Predicativity – Part I

Gerhard Jäger

Institut für Informatik und angewandte Mathematik
Universität Bern

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1. Historical background

2. Feferman – Schütte – Γ_0

Russell and Poincaré

Russell (around 1901 – 1906)

A propositional function $\varphi[x]$ is called *predicative* if it defines a class, i.e. if the class $\{x : \varphi[x]\}$ exists, and impredicative otherwise. For example, the propositional function $(x \notin x)$ is impredicative.

Poincaré (around 1906)

- The *vicious circle principle (VPC)*: A definition of an object S is *impredicative* if it refers to a totality to which S belongs.
- VPC is the essential source of inconsistencies.
- The structure of the natural numbers and the principle of complete induction do not require foundational justification; further sets have to be introduced by purely predicative means.

Typical impredicative definitions

- $S \;=\; \{\, n \in \mathbb{N} : (\forall X \subseteq \mathbb{N})\varphi[X, n] \,\}$

  $$?:\quad m \in S \;\rightsquigarrow\; (\forall X \subseteq \mathbb{N})\varphi[X, m] \;\rightsquigarrow\; \varphi[S, m] \;\rightsquigarrow\; m \in S.$$

- The least upper bound principle of classical analysis

  We identify rational numbers with certain natural numbers and real numbers with the upper parts of Dedekind sections. Then the least upper bound S of a bounded non-empty set $\mathcal{R} = \{X \subseteq \mathbb{N} : \varphi[X]\}$ of reals is given by

  $$S \;=\; \bigcap_{M \in \mathcal{R}} M \;=\; \{n \in \mathbb{N} : (\forall X \subseteq \mathbb{N})(\varphi[X] \rightarrow n \in X)\}.$$

Typical impredicative definitions (cont.)

- Accessible parts and well-orderings

Let $\prec$ be a (primitive recursive) linear ordering on $\mathbb{N}$.

$$\mathrm{Closed}[\prec, X] \;:\Leftrightarrow\; (\forall m \in \mathbb{N})((\forall n \prec m)(n \in X) \to m \in X),$$

$$\mathrm{Acc}[\prec] \;:=\; \{n \in \mathbb{N} : (\forall X \subseteq \mathbb{N})(\mathrm{Closed}[\prec, X] \to n \in X)\},$$

$$\mathrm{WO}[\prec] \;:\Leftrightarrow\; \mathrm{Acc}[\prec] = \mathbb{N}.$$

Clearly,

$$\mathrm{WO}[\prec] \;\Leftrightarrow\; \neg(\exists F \in \mathbb{N}^{\mathbb{N}})(\forall n \in \mathbb{N})(F(n+1) \prec F(n)).$$

First developments

Hermann Weyl (1885–1955)

- Predicative development of a substantial of analysis.
- Restriction to arithmetically definable sets of natural and rational numbers; real numbers via Cauchy sequences.
- Implicit formal framework equivalent to ACA_0.

Ramified analytic hierarchy

$$R_0 := \emptyset, \qquad R_{\alpha+1} := \mathrm{Def}^{(2)}(R_\alpha), \qquad R_\lambda := \bigcup_{\xi<\lambda} R_\xi \ (\lambda \text{ limit}).$$

Gödel's constructible hierarchy

$$L_0 := \emptyset, \qquad L_{\alpha+1} := \mathrm{Def}(L_\alpha), \qquad L_\lambda := \bigcup_{\xi<\lambda} L_\xi \ (\lambda \text{ limit}).$$

First developments (cont.)

Some properties:

- $\bigcup_{\xi \in \mathrm{On}} R_\xi = \bigcup_{\xi < \beta_0} R_\xi$ for some countable β_0.
- $R_\xi = L_\xi \cap \mathrm{Pow}(\mathbb{N})$ for suitable $\xi < \beta_0$.
- The steps $R_\alpha \mapsto R_{\alpha+1}$ and $L_\alpha \mapsto L_{\alpha+1}$ are justified from a predicative perspective.

Kleene, Spector et al.

$$\mathrm{HYP} = \Delta^1_1 = R_{\omega_1^{CK}} = L_{\omega_1^{CK}} \cap \mathrm{Pow}(\mathbb{N}).$$

First developments (cont.)

Conjecture (Kreisel, Spector, Wang)

$$\text{Predicatively justifiable subsets of } \mathbb{N} \ = \ \mathrm{HYP}.$$

Naive approach via predicatively definable ordinals (PDO):

(1) 0 is a PDO.

