The Big Six and Big Seven of Reverse Mathematics

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Working in Kohlenbach's higher-order RM, we identify two new 'Big' systems.

This is part of my joint project with Dag Normann to investigate the logical and computational properties of the uncountable.

`https://arxiv.org/abs/2102.04787`

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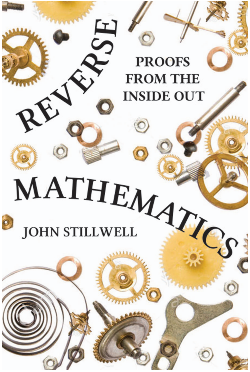

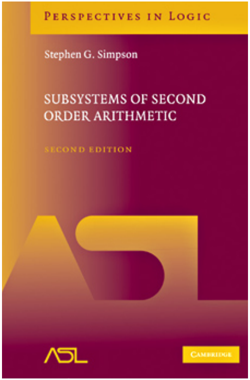

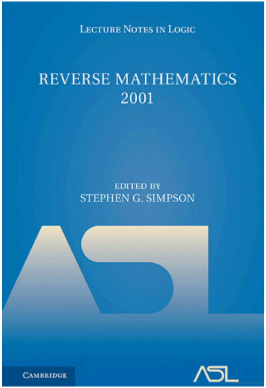

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Most theorems of 'ordinary' mathematics are either provable in RCA_0 or equivalent to one of the 'Big Five' theories.

$=$ Main Theme of RM

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Intuitively, RCA_0 can do computable mathematics (with restricted induction).

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Principle (Weak König's Lemma)

Every infinite binary tree has an infinite path.

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Nonetheless, such maxima and infinite paths are not computable.

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Simpson: connection to Hilbert's program for the FOM. . .

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Similar equivalences for ATR_0 and Π^1_1-CA_0, though some set theory comes to the fore already.

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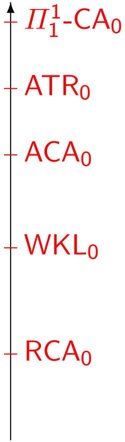

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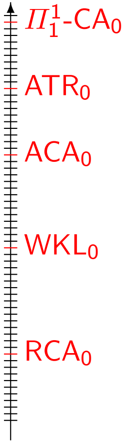

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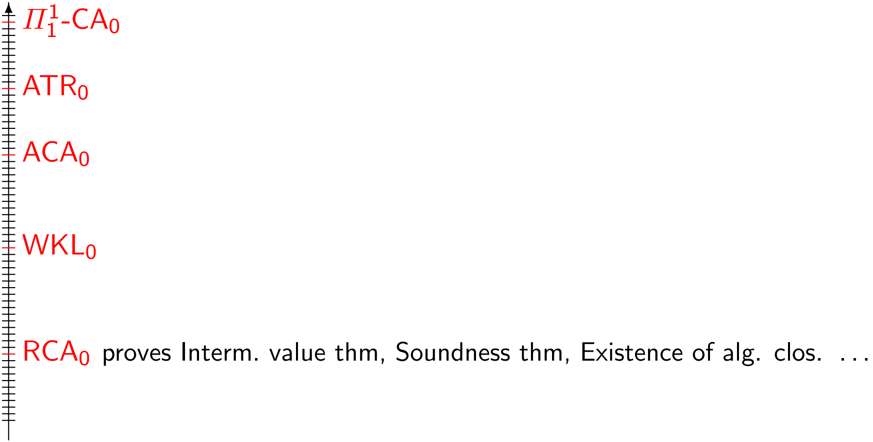

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(Not Absolute: exceptions are in Dzhafarov's RM zoo)

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Distinction between logical formula with mathematical meaning and 'purely logical' formula, i.e. between subject (math) and formalization (logic).

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Our best, most fine-grained foundation of ordinary math?

Representations

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Prime Directive: if one wants to classify theorems as they stand, coding should not change the logical strength of these theorems.

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ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ is defined as follows:

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These two definitions are equivalent in a weak higher-order system based on WKL (Kohlenbach/Kleene).

Hence, coding does not change the logical strength of theorems about continuous functions (assuming WKL is available).

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Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. *Each* f_n *is Riemann integrable on* $[0,1]$.
2. *There is* $M > 0$ *such that* $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.
3. $\lim_{n\to\infty} f_n = f$ *exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

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Massive change of logical strength for a basic theorem about functions that are continuous almost everywhere.

