Failure of interpolation in the intuitionistic logic of constant domains

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Review [RZhMat 7A70,1984] by G. Mints of

E.G.K Lopez-Escobar, On the interpolation theorem for the logic of constant domains, JSL 1983, 48, no.3, 595-599

. . . The first proof of the interpolation theorem for CD by Dov Gabbay (RZhmat 1971, 11A83) contained gaps, and specialists have different opinions on whether these gaps were closed later. The proof from the previos paper by the present author [L.-E.] (RZhmat 1981, 12A40) consisted in essense of the reference to a cut-elimination theorem for some Gentzen-type axiomatization for CD. In the paper under review its author reproduces a counterexample to this claim given by M. Fitting. (Another counterexample: RZhmat 1981,12A40). Therefore the question of validity of interpolation theorem for CD is not clear.

Now we know CD has no good cut-free formulation where

good = a version of a standard deductive proof of interpolation goes through =

each rule is “interpolable” =

there is an explicit definition of the interpolant for the conclusion of the rule from interpolants for the premises.

References:

Gabbay 1977, Craig interpolation theorem for intuitionistic logic and extensions, Part III", JSL,42, 269-271

Gabbay and Maksimova, "Interpolation and Definability: Modal and Intuitionistic Logics", "Oxford University Press", 2005

Lopez-Escobar 1981, "On the interpolation theorem for the logic of constant domains", JSL 46, "87-88",

Lopez-Escobar 1983, A second paper "On the interpolation theorem for the logic of constant domains", JSL, 48, 595-599,

Fine 1979, Failures of the interpolation lemma in quantified modal logics, JSL,44,no 2, 1979, 201-206

Kripke1983, Review of Fine 1979, J. Symbolic Logic Volume 48, Issue 2 486-488.

Timeline

Gabbay	1977	
Fine	1979	
Lopez-Escobar	1981	
Kripke	1983	(review of Fine1979,JSL)
Lopez-Escobar	1983	
Mints	1984	
Mints	2008	
Urquhart	2011	
Olkhovikov	2012	

Kripke 1959, 1965; Grzegorczyk in 1964, inspired by forcing, Beth 1959.

CD= Int +(D):

$$(D) \quad \forall x(A \vee B) \rightarrow (A \vee \forall x B),$$

where x is not free in A.

Completeness: Sabine Görnemann 1971

Klemke1971 and Gabbay1969 (Technical report).

Dov M. Gabbay 1977 gives a model-theoretic proof.

The second is by López-Escobar 1981, using proof theory.

In his paper of 1977, Gabbay shows that the strong Robinson consistency theorem does not hold for **CD**; however, he goes on to claim that the weak Robinson consistency theorem holds for **CD**, and from this deduces the interpolation theorem.

The first (negative) result appears in the comprehensive monograph co-authored by Gabbay and Maksimova 2005, but the proof of interpolation does not appear.

The language L of the logic **CD** :

$\wedge,\ \vee,\ \rightarrow, \bot, \forall, \exists$.

$\neg A :\equiv\ (A \rightarrow \bot)$

0-ary predicate (propositional) symbols are allowed.

Models

$\mathcal{M} = \langle W, \leq, D, \phi \rangle$,

where $W \neq \emptyset$, $\leq$ is a quasi-ordering on W (a reflexive transitive relation), $D \neq \emptyset$

$\phi(P)$ is a $k+1$-ary relation contained in $W \times D^k$ that satisfies the monotonicity condition

$[v \leq w \wedge \langle v, a_1, \ldots, a_k \rangle \in \phi(P)] \Rightarrow \langle w, a_1, \ldots, a_k \rangle \in \phi(P)$

1. $v \Vdash_{\mathcal{M}} P\mathbf{a}_1, \ldots, \mathbf{a}_k \Leftrightarrow \langle v, a_1, \ldots, a_k \rangle \in \phi(P)$,

2. $v \Vdash_{\mathcal{M}} A \wedge B \Leftrightarrow (v \Vdash_{\mathcal{M}} A \wedge v \Vdash_{\mathcal{M}} B)$,