(2) If α is a PDO and $\prec$ a primitive recursive linear ordering on $\mathbb{N}$ such that $R_\alpha \models \mathrm{WO}[\prec]$, then $|\prec|$ is a PDO.

However,

$$\alpha < \beta \text{ and } R_\alpha \models \mathrm{WO}[\prec] \quad \not\Rightarrow \quad R_\beta \models \mathrm{WO}[\prec].$$

Feferman-Schütte approach

Systems of ramfied analysis RA_α:

- Variables $X^\beta, Y^\beta, Z^\beta$ for each $1 \leq \beta \leq \alpha$.
- Comprehension principles expressing closure under the appropriate ramified definitions (formal versions of those of R_β and L_β).
- For any ordinal γ: If $R_{\gamma+\beta}$ is the range of the quantifiers QX^β, then

$$\langle R_{\gamma+\beta} \rangle_{\beta \leq \alpha} \models \mathsf{RA}_\alpha,$$

- In particular,

$$\begin{aligned} \mathsf{RA}_\alpha \vdash \mathrm{WO}^1[\prec] \quad &\Rightarrow \quad R_\gamma \models \mathrm{WO}[\prec] \ \text{ for all } \gamma, \\ &\Rightarrow \quad \prec \text{ is a well-ordering.} \end{aligned}$$

Feferman-Schütte approach (cont.)

Predicatively provable ordinals (PPO)

(1) 0 is a PPO.

(2) If α is a PPO and $\prec$ a primitive recursive linear ordering on $\mathbb{N}$ such that $\mathsf{RA}_\alpha \vdash \mathrm{WO}^1[\prec]$, then $|\prec|$ is a PPO.

Theorem (Feferman & Schütte, independently)

$$\textit{Limit of predicativity} \;:::\; \Gamma_0$$

$$\textit{Predicative mathematics} \;:::\; \mathsf{RA}_{<\Gamma_0} := \bigcup_{\xi<\Gamma_0} \mathsf{RA}_\xi.$$

$\mathsf{RA}_{<\Gamma_0}$ *is called the system of autonomous ramified progressions.*

The ordinal Γ_0

The Veblen hierarchy $\langle\varphi_\alpha\rangle_{\alpha\in\mathrm{On}}$

$$\varphi_0(\xi) := \omega^\xi,$$

$$\varphi_\alpha(\xi) := \xi\text{-th element of } \{\eta : (\forall\beta < \alpha)(\varphi_\beta(\eta) = \eta)\} \quad \text{if } \alpha > 0.$$

Then

$$\Gamma_0 := \text{least } \xi \text{ such that } \varphi_\xi(0) = \xi.$$

Remark

System of terms based on 0, $+$, and φ can be used to build a recursive notation system $(\mathrm{OT}, \lhd)$ for all ordinals less than Γ_0.

$$\mathrm{WO}[n] :\Leftrightarrow \mathrm{WO}[\lhd{\restriction}n].$$

Determining the limit of predicativity

Upper bound

- Cut elimination: $\mathrm{RA}_\alpha \vdash^\sigma_\tau A \quad \Rightarrow \quad \mathrm{RA}_\alpha \vdash^{\varphi_\tau(\sigma)}_0 A.$
- Boundedness: $\mathrm{RA}_\alpha \vdash^\sigma_0 \mathrm{WO}^1[\prec] \quad \Rightarrow \quad |\prec| \leq \omega \cdot \sigma.$

Lower bound

- For suitable α, β: $\mathrm{RA}_\alpha \vdash (\forall x \lhd \beta)(\mathrm{WO}[x] \rightarrow \mathrm{WO}[\varphi_x(0)]).$
- $\Gamma_0 = \sup\{\sigma_n : n < \omega\}, \quad$ where $\sigma_0 := \omega, \ \sigma_{n+1} := \varphi\sigma_n 0.$

Predicatively justifiable principles

Arithmetic comprehension

For all formulas $A[x]$ of second order arithmetic without bound set quantifiers:

$$\exists X \forall n(n \in X \leftrightarrow A[n]). \qquad (\Pi^0_\infty\text{-CA})$$

Hyperarithmetic comprehension rule

For all Σ^1_1 formulas $A[x]$ and $B[x]$ of second order arithmetic:

$$\frac{\forall n(A[n] \leftrightarrow \neg B[n])}{\exists X \forall n(n \in X \leftrightarrow A[n])}. \qquad (\Delta^1_1\text{-CR})$$

Predicatively justifiable principles (cont.)

