On an abstract approach to the Reduction Property

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Contents

Reflection principles

- The basic theory is EA (Elementary Arithmetic) with language $\mathcal{L}_{exp} = \{0, S, +, \cdot, \exp, <\}$
- For each theory T, *elementary presented*, we consider formulas
  - $\mathrm{Prf}_T(y, x)$ expresing "y is (codes) a proof of x in T"
  - $\Box_T(x) \equiv \exists y\, \mathrm{Prf}_T(y, x)$
- Local Reflection for T is the following scheme, $\mathrm{Rfn}(T)$,

$$\Box_T(\ulcorner \varphi \urcorner) \rightarrow \varphi$$

  for each sentence φ.
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Partial Reflection

Partial Reflection: Reflection scheme restricted to a class of formulas Σ.

- Partial Local Reflection, $\mathrm{Rfn}_\Sigma(T)$ is given by

$$\Box_T(\ulcorner \varphi \urcorner) \to \varphi$$

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Relativized reflection principles

- For each $n \geq 1$ and each theory T, *elementary presented*, we consider the formulas
- $\mathrm{Prf}^n_T(y, x)$ expressing

  $$\text{“}y \text{ is (codes) a proof of } x \text{ in } T + Th_{\Pi_n}(\mathcal{N})\text{”}$$

- $[n]_T(x) \equiv \exists y \, \mathrm{Prf}^n_T(y, x)$
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Metareflection

- **Metareflection Rule**, $\mathrm{RR}^n(T)$:

$$\frac{\varphi}{\langle n \rangle_T(\varphi)}$$

- Π_m–$\mathrm{RR}^n(T)$ is the rule $\mathrm{RR}(T)$ with the restriction that φ is a Π_m–sentence.
- If T is an elementary presented extension of EA, then for every $m \geq 1$,

$$T^n_m \equiv [T, \Pi_{n+1}\text{–}\mathrm{RR}^n(T)]_m$$

where $T^n_0 = T$ and $T^n_{k+1} = T_k + \langle n \rangle_{T_k} \top$, and

  - $[U, \Pi_{n+1}\text{–}\mathrm{RR}(T)]$ is the closure of U under first order logic and **unnested** applications of Π_{n+1}–$\mathrm{RR}(T)$],
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Conservation properties of reflection principles

- (Beklemishev) If U is a Π_{n+2}–axiomatized extension of EA then
  - $U + \mathrm{RFN}_{\Sigma_{n+1}}(T)$ is Π_{n+1}–conservative over $U + \Pi_{n+1}\text{–}\mathrm{RR}^n(T)$.
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- (Goryachev, Beklemishev) Let $\varphi_1, \ldots, \varphi_m$ a finite set of sentences. Then for every $\psi \in \Pi_{n+1}$,

$$T + \mathrm{Rfn}^n_{\varphi_1}(T) + \cdots + \mathrm{Rfn}^n_{\varphi_m}(T) \vdash \psi \quad \Longrightarrow \quad T^n_m \vdash \psi$$

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Σ_{n+1}–closed models

- Σ_{n+1}–closed models provide a simple and clear method to obtain conservation results.
- **Definition**. Let T be a theory. We say that $\mathfrak{A} \models T$ is a Σ_{n+1}–closed model of T if for each $\mathfrak{B} \models T$,

$$\mathfrak{A} \prec_n \mathfrak{B} \quad \Longrightarrow \quad \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

- It generalizes the notion of an *existentially closed model*.
- **Proposition**. (Existence)
  Let T be a Π_{n+2}–axiomatizable theory and $\mathfrak{A} \models \mathbf{T}$ countable. Then there exists $\mathfrak{B} \models T$ such that $\mathfrak{A} \prec_n \mathfrak{B}$ and $\mathfrak{B}$ is Σ_{n+1}–closed for T.
- **Corollary**. Every consistent and Π_{n+2}–axiomatizable theory has a Σ_{n+1}–closed model.

