Decidability of the elementary theory of the free **0**-generated **GLP**-algebra and some related questions.

Fedor Pakhomov

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GLP-Algebras

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$\mathbf{GLP}_1$-algebras are known as Magari algebras.

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Semilattices

$$\langle k \rangle x \leftrightharpoons \sim[k]\sim x$$

$W \subset \mathfrak{G}$ is the set generated from $\mathbf{1}$ by $\langle 0 \rangle, \langle 1 \rangle, \ldots$
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$\mathrm{Th}(W_n, \wedge)$ *is decidable iff* $n \leq 2$.

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Theorem (J.R. Büchi '60)

Suppose α *is an ordinal. Then* $\mathrm{Th}(\alpha, <)$ *is decidable. Practically weak monadic theory of* $(\alpha, <)$ *is decidable*

Ordinal notations up to ε_0

Some ordinals:

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- $\mathrm{Th}(W_3, <_0, \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle)$ *is decidable.*

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Some Functors and Morphisms

$\mathbf{O}_n^m \colon \mathcal{M}_n \hookrightarrow \mathcal{M}_m$ is the forgetful functor ($m \leq n$).

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We consider the family of morphisms: $\mathfrak{G}_n^m \xrightarrow{e_n^m} \mathfrak{G}_{n+1}^m \in \mathrm{Hom}(\mathcal{M}_m)$,
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Lemma

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$$\mathcal{M}_n: \quad \mathfrak{G}^n_n \xrightarrow{e^n_n} \mathfrak{G}^n_{n+1} \xrightarrow{e^n_{n+1}} \mathfrak{G}^n_{n+2} \xrightarrow{e^n_{n+2}} \mathfrak{G}^n_{n+3} \xrightarrow{e^n_{n+3}} \cdots \qquad \mathfrak{G}^n_\infty$$

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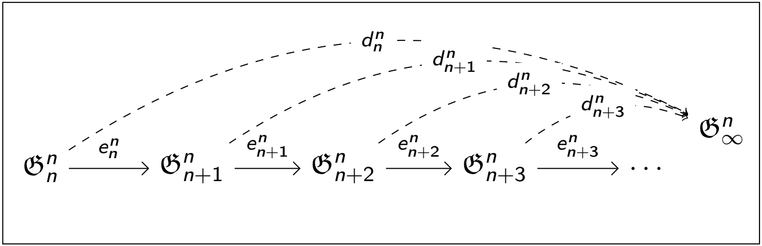

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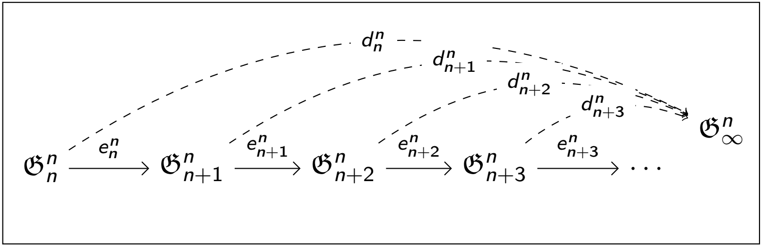
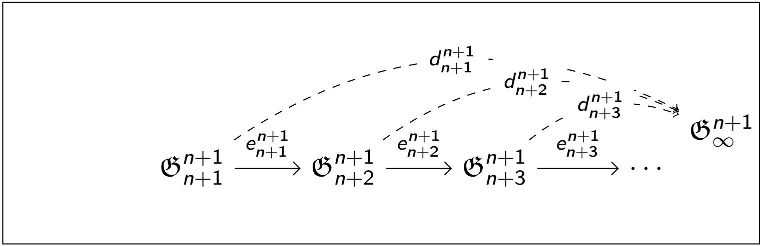


$\mathcal{M}_{n+1}$:



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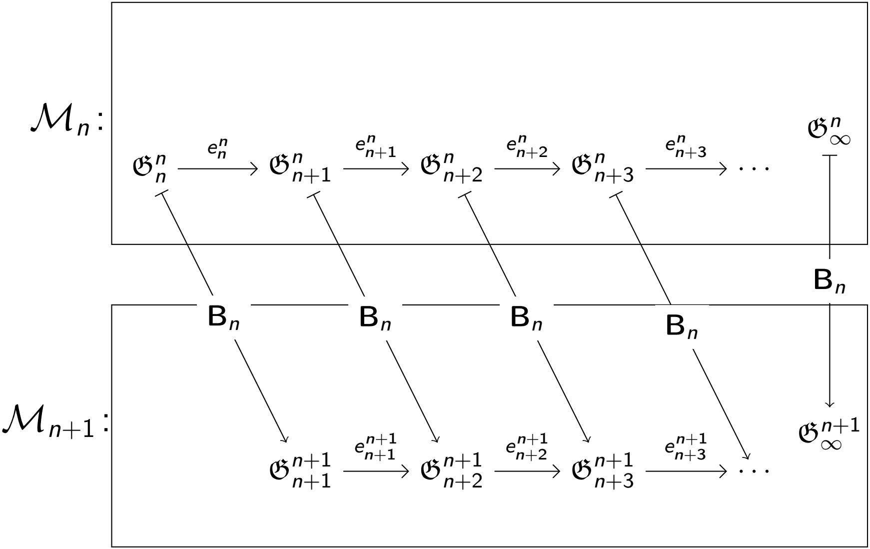

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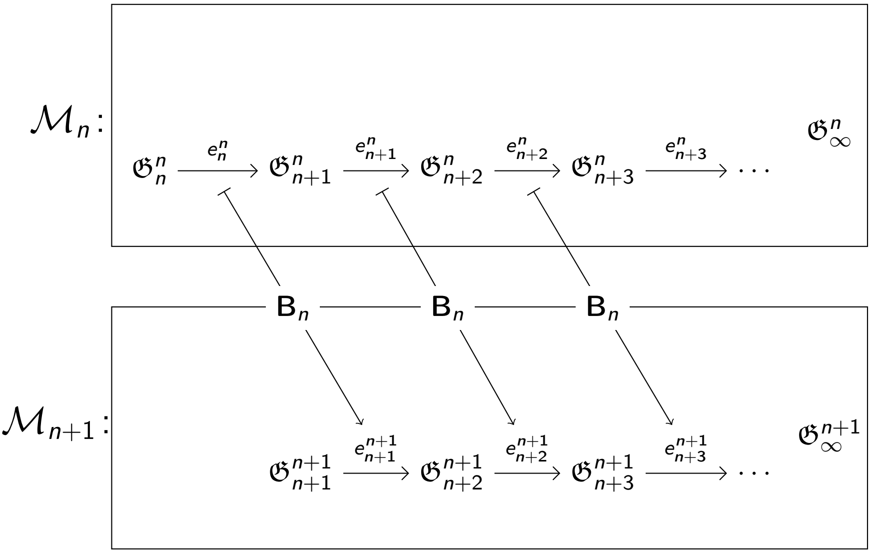