3. $v \Vdash_{\mathcal{M}} A \vee B \Leftrightarrow (v \Vdash_{\mathcal{M}} A \vee v \Vdash_{\mathcal{M}} B)$,

4. $v \Vdash_{\mathcal{M}} A \rightarrow B \Leftrightarrow$

   $(\forall w \geq v)(w \Vdash_{\mathcal{M}} A \rightarrow w \Vdash_{\mathcal{M}} B)$,

5. $v \Vdash_{\mathcal{M}} \bot$ never holds,

6. $v \Vdash_{\mathcal{M}} \exists x A \Leftrightarrow \exists a \in D(v \Vdash_{\mathcal{M}} A[\mathbf{a}/x])$,

7. $v \Vdash_{\mathcal{M}} \forall x A \Leftrightarrow \forall a \in D(v \Vdash_{\mathcal{M}} A[\mathbf{a}/x])$ (*).

Assume a base state: an element $v \in W$ such that $v \leq w$ for all $w \in W$.

The semantics is easily extended to a second-order version, where the second-order variables of arity k range over $k+1$-ary relations R over $W \times D^k$ that satisfy a version of the monotonicity condition above:

$$[v \leq w \wedge \langle v, a_1, \ldots, a_k \rangle \in R] \Rightarrow \langle w, a_1, \ldots, a_k \rangle \in R.$$

The Counterexample

$$\Gamma = [\forall x \exists y (Py \wedge (Qy \rightarrow Rx)) \wedge \neg \forall x Rx],$$

$$\Delta = \forall x (Px \rightarrow (Qx \vee S)) \rightarrow S.$$

Lemma 1 *The implication* $\Gamma \rightarrow \Delta$ *is valid in all Grzegorczyk-models.*

$$\alpha :\equiv\ \forall x(Px \to (Qx \vee S)).$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(Qy \to Rx), Qy \Rightarrow Rx \vee S \qquad S \Rightarrow Rx \vee S}{(Qy \to Rx), (Qy \vee S) \Rightarrow Rx \vee S \qquad Py \Rightarrow Py}}{Py, (Qy \to Rx), (Py \to (Qy \vee S)) \Rightarrow Rx \vee S}}{(Py \wedge (Qy \to Rx)), (Py \to (Qy \vee S)) \Rightarrow Rx \vee S}}{(Py \wedge (Qy \to Rx)), \forall x(Px \to (Qx \vee S)) \Rightarrow Rx \vee S}}{\exists y(Py \wedge (Qy \to Rx)), \alpha \Rightarrow Rx \vee S}}{\forall x \exists y(Py \wedge (Qy \to Rx)), \alpha \Rightarrow Rx \vee S}}{\forall x \exists y(Py \wedge (Qy \to Rx)), \alpha \Rightarrow \forall x(Rx \vee S)}$$

$$\forall x(Rx \vee S), \neg\forall x Rx \Rightarrow S$$

Proof.

Assume $v \Vdash_{\mathcal{M}} \Gamma$.

Assume $w \geq v$ and $w \Vdash_{\mathcal{M}} \forall x(Px \to (Qx \vee S))$.

For $a \in D$, since $v \Vdash_{\mathcal{M}} \Gamma$,

$w \Vdash_{\mathcal{M}} \exists y(Py \wedge (Qy \to Ra))$,

so that for some $b \in D$, $w \Vdash_{\mathcal{M}} Pb \wedge (Qb \to Ra))$.

Also $w \Vdash_{\mathcal{M}} Pb \to (Qb \vee S)$, hence $w \Vdash_{\mathcal{M}} (Qb \vee S)$ and so $w \Vdash_{\mathcal{M}} Ra \vee S$.

Since a was arbitrary, it follows that $w \Vdash_{\mathcal{M}} \forall x(Rx \vee S)$.

Hence, by (D), $w \Vdash_{\mathcal{M}} \forall x Rx \vee S$.