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The basics of the method

The basic device is the following result:

Theorem (Avigad,'02)

Let T_1 be a Π_{n+2}–axiomatizable theory such that every Σ_{n+1}–closed model for T_1 is a model of $\mathbf{T}_2$. Then T_2 is Π_{n+1}–conservative over T_1.

Other key ingredient in most applications:

Lemma

Let $\mathfrak{A}$ be a Σ_{n+1}–closed model for T. Let $\varphi(\vec{x}) \in \Pi_{n+1}$ and $\vec{a} \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(\vec{a})$. Then there exist $\theta(v, \vec{x}) \in \Pi_n$ and $b \in \mathfrak{A}$ such that

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$$\mathfrak{A} \models \theta(b,\vec{a}) \quad \textit{and} \quad T \vdash \theta(v,\vec{x}) \rightarrow \varphi(\vec{x})$$

Parsons's Theorem (I)

Theorem (Parsons, '72)

$\mathbf{I}\Sigma_1$ *is* Π_2*–conservative over* $\mathbf{I}\Delta_0 + \Sigma_1$*–IR.*

- Σ_1–Induction Rule, Σ_1–IR:

$$\frac{\varphi(0) \wedge \forall x\,(\phi(x) \rightarrow \phi(x+1))}{\forall x\,\phi(x)}$$

- Let T be a Π_2–axiomatizable finite extension of $\mathbf{I}\Sigma_0$, $T = I\Sigma_0 + \forall x \exists y \forall u \leq x \exists v \leq y\, \sigma(u,v)$ for some $\sigma(x,y) \in \Delta_0$. Then, for each $m \geq 1$,

$$[T, \Sigma_1\text{–IR}]_m \quad \equiv \quad T + \forall x\,\exists y\,(F_m(x) = y) \vdash \psi$$

where $F_0(x) = (x+1)^2 + (\mu y)(\sigma(x,y))$, $F_{k+1}(x) = F_k(x)^{x+1}$.

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Parsons' Theorem (II)

Theorem

Let $\mathfrak{A}$ be a Σ_{n+2}–closed model of $T + \Sigma_{n+1}$–IR. Then $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$.

Proof. Let $\varphi(x) \in \Sigma_{n+1}$ such that

$$\mathfrak{A} \models \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$$

Then there exist $a \in \mathfrak{A}$ and $\theta(u) \in \Pi_n$ such that

$$T + \Sigma_{n+1}\text{–IR} \vdash \theta(u) \rightarrow \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$$

and $\mathfrak{A} \models \theta(a)$. Put $\psi(x,u) := \theta(u) \rightarrow \phi(x)$, Then

$$T + \Sigma_{n+1}\text{–IR} \vdash \psi(0,u) \wedge \forall x\,(\psi(x,u) \rightarrow \psi(x+1,u))$$

Hence, $T + \Sigma_{n+1}\text{–IR} \vdash \forall x\,\psi(x,u)$. As a consequence $\mathfrak{A} \models \forall x\,\psi(x,u)$. In particular, $\mathfrak{A} \models \theta(a) \rightarrow \forall x\,\varphi(x)$ and it follows that $\mathfrak{A} \models \forall x\,\varphi(x)$.

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Conditional axioms

Let L denote the language of First Order Arithmetic.

Definition

A set of L–formulas, E, is a set of **conditional axioms** if each element of E is a formula of the form $\alpha(\vec{v}) \to \beta(\vec{v})$.

Let T be an L–theory and E be a set of conditional axioms.

