Modal Completeness of **GL** via the step-by-step method

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1 Some definitions

In this note we will provide a proof of the modal completeness of **GL** which is different from that of Boolos.

First we define a new modal consequence relation.

Definition 1.1 *Let Γ be a set of **GL** formulas. If A is derivable from Γ using only theorems of **GL** and Modus Ponens, we write $\Gamma \vdash_{\boldsymbol{GL}} A$.*

It is well known that we have the deduction theorem for this notion of inference.

Theorem 1.2 Deduction Theorem $\Gamma, A \vdash_{\boldsymbol{GL}} B \Leftrightarrow \Gamma \vdash_{\boldsymbol{GL}} A \to B$.

Clearly if $\vdash_{\mathbf{GL}} A$, also $\mathbf{GL} \vdash A$.

Definition 1.3 *We call a set Γ* consistent *if $\Gamma \nvdash_{\boldsymbol{GL}} \bot$.*

We suppose our language is countably infinite. The following lemma is reminiscent to that of Lindenbaum and as a matter of fact, is proved in precisely the same manner.

Lemma 1.4 *Every **GL**-consistent set of sentences $\mathcal{A}$ can be extended to a maximal **GL**-consistent one.*

The main idea of proving completeness is the same as always. We suppose $\mathbf{GL} \nvdash A$ and we are going to make a (tree) model M such that $M, r \Vdash \neg A$. We will work with infinite sets of sentences but actually we are only interested in a finite part of it. Of course this part is precisely the part containing A and all of its subformulas and single negations. The single negation of a formula of the form $\neg A$ is defined to be just A. The single negation of a formula that is not of the form $\neg A$ is just the regular negation.

Definition 1.5 *A labeled **GL** frame is a triple $\langle M, R, \nu\rangle$. Here $\langle M, R\rangle$ is a **GL** frame and ν is a map that assigns to each world w in M a maximal **GL**-consistent set $\nu(w)$.*

Definition 1.6 *For two sets of modal sentences Γ and Δ we define $\Gamma \prec_\Box \Delta$ to hold iff $\Box A \in \Gamma \Rightarrow \Box A, A \in \Delta$.*

Definition 1.7 *A labeled frame $\langle M, R, \nu\rangle$ is* adequate *if R is a transitive relation and $xRy \Rightarrow \nu(x) \prec_\Box \nu(y)$.*

Every adequate labeled frame can be considered as a Kripke model by defining $x \Vdash p \Leftrightarrow p{\in}\nu(x)$. Actually we will not make any notational distinction between these two.

Definition 1.8 *Let $\mathcal{D}$ be a set of formulas and let $\langle M, R, \nu\rangle$ be a labeled adequate frame. We say that* a truth lemma holds in $\langle M, R, \nu\rangle$ w.r.t. $\mathcal{D}$ *if*

$$\forall A{\in}\mathcal{D}\ [\langle M, R, \nu\rangle, x \vdash A \Leftrightarrow A{\in}\nu(x)].$$

Definition 1.9 *Let $\mathcal{D}$ be a set of formulas. A* $\mathcal{D}$-problem *in a labeled frame $\langle M, R, \nu\rangle$ is a pair $\langle x, \neg\Box A\rangle$ such that $x{\in}M$ and $\neg\Box A{\in}\nu(x)$ and for no y with xRy we have $\neg A \in \nu(y)$.*

We will assume that phrases like "a problem in x" and the like are completely unambiguous.

2 The construction

It is easy to prove the following lemma.

Lemma 2.1 *Let $\mathcal{D}$ be a set of formulas that is closed under subformulas and single negation. Let $\langle M, R, \nu\rangle$ be a labeled frame. A truth lemma holds in $\langle M, R, \nu\rangle$ w.r.t. $\mathcal{D}$ iff there are no $\mathcal{D}$-problems in $\langle M, R, \nu\rangle$.*

Our strategy to prove completeness is now evident. By successively eliminating problems we finally get a model for which we have a truth lemma with respect to a set of sentences containing the one for which we had to come up with a countermodel. The following lemma will be the main engine behind our construction method.

Lemma 2.2 *Let Γ be a maximal **GL**-consistent set of sentences with $\neg\Box A \in \Gamma$. There exists a a maximal **GL**-consistent set of sentences Δ such that $\Gamma \prec_\Box \Delta$ containing both $\neg A$ and $\Box A$.*

Proof of Lemma 2.2. Suppose the contrary, that is

$$\{B, \Box B \mid \Box B \in \Gamma\} \cup \{\neg A, \Box A\} \vdash \bot$$

In this case we get for some n, that

$$\bigwedge\!\!\!\bigwedge_{i=0}^{n} \boxdot B_i, \neg A, \Box A \vdash \bot.$$

By the Deduction theorem we get

$$\mathbf{GL} \vdash \bigwedge_{i=0}^{n} \boxdot B_i \to (\Box A \to A).$$

Now, as there are no assumptions, we can do a necessitation and a distribution axiom to obtain.

$$\mathbf{GL} \vdash \Box(\bigwedge_{i=0}^{n} \boxdot B_i) \to \Box(\Box A \to A). \quad (*)$$

By an instantiation of $\Box p \to \Box\Box p$ we see that

$$\mathbf{GL} \vdash \Box(\bigwedge_{i=0}^{n} \boxdot B_i) \leftrightarrow \bigwedge_{i=0}^{n} \Box B_i.$$

Combining this with $(*)$ and applying Löb's axiom we get

$$\mathbf{GL} \vdash \bigwedge_{i=0}^{n} \Box B_i \to \Box A.$$

But this clearly violates the assumption that Γ is consistent.

