathbf{GL} \vdash \bot \quad \Rightarrow \quad \mathbf{GL} \vdash A$$

for any formula A.

43. Prove that if $\mathbf{K} \vdash A \rightarrow B$ and $\mathbf{K} \vdash A \rightarrow C$ then $\mathbf{K} \vdash A \rightarrow (B \wedge C)$.

44. Formulate the axiom schemas $\Box A \rightarrow \Box\Box A$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ in terms of the $\Diamond$ modality.

45. Derive the following formulas in the respective logics:

    (a) $\mathbf{K} \vdash \Box(\phi \wedge \psi) \rightarrow \Box\phi$
    (b) $\mathbf{K} \vdash (\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

(c) $\mathbf{K} \vdash (\Box\phi \lor \Box\psi) \to \Box(\phi \lor \psi)$

(d) $\mathbf{K4} \vdash \Box\phi \to \Box\Box\Box\phi$

(e) $\mathbf{K4} \vdash (\Box\phi \land \Box\Box\psi) \to \Box\Box(\phi \land \psi)$

(f) $\mathbf{S4} \vdash \Diamond\phi \leftrightarrow \Diamond\Diamond\phi$

(g) $\mathbf{K} \vdash \Box p \land \neg\Box\bot \to \neg\Box\neg p$

(h) $\mathbf{K4} \vdash \Box p \land \Box\Box(p \to q) \to \Box\Box q$

(i) $\mathbf{K} \vdash \Box\Box p \land \Box\Box(p \to q) \to \Box\Box q$

46. Recast the inductive definition of a modal sentence as an explicit one employing finite sequences.

47. State the rule of substitution informally.

48. Prove that **GL** is closed under the rule of substitution.