Since $v \Vdash_{\mathcal{M}} \Gamma$, it follows that $w \Vdash_{\mathcal{M}} \neg \forall x Rx$, so $w \Vdash_{\mathcal{M}} S$, showing

that $v \Vdash_{\mathcal{M}} \Delta$. $\square$

Corollary 1 *The second-order implication*

$\exists R\Gamma \to \forall S\Delta$

is valid in all G-models.

Lemma 2 *Let $\mathcal{M}$ be a G-model with base point w_0.*

1. $w_0 \Vdash_{\mathcal{M}} \exists R\Gamma$ *iff*

   $$\forall w \exists a (w_0 \Vdash_{\mathcal{M}} Pa \wedge w \nVdash_{\mathcal{M}} Qa). \qquad [I(P,Q)]$$

2. $w_0 \Vdash_{\mathcal{M}} \forall S\Delta$ *iff*

   $$\forall w \exists a (w \Vdash_{\mathcal{M}} Pa \wedge w \nVdash_{\mathcal{M}} Qa). \qquad [J(P,Q)]$$

   $$\Box \exists x (\Box Px \wedge \overline{\Box Q} x)$$

   where $^{-}$ is classical negation.

Substitutions for the second-order quantifiers $\exists R, \forall S$ used in this proof proved to be important to the choice of the model proving non-interpolation.

The *standard translation* of a formula of **CD**.

The *forcing language* L_f associated with **CD** is a two-sorted first-order language.

The *state variables*, $u, v, w, u_0, v_0, w_0, u_1, v_1, \ldots$.

The *individual variables* $x, y, z, x_0, y_0, z_0, x_1, y_1, \ldots$.

For every k-place predicate $Px_1, \ldots, x_k$ in **CD**,

a $k+1$-place predicate $P^*w, x_1, \ldots, x_k$;

In addition, it contains the two-place predicate $\leq$.

Definition 1 *The standard translation of a formula A of* **CD** *is a formula $ST_w(A)$ of the forcing language L_f, where w is a state variable.*

$$
\begin{aligned}
ST_w(Px_1, \ldots, x_k) &= P^*w, x_1, \ldots, x_k, \\
ST_w(A \wedge B) &= ST_w(A) \wedge ST_w(B), \\
ST_w(A \vee B) &= ST_w(A) \vee ST_w(B), \\
ST_w(\bot) &= \bot, \\
ST_w(\exists x A &= \exists x ST_w(A), \\
ST_w(\forall x A(x) &= \forall x ST_w(A), \\
ST_w(A \to B) &= \forall v \geq w[ST_v(A)) \Rightarrow ST_v(B)]
\end{aligned}
$$

where v is a fresh state variable.

Lemma 3 *If* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathcal{D}, \phi \rangle$ *is a G-model, and* A *is a sentence of L(D), then:*

1. *For any state* $u \in W$,

   $u \Vdash_{\mathcal{M}} A$ *if and only if* $\mathcal{M} \vDash ST_w(A)[\mathbf{u}/w]$.

2. $\Vdash_{\mathcal{M}} A$ *if and only if* $\mathcal{M} \vDash ST_w(A)[\mathbf{w}_0/w]$.

Proof. By induction on the complexity of the formula A. $\square$

Lemma 4 *If Θ is an interpolant for $\Gamma \to \Delta$, then $ST_w(\Theta)$ is a semantical interpolant for $I(P,Q)$ and $J(P,Q)$. That is, for any G-model $\mathcal{M}$,*

1. $\mathcal{M} \models I(P,Q) \Rightarrow \mathcal{M} \models ST_w(\Theta)[\mathbf{w}_0/w]$;

2. $\mathcal{M} \models ST_w(\Theta)[\mathbf{w}_0/w] \Rightarrow \mathcal{M} \models J(P,Q)$.

Asimulations

The basic concept of *asimulation* due to Grigory Olkhovikov.

An asymmetric counterpart of the concept of bisimulation in modal logic.

