Interpolation Properties for Provability Logics GL and GLP

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Craig interpolation

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If $\mathsf{L} \vdash \varphi \to \psi$, then there is a formula θ containing only common variables of φ and ψ such that $\mathsf{L} \vdash \varphi \to \theta$, $\mathsf{L} \vdash \psi \to \theta$.

Lyndon interpolation

is a strengthening of the Craig one by the additional requirement:

every propositional variable that has a positive (negative) occurrence in θ must also have positive (negative) occurrences both in φ and ψ.

Uniform interpolation

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For any formula φ and any propositional variable p, there is a formula θ such that $\mathsf{PV}(\theta) \subset \mathsf{PV}(\varphi) \setminus \{p\}$ and for all formulas ψ with $p \notin \mathsf{PV}(\psi)$ one has $\mathsf{L} \vdash \varphi \to \psi \iff \mathsf{L} \vdash \theta \to \psi$.

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Theorem (Pitts 1992)

Intuitionistic propositional logic enjoys the uniform interpolation.

Examples

The uniform interpolation holds for:

- Classical propositional logic
  the uniform interpolant for $\varphi(p)$ is $\varphi(\bot) \vee \varphi(\top)$
- K, T, GL, Grz, S5
- μK

Interpolation is not uniform for:

- K4, S4
- First order logic
- J

Gödel-Löb logic

Axioms

- Boolean tautologies
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

Rules

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{\Box\varphi}$$

Interpolation properties

- Craig (Smoryński 1978, Boolos 1979)

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Interpolation properties

- Craig (Smoryński 1978, Boolos 1979)
- Uniform (Shavrukov 1993)
- Lyndon

Japaridze polymodal logic GLP

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- $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$
- $[n]([n]\varphi \to \varphi) \to [n]\varphi$ (Löb's axiom)
- $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$ for $m < n$
- $[m]\varphi \to [n]\varphi$ for $m < n$ (monotonicity axiom)

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$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{[n]\varphi}$$

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	GL	GLP
Craig	Smoryński 1978 Boolos 1979	Ignatiev 1993 Beklemishev 2010
Lyndon	$+$	
Uniform	Shavrukov 1993	$+$

Uniform interpolation for GLP:

proof sketch

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Equivalent formulation

For any formula φ and any propositional variable p, there is a formula θ such that $\mathrm{PV}(\theta) \subset \mathrm{PV}(\varphi) \setminus \{p\}$ and

- $\mathsf{L} \vdash \varphi \to \theta$;
- if $\mathsf{L} \vdash \varphi \to \psi$ and $p \notin \mathrm{PV}(\psi)$, then $\mathsf{L} \vdash \theta \to \psi$.

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Equivalent formulation

For any formula φ and any propositional variable p, there is a formula θ such that $\mathrm{PV}(\theta) \subset \mathrm{PV}(\varphi) \setminus \{p\}$ and

- $\mathsf{L} \vdash \varphi \to \theta$;
- if $\mathsf{L} \vdash \varphi \to \psi$ and $p \notin \mathrm{PV}(\psi)$, then $\mathsf{L} \vdash \theta \to \psi$.

The formula θ, if it exists in a logic L, will be denoted by $(\exists p\, \varphi)_{\mathsf{L}}$.

Proof sketch:

- reduce GLP to a logic M
- interpret fragments of M with finite number of modalities in the modal μ-calculus
- show that the uniform interpolation of the modal μ-calculus implies the same property for M (and thus for GLP)

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M is obtained from GLP by excluding the axiom schema:

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Lemma

$$\mathsf{GLP} \vdash \varphi \iff \mathsf{M} \vdash C(\varphi) \to \varphi,$$

where

$$C(\varphi) \equiv \boxdot \bigwedge_{i<s} (\langle m_i \rangle \varphi_i \to [m_i + 1]\langle m_i \rangle \varphi_i)$$

and $\langle m_i \rangle \varphi_i$ for $i < s$ are all subformulas of φ of the form $\langle k \rangle \psi$, $k < m(\varphi)$.

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where $\langle m_i \rangle \varphi_i$ for $i < s$ are all subformulas of φ of the form $\langle k \rangle \psi$ (we don't require the condition $k < m(\varphi)$).

Then $(\exists p\, \varphi)_{\mathrm{GLP}}$ can be defined as $(\exists p\, (\bar{C}(\varphi) \wedge \varphi))_{\mathrm{M}}$.

Fact (interpolation for M)

If M $\vdash \varphi \to \psi$, then there is a Lyndon interpolant θ such that $m(\theta) \leqslant min\{m(\varphi), m(\psi)\}$.

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$(\exists p\ \varphi)_{\mathsf{M}}$ is simply $(\exists p\ \varphi)_{\mathsf{M}_{m(\varphi)+1}}$.

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Introduction to the μ-calculus

The modal μ-calculus extends the language of modal logic with

$$\mu p.\, \varphi, \quad \nu p.\, \varphi \qquad (\varphi \text{ is positive in } p)$$

Semantics: we interpret a modal formula $\varphi(p)$ over a Kripke model $\mathcal{M} = (W, \vDash)$ as a monotone operator sending

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$$\mathcal{M}, x \vDash \mu p.\,\varphi \quad \Longleftrightarrow \quad x \in LFP(\varphi)$$

Examples

- Reachability: $\mu q. (p \vee \Diamond q)$
- Existence of an infinite path: $\nu q. \Diamond q$
- Existence of a path where p appears infinitely often: $\nu q. \mu r. ((p \wedge \Diamond q) \vee \Diamond r)$

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Example

K4 and S4 fail to have the uniform interpolation but μK4 and μS4 enjoy this property.

