Provability Logics and Applications
Day 1
Provability as modality

David Fernández Duque\textsuperscript{1} and Joost J. Joosten\textsuperscript{2}

1: Universidad de Sevilla;
2: Universitat de Barcelona

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ESSLLI Tutorial, Opole
From now on: no distinction between $A$, $\neg A$, $\neg\neg A$, etc.
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If the context allows us to
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If we allow ourselves to...
From now on: no distinction between $A$, $\neg\neg A$, $\neg A\neg$, etc.

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Easy to see:

$$\mathbb{N} \models \text{Prv}_{PA}(A \rightarrow B) \land \text{Prv}_{PA}(A) \rightarrow \text{Prv}_{PA}(B)$$
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For example via Hilbert style implementation of $\text{Prv}_{PA}$. 
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We can construct $\text{prv}_{PA}(z, B)$
given $\text{prv}_{PA}(x, A \rightarrow B)$ and $\text{prv}_{PA}(y, A)$
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For example via Hilbert style implementation of $\Prv_{PA}$.
We can construct $\text{prv}_{PA}(z, B)$
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Thus:

$$PA \vdash \Prv_{PA}(A \rightarrow B) \land \Prv_{PA}(A) \rightarrow \Prv_{PA}(B)$$

Provable/Formalized Modus Ponens
▶ Provable/Formalized Modus Ponens

We also have a formalized version of the Deduction Theorem.

Theorem:
\[ N \vdash \Prv_{\mathcal{PA}}(A \rightarrow B) \iff \Prv_{\mathcal{PA}}(A) \rightarrow \Prv_{\mathcal{PA}}(B) \]

Proof:

\( \leftarrow \) is easy;

\( \rightarrow \) follows from induction on the length of a proof;

Hilbert style calculus: only deal with Modus Ponens.

Note, \( \mathcal{PA} \) can perform this induction!

So:
\[ \mathcal{PA} \vdash \Prv_{\mathcal{PA}}(A \rightarrow B) \iff \Prv_{\mathcal{PA}}(A) \rightarrow \Prv_{\mathcal{PA}}(B) \]
Provable/Formalized Modus Ponens

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\[ \text{Proof:} \quad \Downarrow \text{is easy; } \quad \Rightarrow \text{follows from induction on the length of a proof; } \]

Hilbert style calculus: only deal with Modus Ponens. Note, PA can perform this induction!

So:

\[ \text{PA} \vdash \text{Prv}_{\text{PA}} + A (B) \Leftrightarrow \text{Prv}_{\text{PA}} (A \rightarrow B) \]
Provable/Formalized Modus Ponens

We also have a formalized version of the Deduction Theorem

\[
\mathbb{N} \models \text{Prv}_{\text{PA} + A}(B) \iff \text{Prv}_{\text{PA}}(A \rightarrow B)
\]
Provable/Formalized Modus Ponens

We also have a formalized version of the Deduction Theorem

**Theorem**

\[ \mathbb{N} \models \text{Prv}_{PA+A}(B) \iff \text{Prv}_{PA}(A \rightarrow B) \]

**Proof:** \( \leftarrow \) is easy;

\[ \text{Prv}_{PA+A}(B) \iff \text{Prv}_{PA}(A \rightarrow B) \]
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We also have a formalized version of the Deduction Theorem

**Theorem**

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**Theorem**

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**Theorem**

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**Proof:** ← is easy;

→ follows from induction on the length of a proof;

Hilbert style calculus: only deal with Modus Ponens.

Note, PA can perform this induction!

So:

\[ PA \vdash \text{Prv}_{PA+A}(B) \leftrightarrow \text{Prv}_{PA}(A \rightarrow B) \]
Gödel I: PA is incomplete
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That is, there is some true sentence $\pi$ with $\text{PA} \not\vdash \pi$
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We have seen such a $\pi$:
- Gödel I: PA is incomplete
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- We have seen such a $\pi$: The Gödel sentence $\lambda$
Gödel I: PA is incomplete
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We have seen such a $\pi$: The Gödel sentence $\lambda$
Note that $\lambda \in \Pi_1$
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Thus: PA is $\Pi_1$-incomplete
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**Theorem** PA is $\Sigma_1$-complete
Gödel I: PA is incomplete

That is, there is some true sentence $\pi$ with $\text{PA} \not\vdash \pi$

We have seen such a $\pi$: The Gödel sentence $\lambda$

Note that $\lambda \in \Pi_1$

Thus: $\text{PA}$ is $\Pi_1$-incomplete

We shall now see that this is optimal

**Theorem** $\text{PA}$ is $\Sigma_1$-complete

$\mathbb{N} \models \sigma \implies \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$
\[ N \models \sigma \implies \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]
\[ \mathbb{N} \models \sigma \implies \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]

**Proof:** by induction on the complexity of \( \sigma \)
\[ \mathbb{N} \models \sigma \implies \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]

**Proof**: by induction on the complexity of \( \sigma \)

- True atomic sentences can all be proved in \( \text{PA} \)
\[ \mathbb{N} \models \sigma \implies \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]