Let $\mathcal{M}_1 = \langle W_1, \leq_1, D_1, \phi_1 \rangle$ and

$\mathcal{M}_2 = \langle W_2, \leq_2, D_2, \phi_2 \rangle$.

A mapping many valued

$$v,\ d_1, \ldots, d_k \ \mapsto\ w,\ e_1, \ldots, e_k; \qquad v \in \mathcal{M}_i, w \in \mathcal{M}_j.$$

$$v \mapsto w; \qquad d_n \mapsto e_n,\ n \in \{1, \ldots, k\}.$$

An *asimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a relation Z satisfying the following conditions:

Definition 2 *1.* $Z \subseteq \bigcup_{k\geq 0}$

$(W_1 \times D_1^k) \times (W_2 \times D_2^k)]\cup$

$[(W_2 \times D_2^k) \times (W_1 \times D_1^k)$;

2. $\{(v, \vec{d}\ Z\ w, \vec{e}) \wedge v \Vdash_i P[\vec{\mathbf{d}}]\} \Rightarrow w \Vdash_j P[\vec{\mathbf{e}}]$,

for P a k-place predicate;

3. $\{(t, \vec{d}\ Z\ u, \vec{e}) \wedge u \leq_j v\} \Rightarrow (\exists w \in W_i)$

$(t \leq_i w \wedge (w, \vec{d}\ Z\ v, \vec{e}) \wedge (v, \vec{e}\ Z\ w, \vec{d}))$;

4. $\{t \in W_i \wedge (t, \vec{d}\ Z\ u, \vec{e}) \wedge f \in D_i\}$

$\Rightarrow (\exists g \in D_j)(t, \vec{d}, f\ \ Z\ \ u, \vec{e}, g)$;

5. $\{t \in W_i \wedge (t, \vec{d}\ Z\ u, \vec{e}) \wedge g \in D_j\}$

$\Rightarrow (\exists f \in D_i)(t, \vec{d}, f\ \ Z\ \ u, \vec{e}, g)$,

where $\{i, j\} = \{1, 2\}$, and $\Vdash_i$ and $\Vdash_j$ are the forcing relations in $\mathcal{M}_i$ and $\mathcal{M}_j$.

The concept of asimulation is due to G. Olkhovikov, this presentation is due to A. Urquhart.

In the present case, we need only a half.

Lemma 5 *Let* Z *be an asimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*. Then if* $t, \vec{d}\ Z\ u, \vec{e}$ *and* $t \Vdash_i A[\vec{\mathbf{d}}]$*, then* $u \Vdash_j A[\vec{\mathbf{e}}]$*, where* $\{i, j\} = \{1, 2\}$*.*

Proof. By induction on the complexity of the formula A. The steps for atomic formulas, $\wedge$, $\vee$ and $\bot$ are straightforward.

Assume $t, \vec{d}\ Z\ u, \vec{e}$, $\qquad t \Vdash_i A[\vec{\mathbf{d}}] \to B[\vec{\mathbf{d}}]$.

If $u \leq_j v$ and $v \Vdash_j A[\vec{\mathbf{e}}]$, then by the third condition in Definition 2,

$$(\exists w \in W_i)(t \leq_i w \wedge (w, \vec{d}\ Z\ v, \vec{e}) \wedge (v, \vec{e}\ Z\ w, \vec{d})).$$

By inductive assumption, $w \Vdash_i A[\vec{\mathbf{d}}]$, so that $w \Vdash_i B[\vec{\mathbf{d}}]$, since $t \Vdash_i A[\vec{\mathbf{d}}] \to B[\vec{\mathbf{d}}]$. Again, by inductive assumption, $v \Vdash_j B[\vec{\mathbf{e}}]$, showing that $u \Vdash_j A[\vec{\mathbf{e}}] \to B[\vec{\mathbf{e}}]$.