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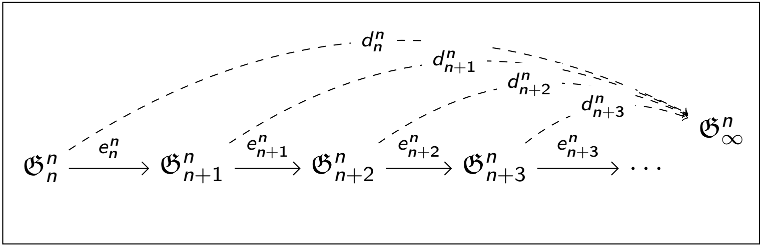
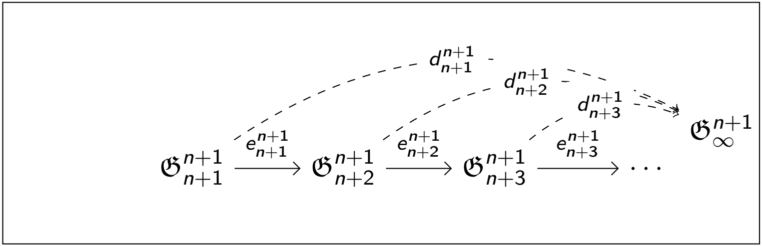


We have $\mathbf{B}_n(d^n_{n+i}) = d^{n+1}_{n+1+i}$ for any $i \geq 0$

$\mathcal{M}_{n+1}$:



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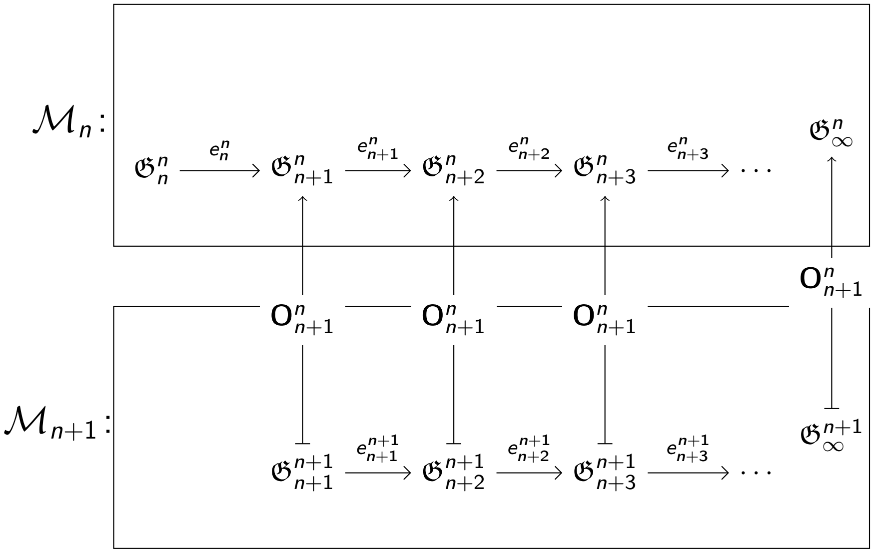

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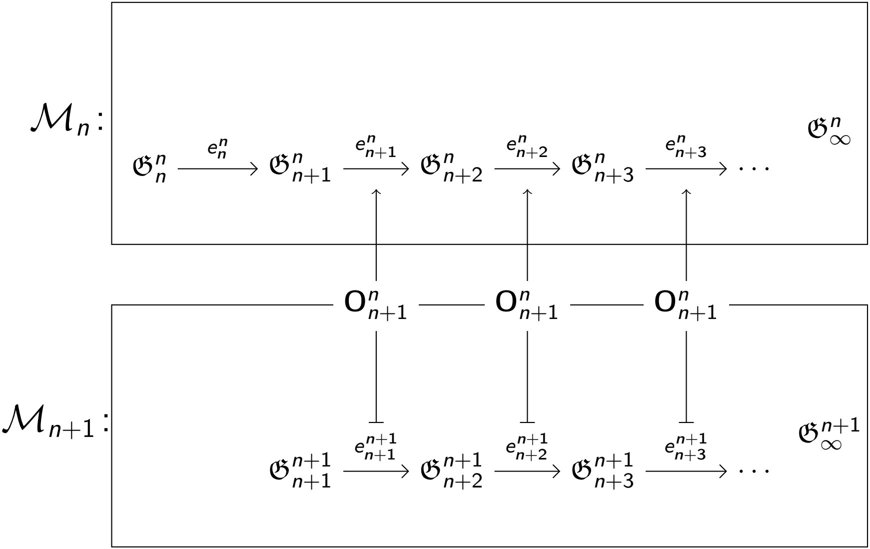

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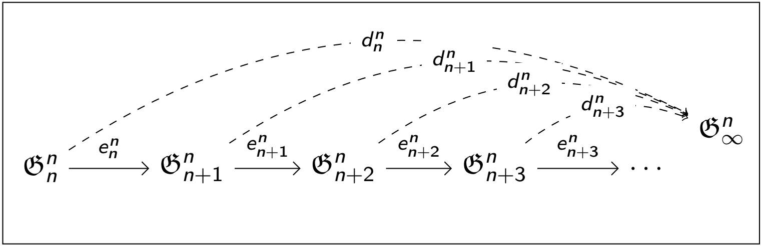