**Proof:** by induction on the complexity of \( \sigma \)

- True atomic sentences can all be proved in \( \text{PA} \)
- \( t_1 = t_2 \) and \( t_1 < t_2 \)
\[ \mathbb{N} \models \sigma \implies \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]

**Proof:** by induction on the complexity of \( \sigma \)

- True atomic sentences can all be proved in PA
- \( t_1 = t_2 \) and \( t_1 < t_2 \)
- By induction on the complexity of \( t_1 \) and sufficient for \( t_1 = \bar{n} \)
\( \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \)

**Proof:** by induction on the complexity of \( \sigma \)

True atomic sentences can all be proved in \text{PA}

\( t_1 = t_2 \) and \( t_1 < t_2 \)

By induction on the complexity of \( t_1 \) and sufficient for \( t_1 = \overline{n} \)

\( n \) times

For example, in \( a + b = S \ S \ldots \ S \ 0 \)
\[ \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \]

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  \[ n \text{ times} \]
- For example, in \( a + b = S S \ldots S 0 \)
  \[ n \text{ times} \]
  - \( b = 0 \), then \( a + 0 = a \) and by induction \( \text{PA} \vdash a = S S \ldots S 0; \)
\( \mathbb{N} \models \sigma \implies PA \vdash \sigma \text{ for } \sigma \in \Sigma_1 \)

**Proof:** by induction on the complexity of \( \sigma \)

True atomic sentences can all be proved in \( PA \)

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For example, in \( a + b = SS \ldots S0 \)

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and using an axiom: \( PA \vdash a + b = SS \ldots S0 \)

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\( \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \)

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\( b = Sb', \) then \( a + Sb' = S(a + b') \) whence by IH

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\( \text{PA} \vdash a + b' = S \ldots S 0 \)
- \( \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \) for \( \sigma \in \Sigma_1 \)

- **Proof**: by induction on the complexity of \( \sigma \)

- True atomic sentences can all be proved in \( \text{PA} \)

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- For example, in \( a + b = SS\ldots S 0 \)

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True atomic sentences can all be proved in PA
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We have seen one simple case
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Many more cases but equally simple
True atomic sentences can all be proved in PA

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Next step: bounded quantification $\forall x<y \psi(x)$
True atomic sentences can all be proved in PA
We have seen one simple case
Many more cases but equally simple
Next step: bounded quantification $\forall x < y \psi(x)$
For each natural number $y$

$$\vdash \forall x < y \psi(x) \leftrightarrow \psi(0) \land \ldots \land \psi(y)$$
True atomic sentences can all be proved in PA

We have seen one simple case

Many more cases but equally simple

Next step: bounded quantification $\forall x < y \psi(x)$

For each natural number $y$

$$PA \vdash \forall x < y \psi(x) \leftrightarrow \underbrace{\psi(0) \land \ldots \land \psi(y)}_{y+1 \text{ conjuncts}}$$

Thus, we can apply our IH
True atomic sentences can all be proved in PA
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Many more cases but equally simple
Next step: bounded quantification $\forall x < y \psi(x)$
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$$\forall x < y \psi(x) \iff \psi(0) \land \ldots \land \psi(y)$$

Thus, we can apply our IH
Likewise for $\exists x < y \psi(x)$
- Boolean connectives: by an easy call to the IH
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Unbounded existential quantification: $\exists x \psi(x)$
- Boolean connectives: by an easy call to the IH
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- also directly from the IH
Boolean connectives: by an easy call to the IH

Unbounded existential quantification: \( \exists x \, \psi(x) \)

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This finishes the proof
Boolean connectives: by an easy call to the IH
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This finishes the proof

$$\mathbb{N} \models \sigma \quad \Rightarrow \quad \text{PA} \vdash \sigma \quad \text{for } \sigma \in \Sigma_1$$
Some remarks on Sigma completeness:

- Goldbach's conjecture: each even number above two is the sum of two prime numbers.
- This is a $\Pi^1_1$ statement.
- Thus: If Goldbach's conjecture is independent of $\text{PA}$, then it is true.

Theorem: If $\text{PA} \vdash \phi$, then $\text{PA} \vdash \text{Prv}_{\text{PA}}(\phi)$

Proof: Remember, representing "provability in $\text{PA}$" implies $p$ is the code of a $\text{PA}$ proof of $\phi$ $\iff N | = \text{prv}_{\text{PA}}(p, \phi)$.

As $\text{Prv}_{\text{PA}}(\phi) \in \Sigma^1_1$ we have $\text{PA} \vdash \phi \Rightarrow N | = \text{Prv}_{\text{PA}}(\phi) \Rightarrow \text{PA} \vdash \text{Prv}_{\text{PA}}(\phi)$.
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**Theorem:** If $\mathsf{PA} \vdash \varphi$, then $\mathsf{PA} \vdash \text{Prv}_{\mathsf{PA}}(\varphi)$
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**Theorem:** If $\text{PA} \vdash \varphi$, then $\text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi)$

**Proof:** Remember, representing “provability in PA” implies

\[ p \text{ is the code of a PA proof of } \varphi \iff \mathbb{N} \models \text{prv}_{\text{PA}}(p, \varphi) \]
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**Proof:** Remember, representing “provability in PA” implies

\[ p \text{ is the code of a PA proof of } \varphi \iff \mathbb{N} \models \text{prv}_{\text{PA}}(p, \varphi) \]