- $T + E$ is obtained by adding to T the universal closure of each formula in E.
- **Example**: $T + E = I\Sigma_1$, for $T = I\Delta_0$ and

$$E = \{ I_{\varphi,x}(\vec{v}) : \ \varphi(x, \vec{v}) \in \Sigma_1 \}$$

where $I_{\varphi,x}(\vec{v})$ is the induction scheme

$$\underbrace{\varphi(0, \vec{v}) \wedge \forall x \, (\varphi(x, \vec{v}) \to \varphi(x+1, \vec{v}))}_{\alpha} \to \underbrace{\forall x \, \varphi(x, \vec{v})}_{\beta}$$

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Conditional axioms (cont'd)

- We can associate to each set of conditional axioms, E, two auxiliary sets of conditional axioms:
  - $E^- = E \cap \mathrm{Sent}$, and
  - $UE = \{\forall \vec{v}\, \alpha(\vec{v}) \to \forall \vec{v}\, \beta(\vec{v}) : \ \alpha(\vec{v}) \to \beta(\vec{v}) \in E\}$
- The theories $T + UE$ and $T + E^-$ are obtained by adding to T the sentences in UE and E^- respectively.
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Conditional axioms: Inference rules

We also define an inference rule, E-Rule, with instances

$$\frac{\alpha(\vec{v})}{\beta(\vec{v})}, \qquad \text{for each } \alpha(\vec{v}) \to \beta(\vec{v}) \in E$$

- $[T, E\text{–Rule}]$ denotes the closure of T under first order logic and *unnested* applications of E–Rule.
- $T + E$–Rule denotes the closure of T under first order logic and (nested) applications of E–Rule.
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Normal conditional axioms

Let us fix a countable first order language L.

- Π will denote a fixed set of L–formulas such that:
  1. It contains all atomic formulas and is closed under subformulas.
  2. We assume that (modulo logical equivalence) Π is closed under **disjunctions** and **conjunctions**, and
  3. (modulo logical equivalence) $\neg\Pi \subseteq \exists\Pi$, (here $\neg\Pi$ is $\{\neg\varphi : \varphi \in \Pi\}$ and $\exists\Pi$ is $\{\exists\vec{x}\varphi(\vec{x}) : \varphi(\vec{x}) \in \Pi\}$).
- We say that a formula $\alpha(\vec{v}) \rightarrow \beta(\vec{v})$ is a *normal conditional axiom w.r.t.* Π if
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The basic reduction

- For each set of formulas Π, we introduce the rule E^{Π}–Rule given by the instances

$$\frac{\theta(\vec{v},\vec{z}) \rightarrow \alpha(\vec{v})}{\theta(\vec{v},\vec{z}) \rightarrow \beta(\vec{v})}$$

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Lemma

Let T be a theory and E a set of conditional axioms such that

(S1) *For every $\alpha(\vec{v}) \rightarrow \beta(\vec{v}) \in E$, $\quad \alpha(\vec{v}) \in \exists\forall\neg\Pi$.*

(S2) *$T + E^{\Pi}$–Rule is $\forall\exists\Pi$–axiomatizable.*

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The basic reduction (cont'd)

- It holds that $[U, E\text{–Rule}] \subseteq [U, E^{\Pi}\text{–Rule}]$.
- E is Π–**reducible modulo** T if for every theory U extending T, it holds

$$[U, E^{\Pi}\text{–Rule}] \equiv [U, E\text{–Rule}]$$

Theorem

Let T be a $\forall\exists\Pi$–axiomatizable theory and E a set of normal conditional axioms w.r.t. Π. Assume that E is Π–reducible modulo T. Then

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2. *$T + E$ is $\exists\forall\neg\Pi$–conservative over $T + UE$.*
3. *If every $\forall\exists\Pi$–axiomatizable extension of $T + E^-$ is closed under E–Rule, then $T + E$ is $\exists\forall\neg\Pi$–conservative over $T + E^-$*

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Let T be a $\forall\exists\Pi$–axiomatizable theory and E a set of normal conditional axioms w.r.t. Π. Assume that E is Π–reducible modulo T. Then

1. *$T + E$ is $\forall\neg\Pi$–conservative over $T + E$–Rule.*
2. *$T + E$ is $\exists\forall\neg\Pi$–conservative over $T + UE$.*
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Examples

For Σ_{n+1}–induction:

- $I\Sigma_{n+1}$ is a set of normal conditional axioms w.r.t. Π_{n+1} and it is Π_{n+1}–reducible modulo $I\Delta_0$. Hence, for every Π_{n+2}–axiomatizable theory T extending $I\Delta_0$:
  1. $Th_{\Pi_{n+2}}(T + I\Sigma_{n+1}) \equiv T + \Sigma_{n+1}$–IR.
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The finite case

Theorem

Let F be a set of normal conditional sentences w.r.t. Π. Then, for every $\forall\exists\Pi$–axiomatizable theory T it holds that

$$Th_{\forall\neg\Pi}(T + F) \subseteq [T, F^{\Pi}\text{–}Rule]_m$$

where m is the number of elements of F.

Corollary

Let E be a set of normal conditional sentences w.r.t. Π. Assume that E is Π–reducible modulo T. Then for every finite set of sentences $F \subseteq E$ with m elements, it holds that

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The finite case (proof)

Lemma

Let $E = \{\psi_1, \ldots, \psi_m\}$ *a finite set of closed conditional axioms w.r.t.* Π*. Then*

$$T + E^{\Pi}\text{–}Rule \equiv [T, E^{\Pi}\text{–}Rule]_m$$

- If ψ is a sentence of the form $\alpha \rightarrow \beta$, with $\alpha \in \forall\neg\Pi$ and $\beta \in \forall\exists\Pi$, we define the rule

$$\psi^{\Pi}\text{–Rule} : \quad \frac{\theta(u) \rightarrow \alpha}{\theta(u) \rightarrow \beta}, \qquad (\theta(u) \in \Pi).$$

- $T + \psi^{\Pi}$–Rule $\equiv [T, \psi^{\Pi}$–Rule$]$.
- It holds that for each sentence $\varphi \in \forall\neg\Pi$, a proof of φ in $T + E^{\Pi}$–Rule only requires one application of each rule ψ_j^{Π}–Rule.

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Kaye's Theorem

Theorem

For every theory T *extension of* $I\Sigma_n$, $m \geq 1$ *and* $\varphi_1(x), \dots, \varphi_m(x) \in \Sigma^-_{n+1}$,

$$Th_{\Pi_{n+2}}(T + I_{\varphi_1} + \cdots + I_{\varphi_m}) \subseteq [T, \Sigma_{n+1}\text{–}IR]_m$$

- $I\Sigma^-_{n+1}$ is a set of normal conditional sentences w.r.t. Π_{n+1} .
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Δ_1–Induction

- $UI\Delta_1$ is a set of normal conditional axioms w.r.t. Π_1.
- $UI\Delta_1$ is Π_1–reducible modulo $I\Sigma_0$.

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Let T a Π_3–axiomatizable extension of $I\Sigma_0$. Let $\varphi_1(x,u),\dots,\varphi_m(x,u) \in \Sigma_1$, $\psi_1(x,u),\dots,\psi_m(x,u) \in \Pi_1$, and $\theta \in \Pi_2$ such that

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Local reflection

- Let T an elementary presented theory. For each sentence σ, the **local relativized reflection principle** for σ wrt T, $\mathrm{Rfn}^n_\sigma(T)$ is equivalent to the sentence

$$\neg\sigma \rightarrow \langle n \rangle_T(\neg\sigma)$$

- $\mathrm{Rfn}^n_{\Sigma_{n+1}}(T)$ can be axiomatized by a set of normal conditional sentences w.r.t. Π_n, and
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Goryachev's theorem

Theorem

Let T be an elementary presented extension of EA *and* $\varphi_1, \ldots, \varphi_m$ *a finite set of* Σ_1*–sentences. Then for every* Π_2*-axiomatized extension of* EA $\psi \in \Pi_1$,

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- This provides a weak version of Goryachev's theorem.
- The full result can be obtained from a result by L. Beklemishev: For each sentence $\psi \in \Pi_1$, and sentences $\varphi_1, \ldots, \varphi_m$ such that

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Induction up to Σ_1–definable elements