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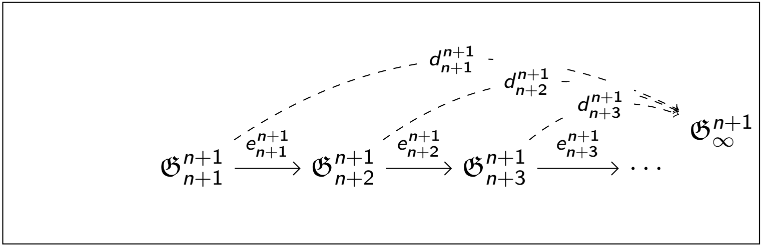
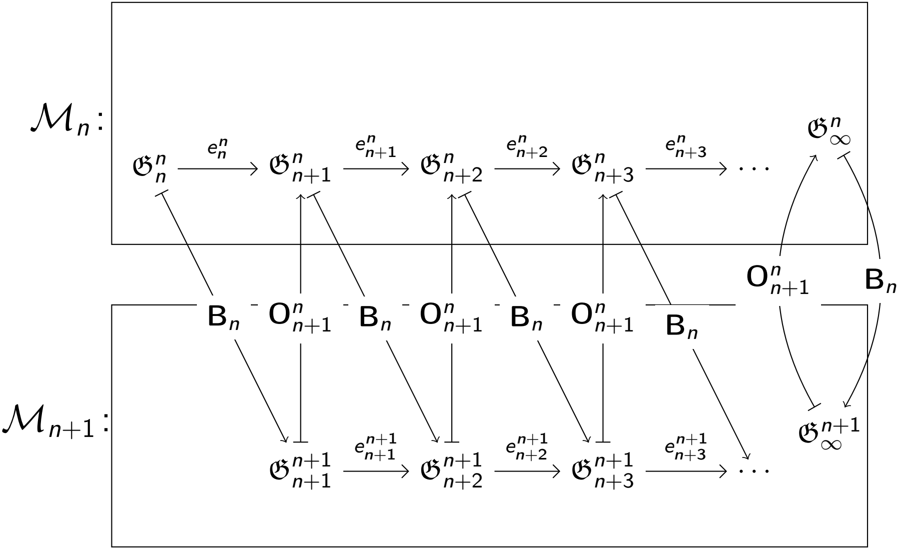


Correspondence between Diagrams



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$$\mathcal{M}_1: \quad \mathfrak{G}_1^1 \xrightarrow{e_1^1} \mathfrak{G}_2^1 \xrightarrow{e_2^1} \mathfrak{G}_3^1 \xrightarrow{e_3^1} \mathfrak{G}_4^1 \xrightarrow{e_4^1} \cdots \qquad \mathfrak{G}_\infty^1$$

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$$\cdots$$

$$\mathcal{M}_\infty: \quad \mathfrak{G}_\infty^\infty$$

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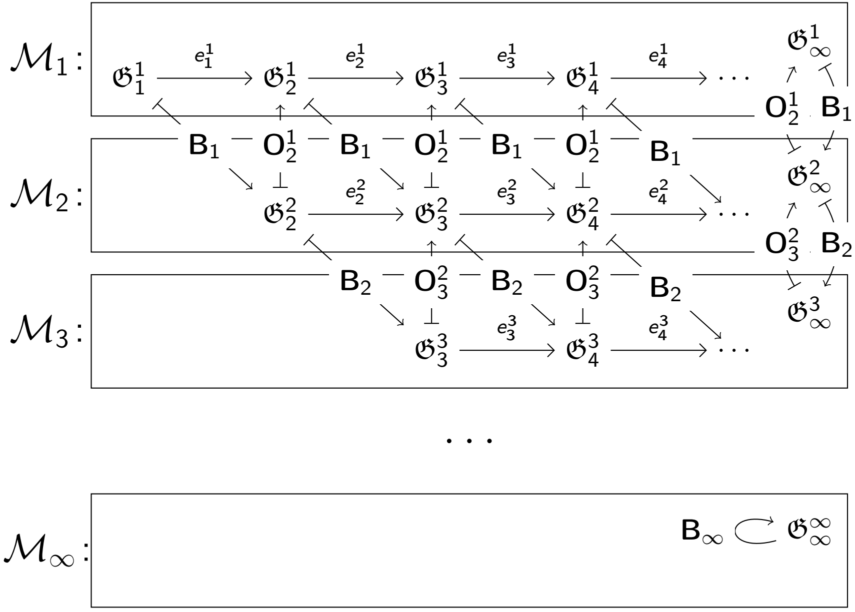

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$\mathcal{L}_n$ is the set of all first order propositions in signature $\mathbf{1}, \mathbf{0}, \wedge, \vee, \sim, [0], \ldots, [n-1]$.

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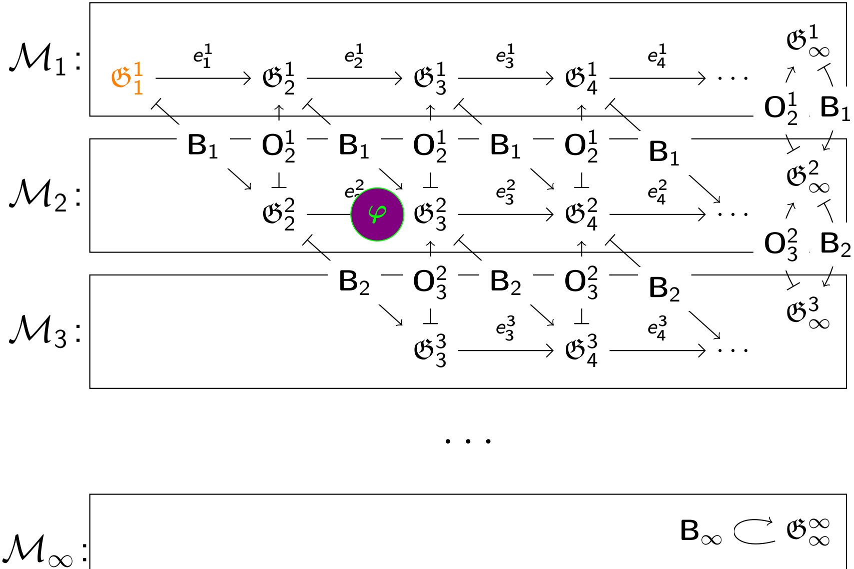
Recall

Theorem (S. Artemov, L. Beklemishev)

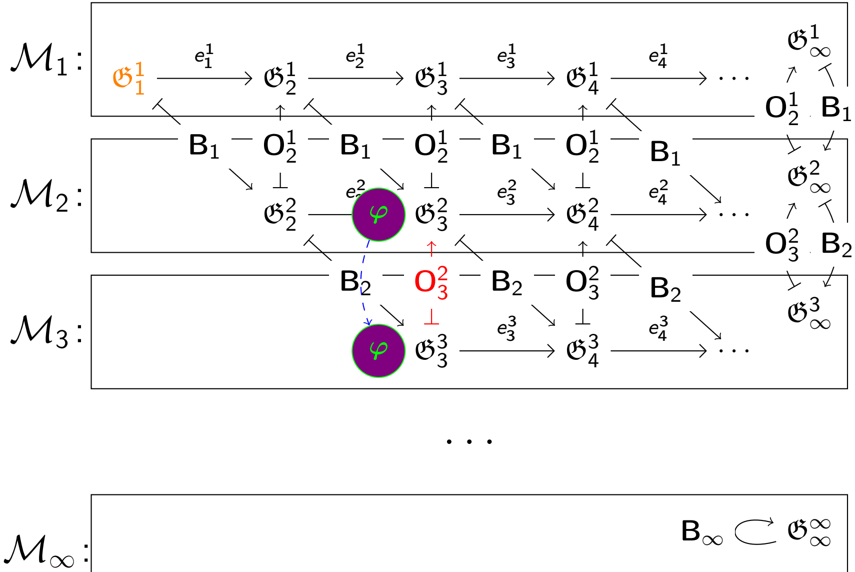
$\mathrm{Th}(\mathfrak{G}_1^1)$ *is decidable.*

Proof of the Theorem



For example we will prove decidability of $\mathrm{Th}(\mathfrak{G}_3^2)$.