As $\text{Prv}_{\text{PA}}(\varphi) \in \Sigma_1$ we have

\[ \text{PA} \vdash \varphi \Rightarrow \mathbb{N} \models \text{Prv}_{\text{PA}}(\varphi) \Rightarrow \text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi) \]
Sigma completeness: \( \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \) for \( \sigma \in \Sigma_1 \)
Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

The proof gives us slightly more:

$\text{PA} \vdash \sigma \rightarrow \text{Prv}_{T}(\sigma)$ for $\sigma \in \Sigma_1$
Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

The proof gives us slightly more:

**Theorem** $\text{PA} \vdash \text{“}\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma''\text{”}$ for $\sigma \in \Sigma_1$
Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

The proof gives us slightly more:

**Theorem** $\text{PA} \vdash \text{``} \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma'' \text{''}$ for $\sigma \in \Sigma_1$

That is: $\text{PA} \vdash \sigma \rightarrow \text{Prv}_T(\sigma)$ for $\sigma \in \Sigma_1$
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That is: $\text{PA} \vdash \sigma \rightarrow \text{Prv}_T(\sigma)$ for $\sigma \in \Sigma_1$

**Proof:** formalizing exactly the previous proof of Sigma-completeness in PA
Sigma completeness: \( \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma \text{ for } \sigma \in \Sigma_1 \)

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**Theorem** \( \text{PA} \vdash \text{"} \mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma'' \text{"} \text{ for } \sigma \in \Sigma_1 \)

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**Proof:** formalizing exactly the previous proof of Sigma-completeness in PA

We have a sufficient amount of induction
Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

The proof gives us slightly more:

**Theorem** $\text{PA} \vdash "\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma''"$ for $\sigma \in \Sigma_1$

That is: $\text{PA} \vdash \sigma \rightarrow \text{Prv}_T(\sigma)$ for $\sigma \in \Sigma_1$

**Proof:** formalizing exactly the previous proof of Sigma-completeness in PA

We have a sufficient amount of induction

Note that we can do bounded quantification as we have the totality of exponentiation
Sigma completeness: $\mathbb{N} \models \sigma \Rightarrow \text{PA} \vdash \sigma$ for $\sigma \in \Sigma_1$

The proof gives us slightly more:

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That is: $\text{PA} \vdash \sigma \rightarrow \text{Prv}_T(\sigma)$ for $\sigma \in \Sigma_1$

**Proof:** formalizing exactly the previous proof of Sigma-completeness in PA

We have a sufficient amount of induction

Note that we can do bounded quantification as we have the totality of exponentiation

**Corollary:** $\text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi) \rightarrow \text{Prv}_{\text{PA}}(\text{Prv}_{\text{PA}}(\varphi))$
Provable $\Sigma_1$-completeness:

$$PA \vdash \Prv_{PA}(\varphi) \rightarrow \Prv_{PA}(\Prv_{PA}(\varphi))$$
Provable $\Sigma_1$-completeness:

$$\text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi) \rightarrow \text{Prv}_{\text{PA}}(\text{Prv}_{\text{PA}}(\varphi))$$

**Corollary:** Gödel II: If a theory is consistent, it will not prove its own consistency.
Provability as modality

Modal logics

Löb revisited

Arithmetical soundness of GL

Formalized Modus Ponens and Deduction

Sigma completeness of PA

Löb’s theorem

Provable $\Sigma_1$-completeness:

$$PA \vdash \Prv_{PA}(\varphi) \rightarrow \Prv_{PA}(\Prv_{PA}(\varphi))$$

**Corollary:** Gödel II: If a theory is consistent, it will not prove its own consistency.

**Proof** We see that $PA \vdash \lambda \leftrightarrow \neg \Prv_{PA}(\bot)$
Provability as modality
Modal logics
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Löb’s theorem

Provable $\Sigma_1$-completeness:

$$PA \vdash \text{Prv}_{PA}(\varphi) \rightarrow \text{Prv}_{PA}(\text{Prv}_{PA}(\varphi))$$

Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.

Proof We see that $PA \vdash \lambda \leftrightarrow \neg \text{Prv}_{PA}(\bot)$

Clearly, inside $PA$ we have $\neg \text{Prv}_{PA}(\lambda) \rightarrow \neg \text{Prv}_{PA}(\bot)$. 
Provable $\Sigma_1$-completeness:

$$\text{PA} \vdash \Prv_{\text{PA}}(\varphi) \rightarrow \Prv_{\text{PA}}(\Prv_{\text{PA}}(\varphi))$$

**Corollary**: Gödel II: If a theory is consistent, it will not prove its own consistency.

**Proof** We see that $\text{PA} \vdash \lambda \leftrightarrow \neg\Prv_{\text{PA}}(\bot)$

- Clearly, inside $\text{PA}$ we have $\neg\Prv_{\text{PA}}(\lambda) \rightarrow \neg\Prv_{\text{PA}}(\bot)$.
- For the other direction reason in $\text{PA}$ and assume $\neg\Prv_{\text{PA}}(\bot)$.
Provable $\Sigma_1$-completeness:

$$PA \vdash \Prv_{PA}(\varphi) \rightarrow \Prv_{PA}(\Prv_{PA}(\varphi))$$

**Corollary:** Gödel II: If a theory is consistent, it will not prove its own consistency.

**Proof** We see that $PA \vdash \lambda \leftrightarrow \neg \Prv_{PA}(\bot)$

Clearly, inside $PA$ we have $\neg \Prv_{PA}(\lambda) \rightarrow \neg \Prv_{PA}(\bot)$.

