Problems collected at the Wormshop 2012 in Barcelona

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Abstract

This list contains a selection of problems that were discussed during the Wormshop 2012 held in Barcelona. The list is mainly based on Lev's presentation on open problems held during the Wormshop. Problems are formulated very briefly and the list is not meant to be exhaustive. Any additional new questions would be welcome.

Problems

1. Does Japaridze's arithmetical completeness theorem for GLP hold for theories weaker than PA, e.g., for Elementary Arithmetic EA? More specifically, is GLP complete if one interprets each modality $\langle n \rangle$ over EA as RFN_{Σ_n}? It is easy to see that GLP is sound w.r.t. this interpretation, however the arithmetical completeness proofs we know so far all seem to require a base theory as strong as PA.

   A simpler version of this question concerns the standard provability logic GL: is it complete w.r.t. the interpretation of $\Diamond$ as RFN_{Σ_1}, aka formalized 1-consistency, over EA? $B\Sigma_1$ seems to be needed for the standard Solovay-type argument.

2. Is there an axiomatization of admissible rules of GLP similar to the one for GL (given by Jeřábek). Is the set of admissible rules of GLP decidable?

3. Is there a nice cut-free calculus for GLP?

4. Develop projective formulas and study the unification problem for GLP along the lines of the work of Ghilardi. It follows from the uniform interpolation theorem for GLP obtained by Shamkanov that every finitely generated subalgebra of a free GLP-algebra (in the language with finitely many modalities) is finitely presented, i.e., given by finitely many relations. Relations of finitely generated subalgebras of free GLP-algebras correspond to projective formulas.

5. Shavrukov has given a full characterization of the r.e. subalgebras of the Magari algebra of PA and Zambella pushed this result down to EA. Is there a characterization of r.e. subalgebras of the GLP-algebra of PA in the style of Shavrukov? (We also want it for EA, but see Question 1.)

6. Consider the standard ordinal GLP-space, that is, a sequence of topologies $(\tau_n)_{n\in\omega}$ on the class of all ordinals obtained from the interval topology by iterating the plus operation. (Recall that the topology τ^+ is generated as a subbase from a given topology τ and all sets of the form $d_\tau(A)$ where A is a subset of the given space.) Is the statement "τ_{n+1} is non-discrete" equiconsistent with the existence of a Π^1_n-indescribable cardinal?

   It is known that, for $n > 0$, Π^1_n-indescribable cardinals guarantee that τ_{n+2} is non-discrete (Philip Schlicht). If $n = 1$ we know that the converse statement also holds: limit points of τ_2 are the so-called doubly reflecting cardinals. By a result of Magidor, in L the latter are the same as weakly compact ($= \Pi^1_1$-indescribable) cardinals.

7. Is GLP complete w.r.t. the standard ordinal GLP-space under the assumptions that $V = L$ and Π^1_n-indescribable cardinals exist, for all $n \in \omega$?

8. Consider the class of GLP-spaces generated from a scattered space, that is, their topologies are obtained from some scattered topology by iterating the plus operation. The standard ordinal space (restricted to any particular ordinal) is an example of such. Does ZFC prove that GLP is complete w.r.t. the class of all such spaces?

9. Can we prove within ZFC a topological completeness theorem for GLP w.r.t. some class of GLP-spaces that are defined 'constructively,' that is, without heavily relying on Zorn's lemma?

10. Are there manageable countable (topology based) models for GLP?

11. Phrase the analogy between large cardinals related to topological completeness and the large cardinals that occur in proof-theoretical ordinals.

12. Are there natural atomless GLP-algebras apart from the provability algebras? (See the next example.)

13. The following Magari algebra appeared in the work of Shelah on the monadic theories of ordinals: $(\mathcal{P}(\omega_2)/\mathsf{NS}, d)$, where NS denotes the nonstationary ideal and d is the Mahlo operation, i.e., the derivative operator for the club topology on ω_2. It is open whether the elementary theory of this algebra in L is decidable (Magidor).

14. It was noticed by Weiermann that the system of ordinal notations coming from GLP is similar to the one considered by Schütte and Simpson. The latter was derived from a system of ordinal notations for Π^1_1-CA + BI by deleting the ordinal addition. Can this be explained from a provability

algebraic point of view? In particular, is there a provability algebraic analog of ordinal addition?

15. Study the scattered topologies on ordinals corresponding to transfinite iterations of a derivative operator.

16. Study the other scattered topologies on ordinals. For example, is GL complete w.r.t. the derivative operator of the topology corresponding to measurable filter, under suitable set-theoretic assumptions? (This question comes from a paper by Blass.)

17. Is there a predicative proof-theoretic interpretation of the system GLP_Λ satisfying the reduction property? Beklemishev and Dashkov obtained a such an interpretation for theories of iterated Tarskian truthpredicates, however to ensure the reduction property the system had to be extended by additional 'limit' modalities. Is it possible to avoid the use of limit modalities?

18. Give a provability-algebraic ordinal analysis of impredicative theories such as ID_1 and Π^1_1-CA. As a first step in this direction, state a reduction property for the operator of ω-model reflection in second-order arithmetic.