Proof of the Theorem



We consider $\varphi \in \mathcal{L}_2$.

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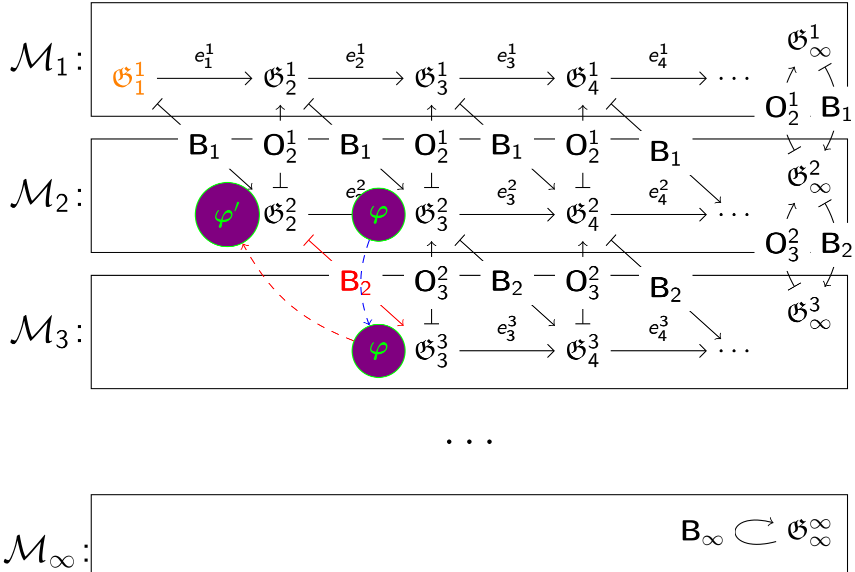

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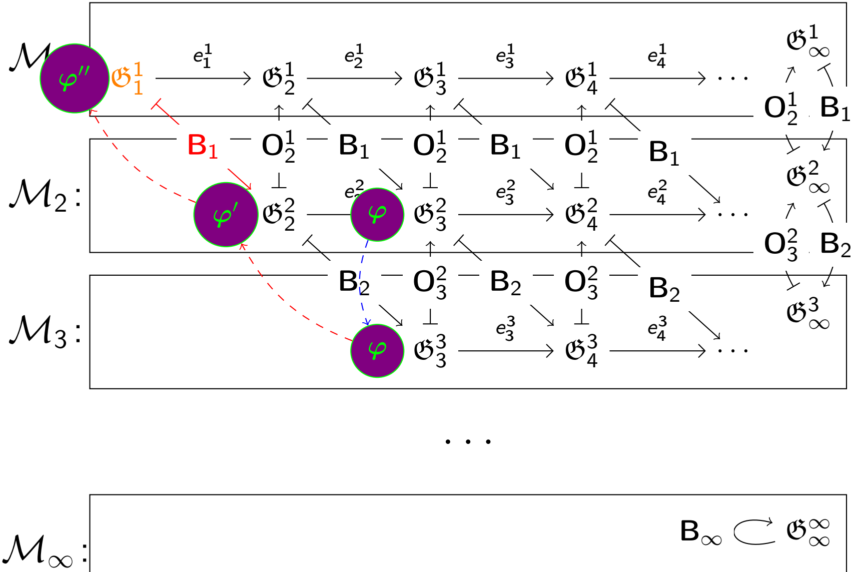

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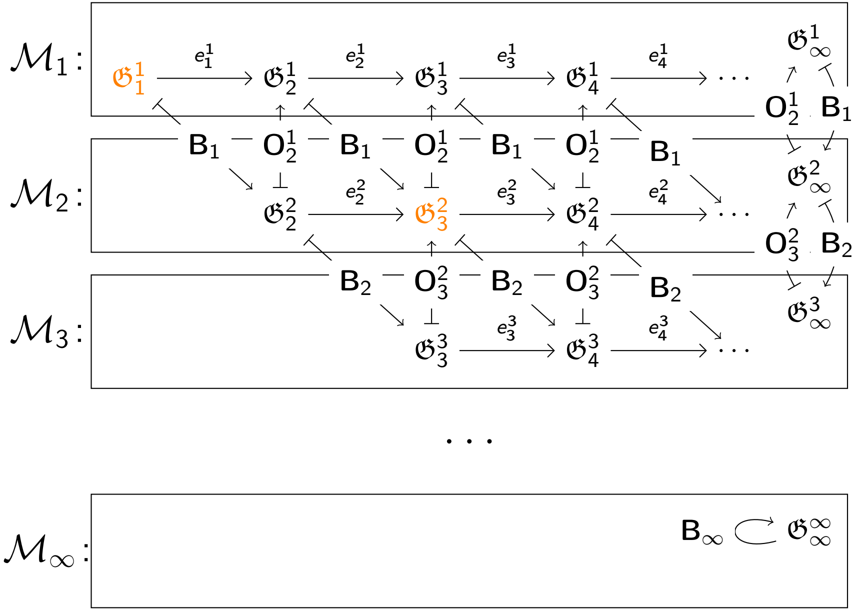

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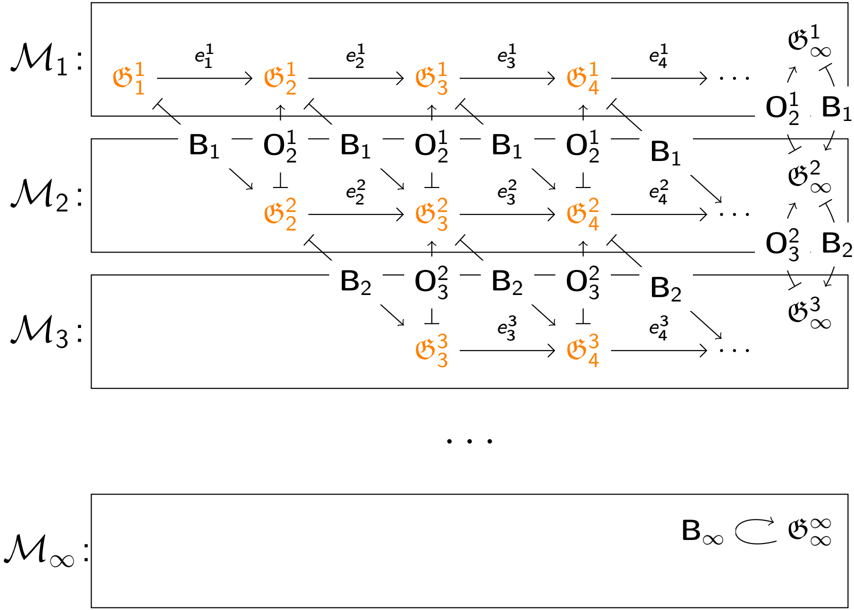

n-Elementary Embeddings

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Theory $\mathrm{Th}(\mathfrak{G}^1_\infty)$ is decidable.