For the other direction reason in $PA$ and assume $\neg \Prv_{PA}(\bot)$

Moreover, for a contradiction assume $\Prv_{PA}(\lambda)$
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Provable $\Sigma_1$-completeness:

$$PA \vdash \text{Prv}_{PA}(\varphi) \rightarrow \text{Prv}_{PA}(\text{Prv}_{PA}(\varphi))$$

Corollary: Gödel II: If a theory is consistent, it will not prove its own consistency.

Proof: We see that $PA \vdash \lambda \leftrightarrow \neg \text{Prv}_{PA}(\bot)$

Clearly, inside PA we have $\neg \text{Prv}_{PA}(\lambda) \rightarrow \neg \text{Prv}_{PA}(\bot)$.

For the other direction reason in PA and assume $\neg \text{Prv}_{PA}(\bot)$

Moreover, for a contradiction assume $\text{Prv}_{PA}(\lambda)$

By provable $\Sigma_1$ completeness: $\text{Prv}_{PA}(\text{Prv}_{PA}(\lambda))$, that is $\text{Prv}_{PA}(\neg \lambda)$
**Theorem** If $\text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi) \rightarrow \varphi$, then $\text{PA} \vdash \varphi$
Theorem If $\text{PA} \vdash \text{Prv}_\text{PA}(\varphi) \rightarrow \varphi$, then $\text{PA} \vdash \varphi$

PA is as modest about it’s own correctness as it could possibly be
Theorem If $PA \vdash \text{Prv}_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$

PA is as modest about its own correctness as it could possibly be

Proof: By contraposition supposing $PA \not\vdash \varphi$
Theorem: If $PA \vdash Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$

PA is as modest about its own correctness as it could possibly be.

Proof: By contraposition supposing $PA \not\vdash \varphi$

Thus $PA + \neg \varphi$ is consistent.
Theorem: If $PA \vdash \text{Prv}_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$

PA is as modest about it’s own correctness as it could possibly be

Proof: By contraposition supposing $PA \not\vdash \varphi$

Thus $PA + \neg \varphi$ is consistent

By Gödel 2 for $PA + \neg \varphi$ we get

$$PA + \neg \varphi \not\vdash \text{Con}_{PA+\neg \varphi}$$
Theorem: If $PA \vdash \Prv_{PA}(\varphi) \rightarrow \varphi$, then $PA \vdash \varphi$

PA is as modest about it's own correctness as it could possibly be.

Proof: By contraposition supposing $PA \nvdash \varphi$

Thus $PA + \neg \varphi$ is consistent.

By Gödel 2 for $PA + \neg \varphi$ we get

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By deduction (and the formalized version)

$$PA \nvdash \neg \varphi \rightarrow \text{Con}_{PA}(\neg \varphi)$$
Theorem If $\text{PA} \vdash \text{Prv}_{\text{PA}}(\varphi) \rightarrow \varphi$, then $\text{PA} \vdash \varphi$

PA is as modest about it’s own correctness as it could possibly be

Proof: By contraposition supposing $\text{PA} \not\vdash \varphi$

Thus $\text{PA} + \neg \varphi$ is consistent

By Gödel 2 for $\text{PA} + \neg \varphi$ we get

$$\text{PA} + \neg \varphi \not\vdash \text{Con}_{\text{PA} + \neg \varphi}$$

By deduction (and the formalized version)

$$\text{PA} \not\vdash \neg \varphi \rightarrow \text{Con}_{\text{PA}}(\neg \varphi)$$

And $\text{Con}_{\text{PA}}(\neg \varphi)$ is just $\neg \text{Prv}_{\text{PA}}(\varphi)$
Provability as modality

Modal logics

Löb revisited

Arithmetical soundness of GL

Syntax of modal logics

Various modal logics

PA ⊢ Prv_{PA}(A → B) ∧ Prv_{PA}(A) → Prv_{PA}(B)

Note, this holds for any (possibly non-standard) formulas A and B.

We would like to collect all such principles.

If possible.

We have to find a suitable signature where to collect such principles.