Assume that $t, \vec{d}\ Z\ u, \vec{e}$ and that $t \Vdash_i \exists x A[\vec{\mathbf{d}}, x]$. Then $t \Vdash_i A[\vec{\mathbf{d}}, \mathbf{f}]$, for some $f \in D_i$. By the fourth condition in Definition 2, there is a g in D_j so that $t, \vec{d}, f\ Z\ u, \vec{e}, g$. By inductive assumption, $u \Vdash_j A[\vec{\mathbf{e}}, \mathbf{g}]$, so that $u \Vdash_j \exists x A[\vec{\mathbf{e}}, x]$.

Assume that $t, \vec{d}\ Z\ u, \vec{e}$ and that $t \Vdash_i \forall x A[\vec{\mathbf{d}}, x]$. Let g be an arbitrary individual in D_j. By the fifth condition in Definition 2,

$(\exists f \in D_i)(t, \vec{d}, f\ \ Z\ \ u, \vec{e}, g)$.

Then $t \Vdash_i A[\vec{\mathbf{d}}, \mathbf{f}]$, so by inductive assumption,

$u \Vdash_j A[\vec{\mathbf{e}}, \mathbf{g}]$, showing that $u \Vdash_j \forall x A[\vec{\mathbf{e}}, x]$. $\square$

Refuting interpolation $\Gamma(P, Q, R) \to \Delta(P, Q, S)$

G. Olkhovikov defined two models for the language P, Q with bases w, v, and asimulation

$\mathcal{M}_1 \; asym_{P,Q} \; \mathcal{M}_2$ plus

$$\mathcal{M}_1^* = (\mathcal{M}_1, R^*), \qquad \mathcal{M}_2^* = (\mathcal{M}_2, S^*), s.t.$$

$$w \vdash_{\mathcal{M}_1^*} \Gamma \qquad v \not\vdash_{\mathcal{M}_2^*} \Delta. \tag{1}$$

Lemma 6 $\Gamma(P, Q, R) \rightarrow \Delta(P, Q, S)$ *does not have an interpolant* $\mathbf{I}(P, Q)$.

Proof. Assume for contradiction that

$$\vdash \Gamma(P, Q, R) \rightarrow \mathbf{I}(P, Q), \qquad \vdash \mathbf{I}(P, Q) \rightarrow \Delta(P, Q, S).$$

Then

$$w \vdash_{\mathcal{M}_1^*} \Gamma \;\Rightarrow\; w \vdash_{\mathcal{M}_1^*} \mathbf{I}(P, Q) \;\Rightarrow_{asym_{P,Q}}$$

$$v \vdash_{\mathcal{M}_2^*} \mathbf{I}(P, Q) \;\Rightarrow v \vdash \mathcal{M}_2^* \Delta$$

a contradiciton with (1). $\square$

Models

$\mathbb{N}$: positive natural numbers,

$k\mathbb{N}+l$: the set $\{kn+l|n \in \mathbb{N}\}$.

Definition 3 *1. A* quasi-partition (A,B,C)*:*

(a) $A \cup B \cup C = \mathbb{N}$*;*

(b) A, B, C *are pairwise disjoint;*

(c) A *and* C *are infinite;*

(d) B *is either empty or infinite.*

2. An order $\trianglelefteq$ *on quasi-partitions:*

$$(A,B,C) \trianglelefteq (D,E,F) \Leftrightarrow [A \subseteq D \wedge F \subseteq C].$$

The base points for $\mathcal{M}_1$ and $\mathcal{M}_2$:

$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (3\mathbb{N}, 3\mathbb{N}+1, 3\mathbb{N}+2)$;

$\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = (2\mathbb{N}, \emptyset, 2\mathbb{N}+1)$.

Definition 4 *The states:*

$W_1 = \{(A, B, C) | \mathbf{v} \trianglelefteq (A, B, C) \wedge B \cap \mathbf{w}_2$ *is infinite*$\}$;

$W_2 = \{\mathbf{w}\} \cup \{(A, B, C) | \mathbf{w} \trianglelefteq (A, B, C) \wedge B \neq \emptyset\}$;

The ordering on $\mathcal{M}_1$ and $\mathcal{M}_2$ is $\trianglelefteq$;

$D_1 = D_2 = \mathbb{N}$; *for* $i = 1, 2$

$$\phi_i(P) = \{\langle v, a\rangle | a \in v_1 \cup v_2\},$$

$$\phi_i(Q) = \{\langle v, a\rangle | a \in v_1\}.$$

$\mathcal{M}_1$ and $\mathcal{M}_2$ are G-models.