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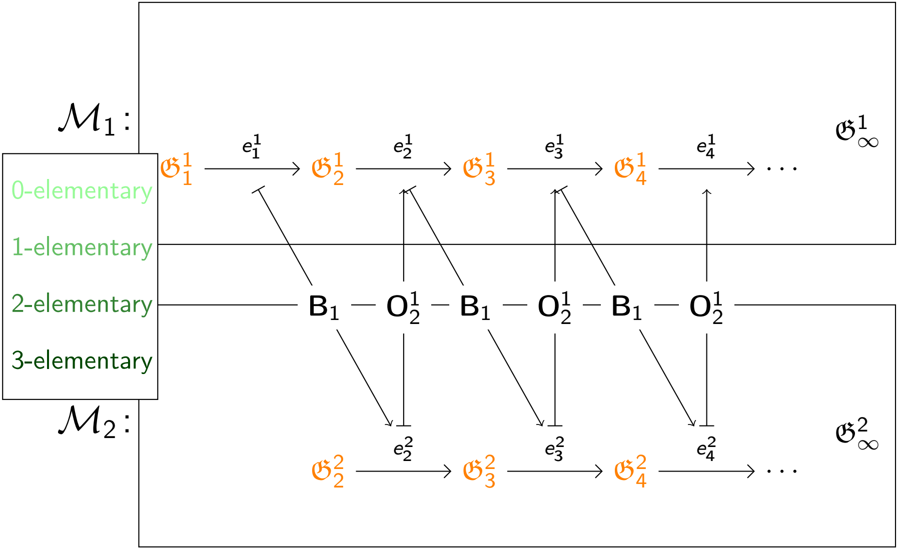

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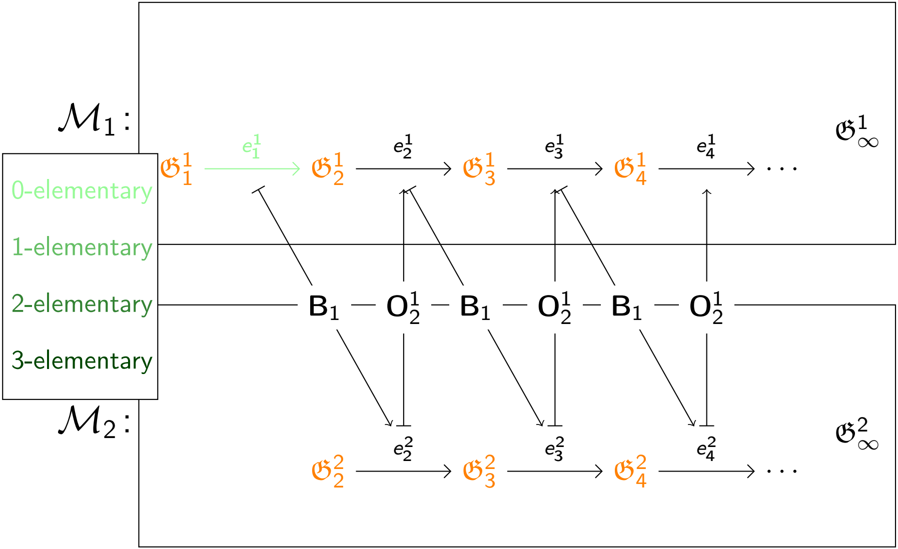

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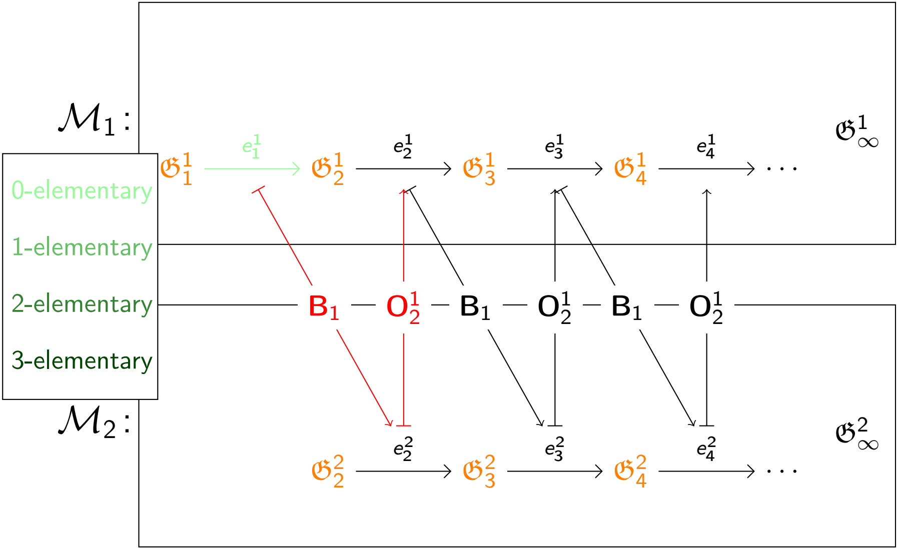