- We denote by $I(\Sigma_1, \mathcal{K}_1)$ the theory given by $I\Delta_0$ together with the induction scheme

$$\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \forall x_1, x_2\,(\delta(x_1) \wedge \delta(x_2) \to x_1 = x_2) \to \forall x\,(\delta(x) \to \varphi(x))$$

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Induction up to Σ_1–definable elements (II)

- $I(\Sigma_1, \mathcal{K}_1)$ is set of normal conditional axioms w.r.t. Π_1.
- $I(\Sigma_1, \mathcal{K}_1)$ is Π_1–reducible modulo $I\Sigma_0$.
- $I\Pi_1^- \equiv I(\Sigma_1^-, \mathcal{K}_1)$
- Let us denote by Π_1^-–IR_0 the rule

$$\frac{\forall x\,(\varphi(x) \to \varphi(x+1))}{\varphi(0) \to \forall x\,\varphi(x)}, \qquad \varphi(x) \in \Pi_1^-$$

- For every Π_2–axiomatizable theory T extending $I\Sigma_0$,

$$[T, (\Sigma_1^-, \mathcal{K}_1)\text{–IR}] \equiv [T, \Pi_1^-\text{–IR}_0]$$

Induction up to Σ_1–definable elements (III)

Theorem

Let T be a $\mathcal{B}(\Sigma_1)$–axiomatizable extension of $I\Delta_0$. Then.

1. $T + I(\Sigma_1, \mathcal{K}_1)$ *is* Π_2–*conservative over* $T + (\Sigma_1, \mathcal{K}_1)$–*IR.*
2. $T + (\Sigma_1, \mathcal{K}_1)$–*IR is* Σ_2–*conservative over* $T + (\Sigma_1^-, \mathcal{K}_1)$–*IR.*
3. $T + (\Sigma_1^-, \mathcal{K}_1)$–*IR* $\equiv [T, (\Sigma_1^-, \mathcal{K}_1)$–*IR*$]$.
4. $T + I(\Sigma_1, \mathcal{K}_1)$ *is* $\mathcal{B}(\Sigma_1)$ *conservative over* $T + I(\Sigma_1^-, \mathcal{K}_1)$.

Corollary

$I\Pi_1^-$ *is* Π_2–*conservative over* $I\Sigma_0 + (\Sigma_1, \mathcal{K}_1)$–*IR.*

Induction up to Σ_1–definable elements (III)

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4. $T + I(\Sigma_1, \mathcal{K}_1)$ *is* $\mathcal{B}(\Sigma_1)$ *conservative over* $T + I(\Sigma_1^-, \mathcal{K}_1)$.

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$I\Pi_1^-$ *is* Π_2*–conservative over* $I\Sigma_0 + (\Sigma_1, \mathcal{K}_1)$*–IR.*

Parameter free Π_1-Induction

- Let $\varphi_1(x), \ldots, \varphi_m(x) \in \Pi_1^-$ and $\psi \in \Pi_2$ such that

$$I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \psi$$

Then $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)\text{–IR}]_m \vdash \psi$.

- If $\psi \in \mathcal{B}(\Sigma_1)$ then $[I\Delta_0, (\mathcal{B}(\Sigma_1)^-, \mathcal{K}_1)\text{–IR}]_m \vdash \psi$.
- If $\psi \in \Pi_1$, then there exist sentences $\pi_1, \ldots, \pi_r \in \Pi_1$ and $\sigma_1, \ldots, \sigma_r \in \Sigma_1$ such that
  - $I\Delta_0 \vdash \bigvee_{j=1}^{r}(\sigma_j \wedge \pi_j)$.
  - For each $j = 1, \ldots, r$, over $I\Delta_0 + S_j$, m unnested applications of $\Pi_1^-\text{–IR}_0$ proves ψ.
  - For each $j = 1, \ldots, r$, $[I\Delta_0 + S_j, \Pi_1\text{–IR}]_m \vdash \psi$.

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