David Fernández Duque and Joost J. Joosten

Provability as modality
PA ⊢ Prv_{PA}(A → B) ∧ Prv_{PA}(A) → Prv_{PA}(B)

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▶ \( \text{PA} \vdash \text{Prv}_{\text{PA}}(A \rightarrow B) \land \text{Prv}_{\text{PA}}(A) \rightarrow \text{Prv}_{\text{PA}}(B) \)

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PA ⊬ Prv_{PA}(A → B) \land Prv_{PA}(A) \rightarrow Prv_{PA}(B)

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We would like to collect all such principles

If possible
PA \vdash \Prv PA (A \rightarrow B) \land \Prv PA (A) \rightarrow \Prv PA (B)

Note, this holds for any (possibly non-standard) formulas $A$ and $B$

We would like to collect all such principles

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Various modal logics

- \( \text{PA} \vdash \Prv_{\text{PA}}(A \rightarrow B) \land \Prv_{\text{PA}}(A) \rightarrow \Prv_{\text{PA}}(B) \)
- Note, this holds for *any* (possibly non-standard) formulas \( A \) and \( B \)
- We would like to collect *all* such principles
- If possible
- We have to find a suitable signature where to collect such principles
- Propositional modal logics
Language of propositional modal logic:
Language of propositional modal logic:
- countable set of propositional variables $\mathbb{P}$;
Language of propositional modal logic:

- countable set of propositional variables \( P \);
- Two logical constants \( \top \) and \( \bot \).
Language of propositional modal logic:
  ▶ countable set of propositional variables $P$;
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Operators of propositional modal logic:
Language of propositional modal logic:
- countable set of propositional variables \( \mathbb{P} \);
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Operators of propositional modal logic:
- Boolean connectives: \( \rightarrow, \wedge \);
Language of propositional modal logic:
- countable set of propositional variables $\mathbb{P}$;
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Operators of propositional modal logic:
- Boolean connectives: $\rightarrow$, $\land$;
- Unary modal operator: $\Box$. 
Language of propositional modal logic:
- countable set of propositional variables \( \mathcal{P} \);
- Two logical constants \( \top \) and \( \bot \).

Operators of propositional modal logic:
- Boolean connectives: \( \rightarrow, \land \);
- Unary modal operator: \( \square \).

All other Boolean connectives are defined as usual:
Language of propositional modal logic:
- countable set of propositional variables $P$;
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Operators of propositional modal logic:
- Boolean connectives: $\to$, $\land$;
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All other Boolean connectives are defined as usual:
- $\neg \psi := \psi \to \bot$;
Language of propositional modal logic:
- countable set of propositional variables \( \mathbb{P} \);
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- Boolean connectives: \( \rightarrow, \land \);
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- \( \neg \psi := \psi \rightarrow \bot \);
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The dual modal operator $\Diamond$ is defined as $\neg \Box \neg$.
Language of propositional modal logic:
- countable set of propositional variables $\mathbb{P}$;
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Operators of propositional modal logic:
- Boolean connectives: $\to$, $\land$;
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All other Boolean connectives are defined as usual:
- $\neg \psi := \psi \to \bot$;
- $\psi \lor \varphi := \neg (\neg \psi \land \neg \varphi)$;
- etc.

The dual modal operator $\diamond$ is defined as $\neg \square \neg$

$\square$ and $\diamond$ bind as $\neg$ and the rest as usual.
Language of propositional modal logic:
- countable set of propositional variables $\mathbb{P}$;
- Two logical constants $\top$ and $\bot$.

Operators of propositional modal logic:
- Boolean connectives: $\to$, $\land$;
- Unary modal operator: $\Box$.

All other Boolean connectives are defined as usual:
- $\neg \psi := \psi \to \bot$;
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- etc.

The dual modal operator $\Diamond$ is defined as $\neg \Box \neg$

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For us:
Language of propositional modal logic:

- countable set of propositional variables $\mathbb{P}$;
- Two logical constants $\top$ and $\bot$.

Operators of propositional modal logic:

- Boolean connectives: $\rightarrow$, $\land$;
- Unary modal operator: $\square$.

All other Boolean connectives are defined as usual:

- $\neg \psi := \psi \rightarrow \bot$;
- $\psi \lor \varphi := \neg (\neg \psi \land \neg \varphi)$;
- etc.

The dual modal operator $\Diamond$ is defined as $\neg \square \neg$

$\square$ and $\Diamond$ bind as $\neg$ and the rest as usual

For us: $\square$ for provable and $\Diamond$ as consistent
Various properties become naturally expressible
Various properties become naturally expressible

Formalized Modus Ponens

\[ \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \]
Various properties become naturally expressible

Formalized Modus Ponens

\[ \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \]

Uniform reflection

\[ \Box p \rightarrow p \]
Various properties become naturally expressible

Formalized Modus Ponens

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

Uniform reflection

$$\Box p \rightarrow p$$

Gödel’s second incompleteness theorem:

$$\Diamond \top \rightarrow \neg \Box \Diamond \top$$
The logic $K$

- All axioms of the form $2(A \rightarrow B) \rightarrow (2A \rightarrow 2B)$
- All propositional tautologies as axioms
- The only rules are Modus Ponens and Necessitation

Non valid reasoning:
- Assume $p$
- Derive $2p$ by Necessitation
- Thus, conclude $p \rightarrow 2p$

Note: $2p \lor \neg 2p$ is also an axiom
The logic $\mathbf{K}$

- All axioms of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
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- All propositional tautologies as axioms
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The logic K

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The only rules are Modus Ponens and Necessitation