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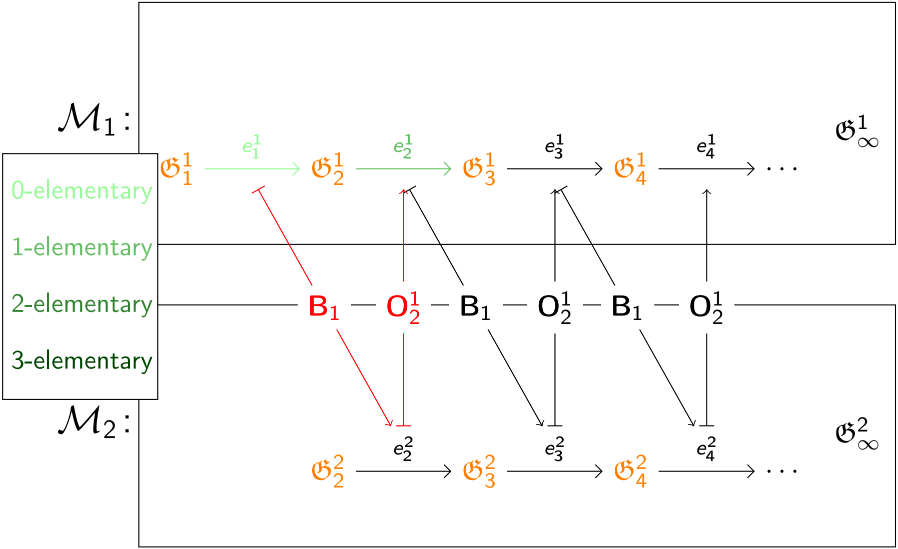

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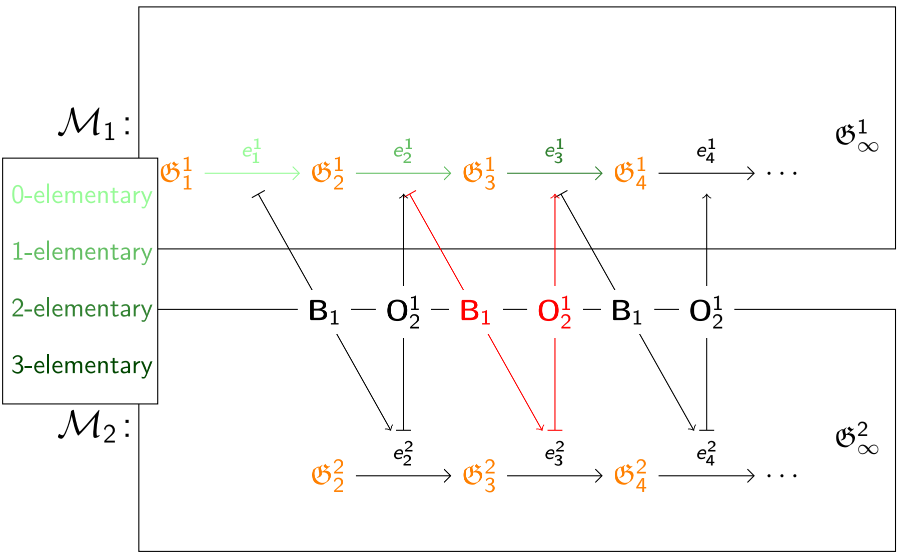

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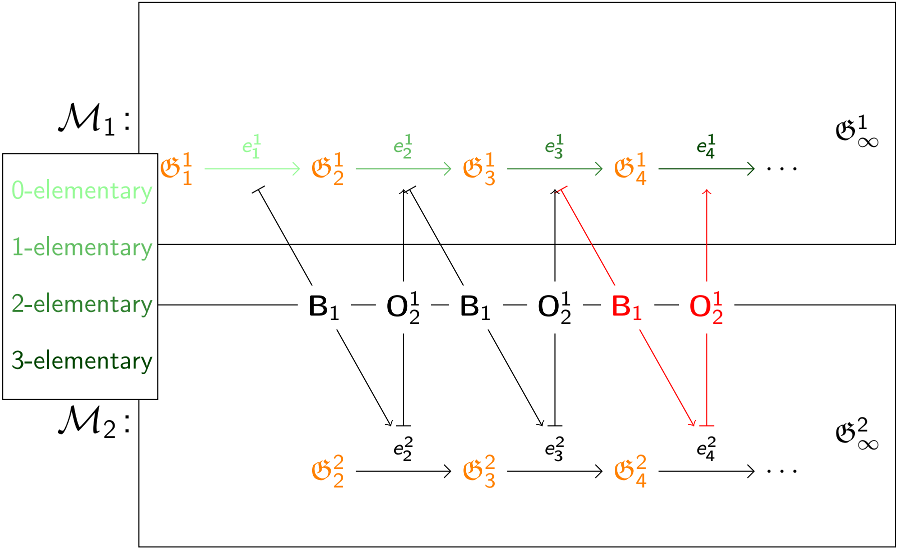

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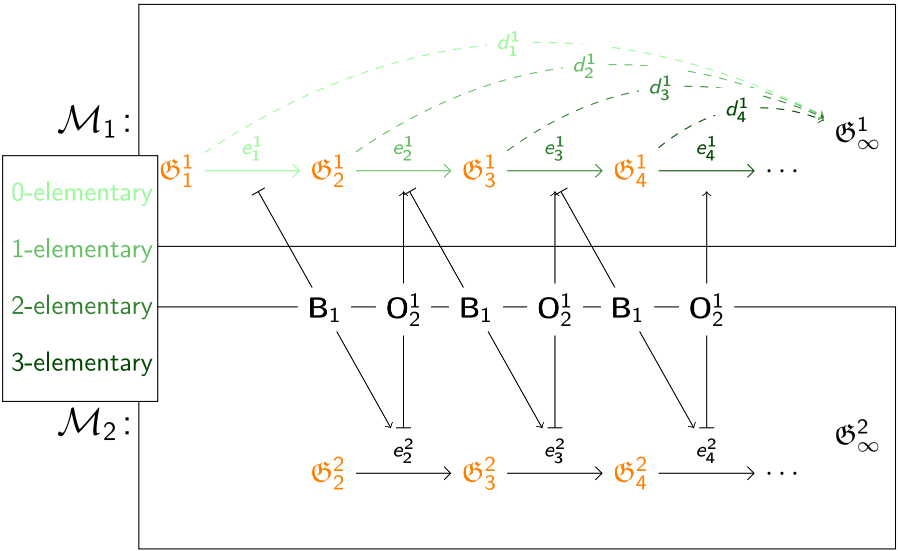

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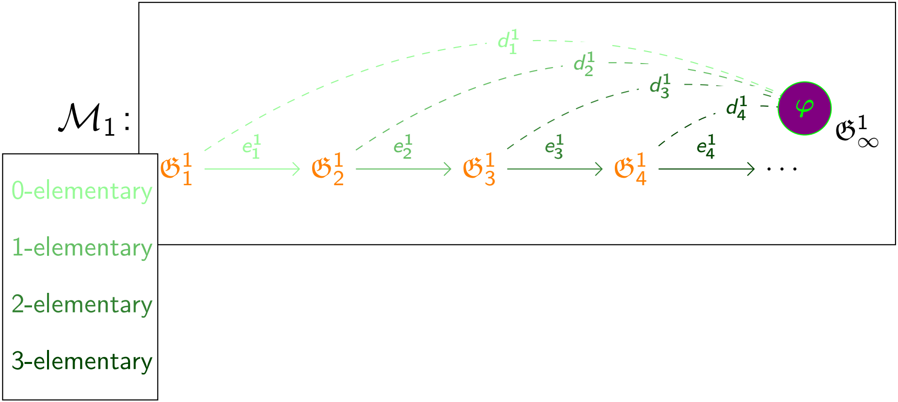