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Similar for other countable objects: they are given by sequences in RM although the original is formulated using sets that are countable (Cantor, König, Ramsey, etc).

Solution

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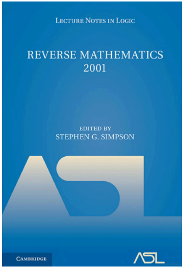
Kohlenbach's higher-order RM, introduced in RM2001.



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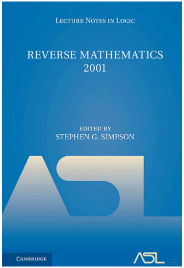

The language of all finite types L_ω has variables for:

$$n \in \mathbb{N}, X \subset \mathbb{N}, F : \mathbb{R} \to \mathbb{R}, \Theta : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}, \ldots$$

The base theory RCA_0^ω proves the same L_2 sentences as RCA_0.

Higher-order counterparts of the Big Five

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Each of the ‘Big Five’ has a higher-order counterpart; we concentrate on the weakest.

$$\ldots \rightarrow \mathsf{ACA}_0 \rightarrow \mathsf{WKL}_0 \rightarrow \mathsf{RCA}_0 \tag{1}$$

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ECF replaces third-order and higher objects by RM-codes (CMTT).

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Thus, theorems about countable sets (injections/bijections to $\mathbb{N}$) have the same first-order strength as RCA_0^ω.

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The following picture emerges:

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Hyper: ACA_0^ω + cocode$_1$ lives as the level of hyperarithmetical analysis. Associated second-order systems are ‘rather logical’

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Principle (cocode_0)

A countable set in $[0,1]$ can be enumerated.

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We observe a certain robustness!

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Item (d) is called the countable chain condition, first formulated by Cantor.

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$\text{CB}\mathbb{N}^+ \leftrightarrow \text{cocode}_0$, where the former expresses that h is locally either f or the inverse of g.

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Principle (HBC_0)

For countable $A \subset \mathbb{R}^2$ with $(\forall x \in I)(\exists (a,b) \in A)(x \in (a,b))$, there is $(a_0,b_0),\dots(a_k,b_k) \in A$ with $(\forall x \in I)(\exists i \leq k)(x \in (a_i,b_i))$.

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Same for many sequential versions, like e.g. sequential ADS, RT22, KL. ...

Separation

The separation axiom as follows

$$(\forall n \in \mathbb{N})(\neg A(n) \vee \neg B(n))$$

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Allowing third-order parameters, there are versions equivalent to HBT and cocode$_i$ for $i = 0, 1$.

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Related results for $\mathbb{R}$ is not the union of countable sets.

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For this concept to make sense, one needs item (c) (and much more). . . .

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The system $\text{ACA}_0^\omega + \text{cocode}_1$ is in the range of hyperarithmetical analysis, and more natural than the known systems.

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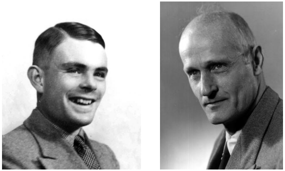
Many equivalences exist and many many more lie in wait.

The future: beyond Kleene and Turing

Our negative results rely on Kleene's S1-S9 computability theory (ITTMs outright compute all the stuff we wish to study).



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 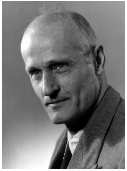

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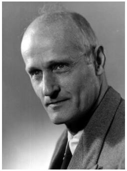

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 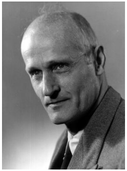

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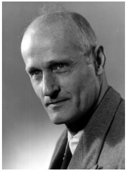

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But we can almost have the best of both worlds!

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Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' means that γ is an RM-code.

Note that $\overline{f}n$ is the finite sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$.

Brouwer to the rescue!

Discontinuous functions (say on $2^{\mathbb{N}}$) are truly third-order, i.e. Turing machines cannot access them in any real/direct way.

But these are the only problematic objects! Intuitively, if a theorem/object does not imply the existence of a discontinuous function (say on $2^{\mathbb{N}}$), then it is provable from a fragment of:

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NFP is a classically equivalent alternative to comprehension from Brouwer's INT. But the '$\gamma \in K_0$' in NFP can be fed to TMs!

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Any (content) questions?