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- Derive $\Box p$ by Necessitation
- Thus, conclude $p \rightarrow \Box p$

Note: $\Box p \lor \neg \Box p$ is also an axiom
The logic $K$

- All axioms of the form $\Box(A \to B) \to (\Box A \to \Box B)$
- All propositional tautologies as axioms

The only rules are Modus Ponens and Necessitation

Non valid reasoning:

- Assume $p$

$\Box p \lor \neg \Box p$ is also an axiom
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The only rules are Modus Ponens and Necessitation

Non valid reasoning:

- Assume $p$
- Derive $\Box p$ by Necessitation
- Thus, conclude $p \rightarrow \Box p$

Note: $\Box p \lor \neg \Box p$ is also an axiom
$\vdash K \vdash \Box A \land \Box B \iff \Box (A \land B)$
\( K \vdash \Box A \land \Box B \leftrightarrow \Box (A \land B) \)

**Proof:** \( A \rightarrow (B \rightarrow A \land B) \) is a tautology
\[
\text{Proof: } A \rightarrow (B \rightarrow A \land B) \text{ is a tautology}
\]

Necessitation and \( K \) axiom twice to obtain
K ⊨ □A ∧ □B ↔ □(A ∧ B)

Proof: A → (B → A ∧ B) is a tautology

Necessitation and K axiom twice to obtain

□A → (□B → □(A ∧ B))
\[
\begin{align*}
\text{\textbf{K}} & \vdash \Box A \land \Box B \iff \Box (A \land B) \\
\text{Proof: } & A \to (B \to A \land B) \text{ is a tautology} \\
\text{Necessitation and } \textbf{K} \text{ axiom twice to obtain} \\
& \Box A \to (\Box B \to \Box (A \land B)) \\
\text{Use the tautology} \\
& (\Box A \to (\Box B \to \Box (A \land B))) \to (\Box A \land \Box B \to \Box (A \land B))
\end{align*}
\]
K ⊨ □A ∧ □B ↔ □(A ∧ B)

Proof: \( A \rightarrow (B \rightarrow A \land B) \) is a tautology

Necessitation and K axiom twice to obtain

\[ \square A \rightarrow (\square B \rightarrow \square (A \land B)) \]

Use the tautology

\[ (\square A \rightarrow (\square B \rightarrow \square (A \land B))) \rightarrow (\square A \land \square B \rightarrow \square (A \land B)) \]

The other direction is similar starting with \( A \land B \rightarrow A \)
The logic \textbf{K4}: as \textbf{K} but now adding all axioms of the form
\[ \Box A \rightarrow \Box \Box A \]
The logic **K4**: as **K** but now adding all axioms of the form
\[ \square A \rightarrow \square \square A \]

The logic **GL**: as **K** but now adding all axioms of the form
\[ \square(\square A \rightarrow A) \rightarrow \square A \]
We shall see that

\[
K + \{ \Box (\Box A \rightarrow A) \rightarrow \Box A \mid A \text{ a modal formula} \} \vdash \Box B \rightarrow \Box \Box B.
\]
We shall see that

\[ K + \{ \Box (\Box A \to A) \to \Box A \mid A \text{ a modal formula} \} \vdash \Box B \to \Box \Box B. \]

Proof:

\[ K \vdash \Box B \to \Box (\Box (\Box B \land B) \to \Box B \land B) \]
We shall see that

$$K + \{\Box(\Box A \rightarrow A) \rightarrow \Box A \mid A \text{ a modal formula}\} \vdash \Box B \rightarrow \Box \Box B.$$ 

Proof:

$$K \vdash \Box B \rightarrow \Box(\Box(\Box B \land B) \rightarrow \Box B \land B)$$

Next, apply Löb to $\Box B \land B$. 
We shall see that

$$K + \{ \Box(\Box A \rightarrow A) \rightarrow \Box A \mid A \text{ a modal formula} \} \vdash \Box B \rightarrow \Box \Box B.$$ 

Proof:

$$K \vdash \Box B \rightarrow \Box(\Box(\Box B \land B) \rightarrow \Box B \land B)$$

Next, apply L"ob to $\Box B \land B$.

From now on we shall sometimes refer to $GL$ as containing the axioms $\Box A \rightarrow \Box \Box A$. 
We shall see that

\[ \text{K} + \{ \Box (\Box A \rightarrow A) \rightarrow \Box A \mid A \text{ a modal formula} \} \vdash \Box B \rightarrow \Box \Box B. \]

**Proof:**

\[ \text{K} \vdash \Box B \rightarrow \Box (\Box (\Box B \land B) \rightarrow \Box B \land B) \]

Next, apply Löb to \( \Box B \land B \).

From now on we shall sometimes refer to \( \text{GL} \) as containing the axioms \( \Box A \rightarrow \Box \Box A \)

and sometimes as not containing those axioms
We considered the liar $\lambda$:

$$\lambda \leftrightarrow \neg \text{Prv}_{PA}(\lambda)$$

The liar is both true and independent.

What about the truth-teller?

$$\tau \leftrightarrow \text{Prv}_{PA}(\tau)$$

Now that we have a link to modal logic, we shall often write $2PA$ for $\text{Prv}_{PA}$.