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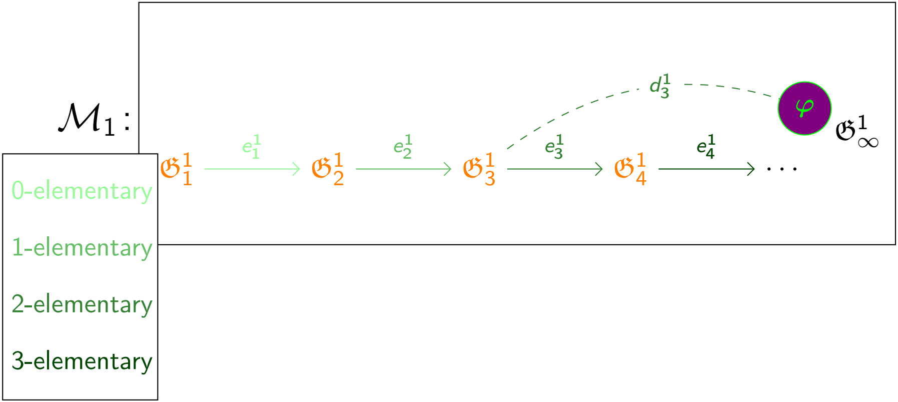

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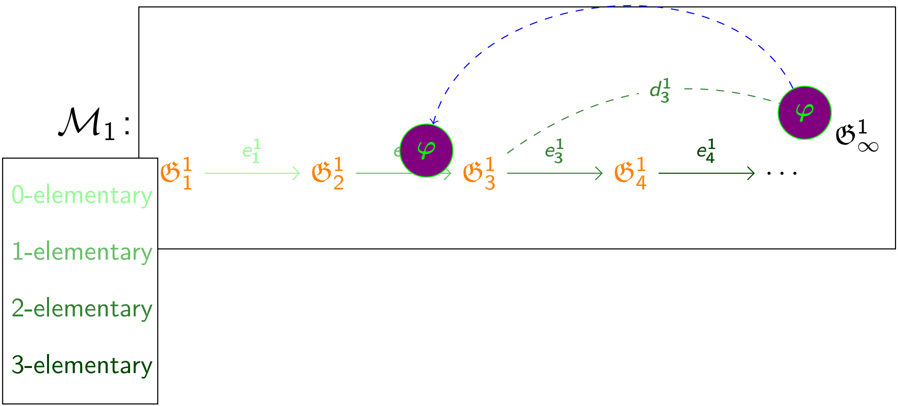

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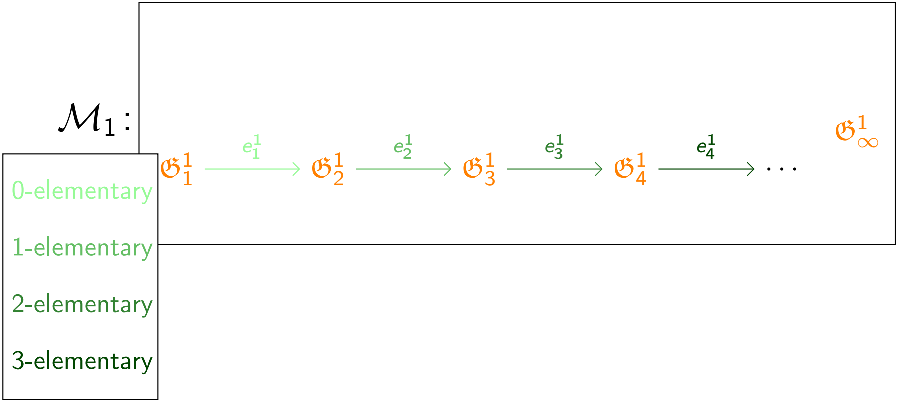

Elementary Theory of $\mathfrak{G}_{\infty}^{\infty}$

Now we will prove our main result

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Theory $\mathfrak{G}_{\infty}^{\infty}$ is decidable.

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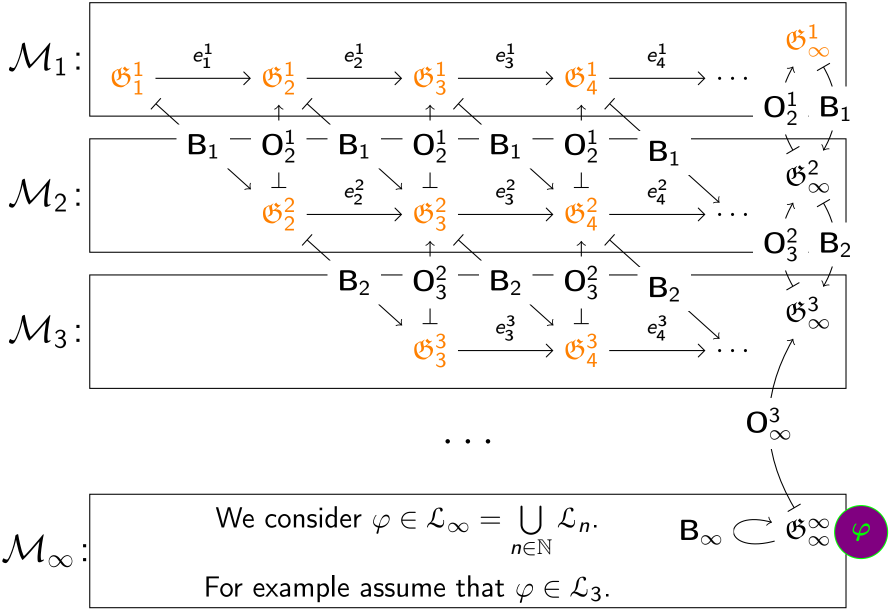