Lemma 7 *1.* $\mathcal{M}_1$ *satisfies* $I(P, Q)$*;*

2. $J(P, Q)$ *fails in* $\mathcal{M}_2$*.*

An asimulation Z between $\mathcal{M}_1$ and $\mathcal{M}_2$.

$D^{(k)}$: the family of all sequences of length k, chosen from the set D, where all k elements of the sequence are distinct.

$D \setminus \vec{u}$

If $\vec{u}, \vec{v} \in \mathbb{N}^{(k)}$, then the mapping $\vec{u}_l \mapsto \vec{v}_l$, for $1 \leq l \leq k$, is a bijection between the elements of $\vec{u}$ and those of $\vec{v}$; we use the notation $[\vec{u} \mapsto \vec{v}]$ for this bijection.

Definition 5 *1.* $Z \subseteq \bigcup_{k \geq 0} [(W_1 \times D_1^{(k)}) \times (W_2 \times D_2^{(k)})] \cup [(W_2 \times D_2^{(k)}) \times (W_1 \times D_1^{(k)})]$*;*

2. If $\langle (A, B, C), \vec{d} \rangle \; Z \; \langle (D, E, F), \vec{e} \rangle$*, where* $\vec{d} \in D_i^{(k)}$*,* $\vec{e} \in D_j^{(k)}$*,* $\{i, j\} = \{1, 2\}$*, then for* $1 \leq l \leq k$*,*

(a) If $\vec{d}_l \in A$ *then* $\vec{e}_l \in D$*;*

(b) If $\vec{d}_l \in B$*, then* $\vec{e}_l \in D \cup E$*.*

Lemma 8 *The relation* Z *is an asimulation between the G-models* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*.*

Review by G. Mints of the paper E..G.K Lopez-Escobar, On the interpolation theorem for the logic of constant domains, JSL 1983, 48, no.3, 595-599

The review was published in Russian mathematical review journal Matematika (abbreviated RZhMat), 7A70, July 1984

Logic CD of constant domains is obtained by adding schema $\forall x(p \vee Bx) \rightarrow (p \vee \forall x Bx)$ to intuitionistic predicate calculus. Corresponding Kripke semantics is a familiar semantics for intuitionistic predicate calculus plus condition that individual domain is one and the same for all worlds.

The first proof of the interpolation theorem for CD given by Dov Gabbay (reviewed in RZhmat 1971, 11A83) contained gaps and specialists have different opinions on whether these gaps were closed later.

The proof from the previos paper by the present author [L.-E.] (reviewed in RZhmat 1981, 12A40) consisted in essense of the reference to a cut-elimination theorem for some Gentzen-type axiomatization for CD. In the paper under review its author reproduces a counterexample to this claim given by M. Fitting. (Another counterexample was given by the present reviewer in RZhmat 1981, 12A40). Therefore the question of validity of interpolation theorem for CD is not clear.

The present author [L.E.] tries to prove that CD does not have any cut-free Genzen-style axiomatization with a finite number of rules where formulas are analyzed only up to finite depth k. He concludes this from the statement of necessity in such axiomatics of a derivable formula

$$\forall x(p \vee B(x)) \to (p \vee (q \to \forall x_1 \ldots x_{k-1} B(x)))$$

However this formula is derivable in the axiomatics obtained by adding the rule

$$\frac{p, \Gamma \Rightarrow C \quad \forall x Bx, \Gamma \Rightarrow C}{\forall x(p \vee Bx), \Gamma \Rightarrow C}$$

to ordinary intuitionistic rules. The error by the author is in the case C2 which is not treated. Signed G. Mints