By Löb we know $PA \vdash 2PA(\tau) \rightarrow \tau = \Rightarrow PA \vdash \tau$.

Thus, the truth-teller is both true and provable.

First proven by Löb.
We considered the liar $\lambda$: $\lambda \leftrightarrow \neg \text{Prv}_{PA}(\lambda)$
We considered the liar \( \lambda : \lambda \leftrightarrow \neg \text{Prv}_{\text{PA}}(\lambda) \)

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What about the truth-teller?: \( \tau \leftrightarrow \Prv_{PA}(\tau) \)
We considered the liar $\lambda$: $\lambda \leftrightarrow \neg \Prv_{PA}(\lambda)$

The liar is both true and independent

What about the truth-teller?: $\tau \leftrightarrow \Prv_{PA}(\tau)$

Now that we have a link to modal logic, we shall often write $\Box_{PA}$ for $\Prv_{PA}$
We considered the liar $\lambda$: $\lambda \leftrightarrow \neg \Prv_{PA}(\lambda)$

The liar is both true and independent

What about the truth-teller? $\tau \leftrightarrow \Prv_{PA}(\tau)$

Now that we have a link to modal logic, we shall often write $\square_{PA}$ for $\Prv_{PA}$

By Löb we know $PA \vdash \square_{PA}(\tau) \rightarrow \tau \implies PA \vdash \tau$
We considered the liar $\lambda$: $\lambda \leftrightarrow \neg \text{Prv}_{PA}(\lambda)$

The liar is both true and independent

What about the truth-teller?: $\tau \leftrightarrow \text{Prv}_{PA}(\tau)$

Now that we have a link to modal logic, we shall often write $\Box_{PA}$ for $\text{Prv}_{PA}$

By Löb we know $\text{PA} \vdash \Box_{PA}(\tau) \rightarrow \tau \implies \text{PA} \vdash \tau$

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What about the truth-teller?: $\tau \leftrightarrow \Prv_{PA}(\tau)$

Now that we have a link to modal logic, we shall often write $\Box_{PA}$ for $\Prv_{PA}$

By L"ob we know $\text{PA} \vdash \Box_{PA}(\tau) \rightarrow \tau \implies \text{PA} \vdash \tau$

Thus, the truth-teller is both true and provable

First proven by L"ob
We shall give Löb's proof for the sake of practicing with fixpoints and for beauty based on a proof of the following theorem.

Theorem: Sinterklaas (Saint Nicholas) exists

Proof:
We shall give Löb’s proof
for the sake of practicing with fixpoints and for beauty
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for the sake of practicing with fixpoints and for beauty
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**Theorem** Sinterklaas (Saint Nicholas) exists
We shall give Löb’s proof
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\textbf{Theorem} Sinterklaas (Saint Nicholas) exists
\textbf{Proof}:
We shall give Löb’s proof
for the sake of practicing with fixpoints and for beauty
based on a proof of the following theorem
Theorem Sinterklaas (Saint Nicholas) exists
Proof:
Sinterklaas (Saint Nicholas) exists
- Sinterklaas (Saint Nicholas) exists
- **Proof:** If this sentence is true, then Sinterklaas exists
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]
Sinterklaas (Saint Nicholas) exists

Proof: If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

Then \( A \rightarrow S \) via Modus Ponens
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

Then \( A \rightarrow S \) via Modus Ponens

Using our assumption again, we get \( S \)
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

Then \( A \rightarrow S \) via Modus Ponens

Using our assumption again, we get \( S \)

We conclude \( A \rightarrow S \) discharging Assumption 1
Provability as modality
Modal logics
Löb revisited
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Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

Then \( A \rightarrow S \) via Modus Ponens

Using our assumption again, we get \( S \)

We conclude \( A \rightarrow S \) discharging Assumption 1

This is just \( A \)
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \leftrightarrow (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

Then \( A \rightarrow S \) via Modus Ponens

Using our assumption again, we get \( S \)

We conclude \( A \rightarrow S \) discharging Assumption 1

This is just \( A \)

Applying twice Modus Ponens we get \( S \)
Sinterklaas (Saint Nicholas) exists

**Proof:** If this sentence is true, then Sinterklaas exists

\[ A \iff (A \rightarrow S) \]

Suppose \( A \) (Assumption 1)

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Using our assumption again, we get \( S \)

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Theorem (Löb) If $\text{PA} \vdash \Box_{\text{PA}} \psi \rightarrow \psi$, then $\text{PA} \vdash \psi$
Theorem (Löb) If $\text{PA} \vdash \Box_{\text{PA}} \psi \rightarrow \psi$, then $\text{PA} \vdash \psi$

Proof We consider $\chi$ with $\text{PA} \vdash \chi \leftrightarrow (\Box_{\text{PA}} \chi \rightarrow \psi)$ and reason in $\text{PA}$
Theorem (Löb) If $\text{PA} \vdash \Box_{\text{PA}} \psi \rightarrow \psi$, then $\text{PA} \vdash \psi$