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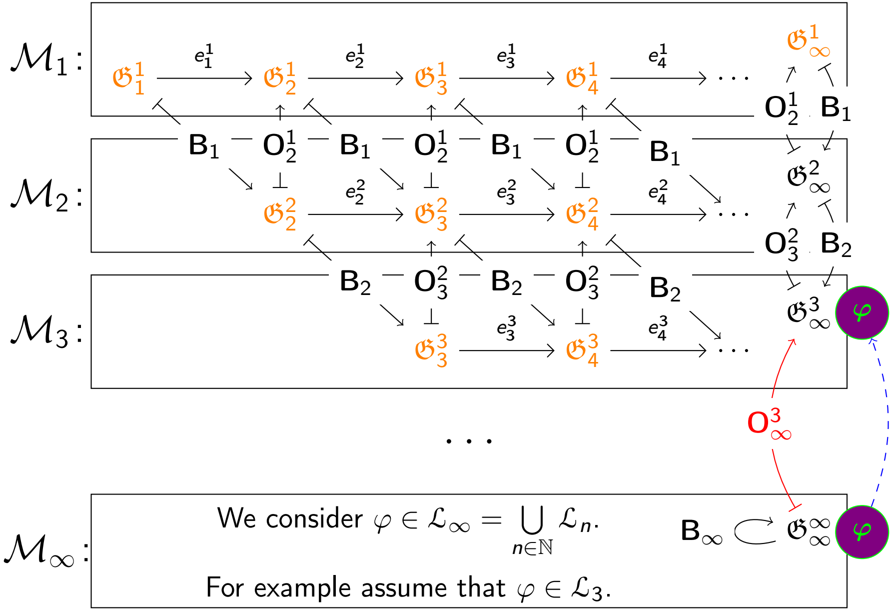

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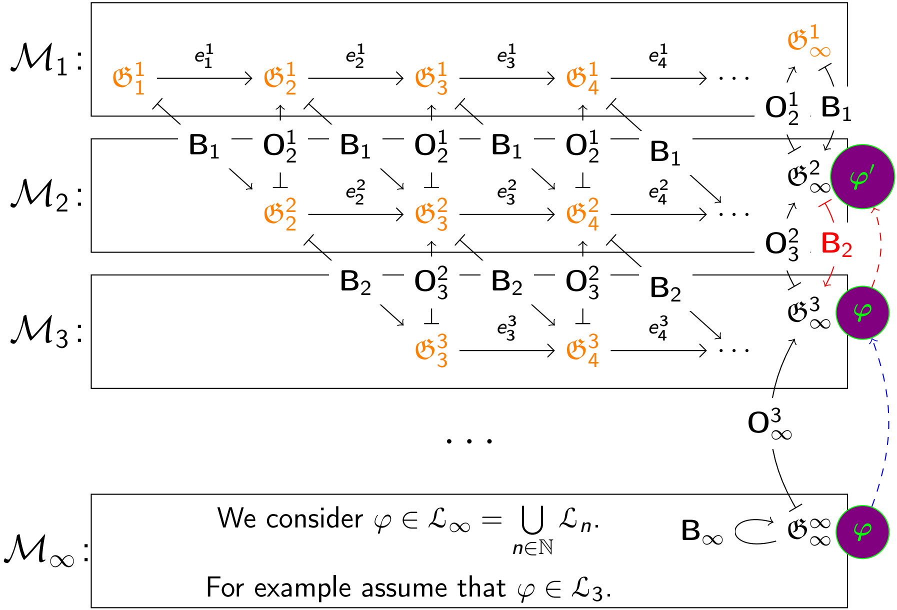

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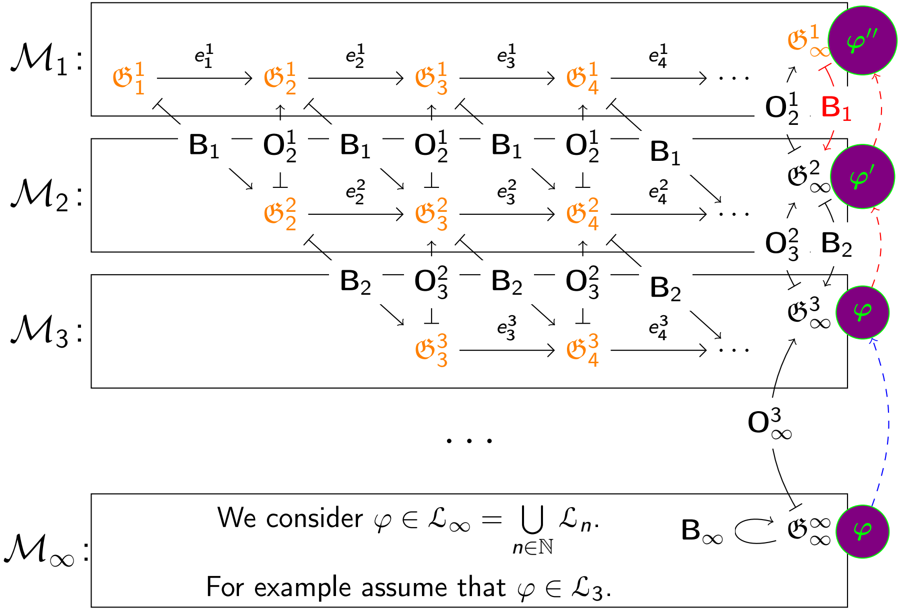

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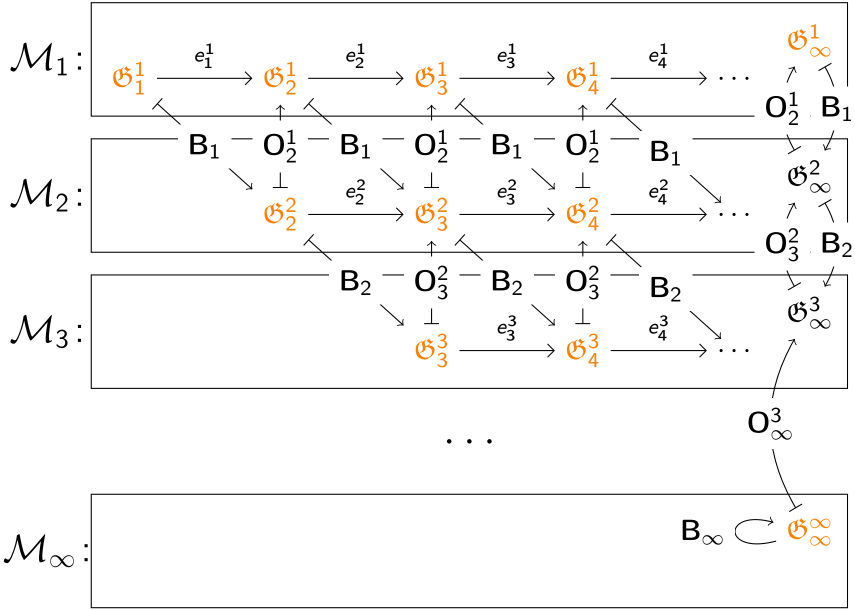

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