Interpretations in μ-calculus (D'Agostino)

Programs

Let φ represents any closed μ-formula, then

$$\alpha \;=\; k \mid \varphi? \mid \alpha \cdot \alpha \mid \alpha \cup \alpha \mid \alpha^\star$$

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Semantics

Given a Kripke frame $(W; \{R_k\})$, and a program α, then R_α is a binary relation defined as follows:

- R_k is already defined
- $R_{\varphi?} = \{(w, w) \in W^2 : w \vDash \varphi\}$
- $R_{\alpha\beta} = R_\alpha \circ R_\beta$
- $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$
- $R_{\beta*} = (R_\beta)^*$.

Now we define an abbreviation $[\alpha]\varphi$:

- $[k]\psi$ is already defined
- $[\varphi?]\psi \equiv \varphi \to \psi$
- $[\alpha\beta]\psi \equiv [\alpha][\beta]\psi$
- $[\alpha \cup \beta]\psi \equiv [\alpha]\psi \land [\beta]\psi$
- $[\alpha^*]\psi \equiv \nu p.(\psi \land [\alpha]p)$, where p doesn't occur in ψ freely.

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Fact

For any Kripke model $\mathcal{M}$ we have

$$\mathcal{M}, x \vDash [\alpha]\varphi \iff \forall y\, (xR_\alpha y \Rightarrow \mathcal{M}, y \vDash \varphi).$$

Interpretation

Let $\pi : \{\textit{modalities}\} \longrightarrow \{\textit{programs}\}$.

For a μ-formula φ, we replace all occurrences of $[k]$ by $[\pi(k)]$ and denote the result by $\pi(\varphi)$.

For a Kripke frame $\mathcal{W} = (W; \{R_k\})$, define $\mathcal{W}_\pi = (W; \{R_{\pi(k)}\})$.
Analogeously for Kripke models.

Fact

For any Kripke model $\mathcal{M}$ we have

$$\mathcal{M}_\pi, x \vDash \varphi \Longleftrightarrow \mathcal{M}, x \vDash \pi(\varphi).$$

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- $\mu\mathsf{S4} \vdash \varphi \leftrightarrow \pi(\varphi)$;
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Now $(\exists p\ \varphi)_{\mu\mathsf{S4}}$ can be defined as $(\exists p\ \pi(\varphi))_{\mu\mathsf{K}}$

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Thus GL has the uniform interpolation

M_n with n modalities $\{0, 1, \ldots, n-1\}$

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Let $\pi(k) = (\mu p.\, [\bar{0}]p)?(\bar{k})^+$,
where $\bar{k} = k \cup (k+1) \cup \ldots \cup (n-1)$.

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Let $\pi(k) = (\mu p. [\bar{0}]p)?(\bar{k})^+$,
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Lemma

We have

- $\mathcal{W}_\pi$ is a M_n-frame for any $\mathcal{W}$;
- $\mathcal{W}$ is a M_n-frame $\Rightarrow \mathcal{W}_\pi = \mathcal{W}$.

Lemma

We have

- $M_n \vdash \varphi \leftrightarrow \pi(\varphi)$;
- $M_n \vdash \varphi \Leftrightarrow \mu K_n \vdash \pi(\varphi)$.

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Fact (μ-elimination)

$\mathsf{M}_n = \mu\mathsf{M}_n$; i.e., every μ-formula φ is equivalent in $\mu\mathsf{M}$ to a μ-free formula ψ such that $\mathsf{PV}(\psi) \subset \mathsf{PV}(\varphi)$ and $m(\psi) \leq m(\varphi)$.

Lemma

For any n the system M_n enjoys uniform interpolation.

Proof

Let us check that $(\exists p\, \varphi)_{\mathsf{M}_n}$ can be defined as $(\exists p\, \pi(\varphi))_{\mu\mathsf{K}_n}$. Here π is the constructed interpretation of M_n in $\mu\mathsf{K}_n$.

$$\mu\mathsf{K}_n \vdash \pi(\varphi) \to (\exists p\, \pi(\varphi))_{\mu\mathsf{K}_n}, \quad \mathsf{M}_n \vdash \varphi \leftrightarrow \pi(\varphi), \quad \mu\mathsf{K}_n \subset \mu\mathsf{M}_n = \mathsf{M}_n.$$

We get

$$\mathsf{M}_n \vdash \varphi \to (\exists p\, \pi(\varphi))_{\mu\mathsf{K}_n}.$$

If $\mathsf{M}_n \vdash \varphi \to \psi$ and $p \notin \mathsf{PV}(\psi)$, then

$$\mu\mathsf{K}_n \vdash \pi(\varphi) \to \pi(\psi), \qquad \mu\mathsf{K}_n \vdash (\exists p\, \pi(\varphi))_{\mu\mathsf{K}_n} \to \pi(\psi).$$

Therefore,

$$\mathsf{M}_n \vdash (\exists p\, \pi(\varphi))_{\mu\mathsf{K}_n} \to \psi.$$

q.e.d.

If $\mathsf{M}_n \vdash \varphi \to \psi$ and $p \notin \mathsf{PV}(\psi)$, then

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Theorem

GLP enjoys uniform interpolation.