Proof We consider $\chi$ with $\text{PA} \vdash \chi \leftrightarrow (\Box_{\text{PA}} \chi \rightarrow \psi)$ and reason in $\text{PA}$

Thus, by necessitation and distribution

$$\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \Box_{\text{PA}} \chi \rightarrow \Box \psi)$$
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$$\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \Box_{\text{PA}} \chi \rightarrow \Box \psi)$$

By transitivity $\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \chi \rightarrow \Box \psi)$
Theorem (Löb) If $\text{PA} \vdash \Box_{PA} \psi \rightarrow \psi$, then $\text{PA} \vdash \psi$

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By transitivity $\Box_{PA} \chi \rightarrow (\Box_{PA} \chi \rightarrow \Box \psi)$

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By transitivity $\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \chi \rightarrow \Box \psi)$

which is just $\Box_{\text{PA}} \chi \rightarrow \Box \psi$

By assumption

$$\Box_{\text{PA}} \chi \rightarrow \psi$$ (1)
Theorem (Löb) If \( \text{PA} \vdash \Box_{\text{PA}} \psi \rightarrow \psi \), then \( \text{PA} \vdash \psi \)

Proof We consider \( \chi \) with \( \text{PA} \vdash \chi \leftrightarrow (\Box_{\text{PA}} \chi \rightarrow \psi) \) and reason in \( \text{PA} \)

Thus, by necessitation and distribution

\[ \Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \Box_{\text{PA}} \chi \rightarrow \Box \psi) \]

By transitivity \( \Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \chi \rightarrow \Box \psi) \)
which is just \( \Box_{\text{PA}} \chi \rightarrow \Box \psi \)

By assumption

\[ \Box_{\text{PA}} \chi \rightarrow \psi \quad (1) \]

Thus \( \chi \), whence by Nec. \( \Box \chi \) and MP on (1) we get \( \psi \)
Theorem (Löb) If $\text{PA} \vdash \Box_{\text{PA}} \psi \rightarrow \psi$, then $\text{PA} \vdash \psi$

Proof We consider $\chi$ with $\text{PA} \vdash \chi \leftrightarrow (\Box_{\text{PA}} \chi \rightarrow \psi)$ and reason in $\text{PA}$

Thus, by necessitation and distribution

$$\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \Box_{\text{PA}} \chi \rightarrow \Box \psi)$$

By transitivity $\Box_{\text{PA}} \chi \rightarrow (\Box_{\text{PA}} \chi \rightarrow \Box \psi)$

which is just $\Box_{\text{PA}} \chi \rightarrow \Box \psi$

By assumption

$$\Box_{\text{PA}} \chi \rightarrow \psi$$  \hspace{1cm} (1)

Thus $\chi$, whence by Nec. $\Box \chi$ and MP on (1) we get $\psi$.
Arithmetical realization: $f : \mathbb{P} \rightarrow \text{Sent}_{PA}$
- **Arithmetical realization**: $f : \mathbb{P} \rightarrow \text{Sent}_{PA}$
- We extend $f$ to be defined on all modal formulas:

  - $f(\top) = \top$
  - $f(\bot) = \bot$
  - $f(\square A) = \square f(A)$

**Theorem**

If $GL \vdash A$, then for any arithmetical realization $f$,

$PA \vdash f(A)$

**Proof**

By induction on the proof $A$ in $GL$

Let Löb’s rule be $(\square A \rightarrow A)$ 

It is easy to show that $K_4 + LR \vdash \square (\square A \rightarrow A) \rightarrow \square A$
- *Arithmetical realization:* \( f : \mathcal{P} \to \text{Sent}_{PA} \)

- We extend \( f \) to be defined on all modal formulas:
  - \( f \) commutes with Boolean connectives;
Arithmetical realization: $f : \mathbb{P} \rightarrow \text{Sent}_{PA}$

- We extend $f$ to be defined on all modal formulas:
  - $f$ commutes with Boolean connectives;
  - In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
Arithmetical realization: $f : \mathbb{P} \rightarrow \text{Sent}_{PA}$

We extend $f$ to be defined on all modal formulas:

- $f$ commutes with Boolean connectives;
- In particular, $f(\top) = \top$ and $f(\bot) = \bot$;
- $f(\square A) = \square_{PA} f(A)$. 

Theorem
If $GL \vdash A$, then for any arithmetical realization $f$,

$PA \vdash f(A)$

Proof
By induction on the proof $A$ in $GL$.

Let Löb’s rule be $(\square A \rightarrow A) / A$.

It is easy to show that $K4 + LR \vdash (\square A \rightarrow A) \rightarrow \square A$.
Arithmetical realization: $f : \mathbb{P} \rightarrow \text{Sent}_{PA}$

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Theorem: If \( \text{GL} \vdash A \), then for any arithmetical realization \( f \), \( \text{PA} \vdash f(A) \)

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Arithmetical realization: $f : \mathbb{P} \rightarrow \text{Sent}_{\text{PA}}$

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Theorem If $\text{GL} \vdash A$, then for any arithmetical realization $f$, $\text{PA} \vdash f(A)$

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**Theorem** If $GL \vdash A$, then for any arithmetical realization $f$, $PA \vdash f(A)$

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