QED

We now prove the completeness theorem.

Theorem 2.3 Completeness *Let A be such that $\mathbf{GL} \nvdash A$. We can find a tree model M such that at the root r, $\neg A$ holds.*

Proof of Theorem 2.3. Let $\mathcal{D}$ be the smallest set of formulas containing both all the subformulas of A and their (single) negations.

Start with a labeled $\mathbf{GL}$-frame F_0, consisting of just one point (the root) r. We set $\nu(r) = \Gamma$ where Γ is some maximal consistent set containing $\neg A$.

We define $\mathsf{level}(y)$, for points y living in some extension of F_0 to be the length of the longest R-chain starting in r and ending in y. By definition we have $\mathsf{level}(r)$=0.

Lemma 2.2 provides us a way to eliminate a $\mathcal{D}$-problem in an arbitrary level of some labeled adequate frame F_n by adding a new world together with its label to F_n. Although the idea is completely clear, let us spell out how this eliminations runs.

Let therefor F_n be given by $\langle M_n, R_n, \nu_n \rangle$ and let $\langle x, \neg\Box A \rangle$ be some $\mathcal{D}$-problem. Let y be some object not yet in M_n. Let Δ be the maximal $\mathbf{GL}$-consistent set that is provided by Lemma 2.2 by taking Γ to be $\nu_n(x)$. We define $F_{n+1} := \langle M_n \cup \{y\}, \overline{R_n \cup \{\langle x, y \rangle\}}^{\mathsf{Tr}}, \nu_n \cup \{\langle y, \Delta \rangle\} \rangle$. Here $\overline{S}^{\mathsf{tr}}$ denotes the transitive closure the relation S, that is, the smallest relation containing S that contains $\langle x, z \rangle$ whenever it contains both $\langle x, y \rangle$ and $\langle y, z \rangle$. It is now also immediately clear that the label of the newly added world y contains stricly less problems than x does. This is the key observation to obttain termination of our elimination process. Also it is clear that the newly obtained labeled frame is again adequate.

It is clear that the number of $\mathcal{D}$-problems in r is finite. Now we can eliminate all problems at a certain level by the process we described above. As the newly added worlds have strictly less problems, the process finally terminates.

QED

We note that changing the order in which you eliminate problems can yield different counter models. The "minimal countermodel" is obtained by always first eliminating problems at the highest level.

3 Applications

The step-by-step method can now be used for example to construct counter models. The method in full detail works with maximal consistent sets. Of course we will not write these down in there full description. Rather we will just write a finite part of information of this set. As we will later see, the range of applications is rather wide. We will now just give one worked out example of this step-by-step method.

Lemma 3.1 $\textbf{\textit{GL}} \nvdash A \to \Box A \quad (*)$

Proof of Lemma 3.1. If $(*)$ were indeed the case we would be able to start the construction. So, let us do so.

We have assumed that $(*)$ as a scheme. Thus, we are to come up with an instantiation of the scheme that is not provable. We take as our instantiation $A := p$ and we commit ourselves to show that $\mathbf{GL} \nvdash p \to \Box p$. Supposing this is so, we can find a maximal $\mathbf{GL}$ consistent set Γ that contains $\neg(p \to \Box p)$. We form our set $\mathcal{D}$ relative to $\neg(p \to \Box p)$. That is, $\mathcal{D} := \{\neg(p \to \Box p), (p \to \Box p), \Box p, \neg\Box p, p, \neg p\}$.

We are going to make a labeled frame for which a truth lemma w.r.t. $\mathcal{D}$ holds. In view of Lemma 2.1 we are to give a labeled frame without $\mathcal{D}$-problems. Our first approach of a countermodel is the labeled frame consisting of just one world r with label Γ.

As $\neg(p \to \Box p) \in \Gamma$ and as Γ is a maximal consistent set, both p and $\neg\Box p$ are in Γ. Clearly $\langle r, \neg\Box p\rangle$ is a problem. Lemma 2.2 gives us a Δ such that $\Gamma \prec_\Box \Delta$ and $\neg p, \Box p \in \Delta$. As $\Box p \in \Delta$, $\neg\Box p \notin \Delta$ and so Δ is problem free. Our final labeled frame is thus $\langle M, R, \nu\rangle$ with $M = \{r, y\}$, $R = \{\langle r, y\rangle\}$ and $\nu = \{\langle r, \Gamma\rangle, \langle y, \Delta\rangle\}$. The corresponding model is obtained by just reading of the values of the propositional variables. Of course the only salient facts are those concerning p. Thus $r \Vdash p$ and $y \Vdash \neg p$. We now easily check that this is indeed a countermodel, that is, that indeed at the root r we have what we wanted. In symbols $r \Vdash \neg(p \to \Box p)$. Hereby we conclude (invoking the soundness of $\mathbf{GL}$) that $\mathbf{GL} \nvdash p \to \Box p$.

QED