Exercises week 2.

1. Provide a reasoning to the effect that PA $\vdash 3 \times 2 = 6$.

2. Show by means of the explicit definition of freeness on page 19 that v_3 is free in $v_3 = v_4 \to \forall v_2\ (v_2 = v_4)$, and in $v_3 = 4 \to \forall v_3\ (v_3 = 3)$.

3. Explicitely spell out the definition of $F_x(t)$.

4. Reason to the effect that
PA $\vdash \forall \vec{x} \exists ! y\ F(\vec{x}, y) \to \forall \vec{x}\ (\exists y\ (F(\vec{x}, y) \wedge A(y)) \leftrightarrow \forall y\ (F(\vec{x}, y) \to A(y)))$.

5. Dissabreviate, that is to say, write out, $B(h(g(x)), g(x))$ in two different ways.

6. Show that $\neg(\forall y < x, A(y)) \leftrightarrow \exists y < x, A(y)$,
and $\neg(\exists y < x, A(y)) \leftrightarrow \forall y < x, \neg A(y)$.

7. Recast the definition on Page 25 of strict Σ-formula in terms of finite sequences as was done for the definition of terms of PA.

8. Prove that every atomic formula is a Σ-formula.

9. Prove that the negation of an atomic formula is a Σ-formula.

10. Express Goldbach's conjecture by a Π_1-sentence. Express its negation by a strict Σ-sentence.

11. Prove the following statement: Let π be a Π^0_1-statement. If π is an undecidable statement of PA, then $\mathbb{N} \models \pi$.

12. Give a proof strategy to show Lemma (21) on Page 27. We do not ask for full proofs here.

13. Give an inductive definition of the set of closed terms. Show that this set is equal to the set of terms where 0 is substituted for all the free variables.

14. Write down some different closed terms that all denote the number 10. What is the shortest one?

15. Show that v_i is free in $\exists x\ ((x + v_i) = x \cdot x)$ conform our definition on page 19.
(Recall that $\exists$ is an abbreviation!)

16. Show that $(x + v_i)_x(v_i + 3) = ((v_i + 3) + v_i)$ and that $(x \cdot x)_x(v_i + 3) = ((v_i + 3) \cdot (v_i + 3))$ by applying our definition of substitution on page 19(footnote). Argue (informally) that we would like $(x + v_i = x \cdot x)_x(v_i + 3) = (v_i + 3 + v_i = (v_i + 3) \cdot (v_i + 3))$.

17. Provide proofs in PA of Lemmas (3) and (4) of Chapter 2.

18. Provide proofs in PA of Lemmas (5) and (6) of Chapter 2.

19. Prove Lemma (15) on Page 23.

20. Give a deatailed proof of (8) on Page 21 of Chapter 2.

21. We have seen the proof of the following fact. For every closed term t, for all i, if $t = i$, then $\mathrm{PA} \vdash t = \mathbf{i}$. As the set of closed terms is an inductively defined set, we could do our proof by closed term induction. We omitted the following inductive step from this proof.

    If $t = t_1 \times t_2$ and $t = i$, then $\mathrm{PA} \vdash t = \mathbf{i}$. Of course you work under the assumptions that $\forall i\ (t_1 = i \rightarrow \mathrm{PA} \vdash t_1 = \mathbf{i})$ and $\forall i\ (t_2 = i \rightarrow \mathrm{PA} \vdash t_2 = \mathbf{i})$.

    Write out the proof of this step.

22. Formulate Theorem (17) on Page 25, Σ_1 completeness in such a way that the inductive structure of it becomes evident.