Bar Rule

Let $\prec$ be any primitive recursive linear ordering (well-ordering) on $\mathbb{N}$ and $A[x]$ any formula of second order arithmetic. We set

$$\begin{aligned} \mathrm{Prog}[\prec, A] &:\Leftrightarrow \forall m((\forall n \prec m)A[n] \to A[m]), \\ \mathrm{TI}[\prec, A] &:\Leftrightarrow \mathrm{Prog}[\prec, A] \to \forall m A[m], \\ \mathrm{WF}[\prec] &:\Leftrightarrow \forall X \mathrm{TI}[\prec, X]. \end{aligned}$$

Then the *Bar Rule* is the rule of inference

$$\frac{\mathrm{WF}[\prec]}{\mathrm{TI}[\prec, B]} \qquad \text{(BR)}$$

for all primitive recursive linear orderings $\prec$ and all formulas $B[x]$ of second order arithmetic.

A warning

The Bar Rule must not be confused with schema of *Bar Induction* which is the implication

$$\forall R(\mathrm{WF}[R] \to \mathrm{TI}[R, A]) \qquad \text{(BI)}$$

for all formulas $A[x]$ of second order arithmetic. (BI) cannot be justified predicatively. On the contrary, (BI) implies strong comprehensions.

Lemma

For all arithmetic formulas $A[X]$ and arbitrary formulas B of second order arithmetic we have

$$\mathsf{ACA}_0 + (\mathsf{BI}) \vdash A[\{n : B[n]\}] \to \exists X A[X].$$

ACA_0: second order arithmetic with the axiom of complete induction and $(\Pi^0_\infty\text{-CA})$; conservative extension of PA.

Some first predicative formal systems

- Feferman, 1964
  - Systems HC_α for suitable iterations of hyperarithmetic comprehension up to α; $\bigcup_{\alpha<\Gamma_0} \mathsf{HC}_\alpha$ is of the same strength as $\mathsf{RA}_{<\Gamma_0}$.
  - The system IR of induction-recursion; corresponds, more or less, to $\mathsf{ACA}_0 + (\Delta^1_1\text{-}\mathsf{CR}) + (\mathsf{BR})$.
- The system PS_1 of set theory
  - usual language of set theory, axioms of Kripke-Platek set theory with infinity but without Δ_0 collection;
  - rules for definition by $\in$-recursion, Σ reflection, and sufficiently many ordinals.
  - PS_1 is a conservative extension of IR.

Some first predicative formal systems (cont.)

- Internalizing autonomous progressions: the theory $\mathsf{AUT}(\Pi^0_1)$
  - Given a Π^0_1 formula $A[X, y]$ and a (primitive recursive) well-ordering $\prec$ write $\mathrm{Hier}_A[\prec, U, V]$ to express that for all elements of the field of $\prec$,

$$
\begin{aligned}
(V)_0 &= U, \\
(V)_{m \oplus 1} &= \{n : A[(V)_m, n]\}, \\
(V)_m &= \biguplus\{(V)n : n \prec m\} \quad \text{for } m \text{ a limit},
\end{aligned}
$$

  - $\mathsf{AUT}(\Pi^0_1)$ is the extension of ACA_0 by the Bar Rule (BR) and, for all primitive recursive well-orderings and all Π^0_1 formulas $A[X, y]$,

$$
\frac{\mathrm{WF}[\prec]}{\forall X \exists Y \mathrm{Hier}_A[\prec, X, Y]} \,.
$$

Some first predicative formal systems (cont.)

Theorem (Feferman, Jä)

1. $\mathsf{AUT}((\Pi^0_1) \equiv \mathsf{RA}_{<\Gamma_0}$.
2. $(\Sigma^1_1\text{-DC}) + (\mathsf{BR})$, $(\Sigma^1_1\text{-AC}) + (\mathsf{BR})$, *and* $(\Delta^1_1\text{-CA}) + (\mathsf{BR})$ *are conservative extensions of* $\mathsf{AUT}((\Pi^0_1)$ *for* Π^1_2 *sentences.*

For all Σ^1_1 formulas $A[X,Y]$, $B[x,Y]$, $C[x]$, and $D[x]$,

$$\forall X \exists Y A[X,Y] \to \forall X \exists Z ((Z)_0 = X \land \forall n A[(Z)_n, (Z)_{n+1}], \qquad (\Sigma^1_1\text{-DC})$$

$$\forall n \exists X B[n,X] \to \exists Z \forall n B[n,(Z)_n], \qquad (\Sigma^1_1\text{-AC})$$

$$\forall n (C[n] \leftrightarrow \neg D[n]) \to \exists X \forall n (n \in X \leftrightarrow C[n]). \qquad (\Delta^1_1\text{-CA})$$