Models of provability logic

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Gödel-Löb logic

Language:

$$p \qquad \neg\varphi \qquad \varphi \wedge \psi \qquad \Box\varphi$$

Axioms:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

Second incompleteness theorem:

$$\Box\Diamond\top \to \Box\bot$$

Arithmetic interpretation

An arithmetic interpretation assigns a formula p^* in the language of arithmetic to each propositional variable p.

- $p \mapsto p^*$
- $\Box \mapsto Prv_{\mathrm{PA}}$

Theorem (Solovay)

If $\mathrm{GL} \nvdash \varphi$ *then for some arithmetic interpretation* $*$, $\mathrm{PA} \nvdash \varphi^*$.

Relational semantics

Kripke models:

- Frames: Well-founded partial orders $\langle W, < \rangle$
- Valuations: $[\![\varphi]\!] \subseteq \mathcal{P}(W)$,

$$w \in [\![\Box\varphi]\!] \Leftrightarrow \forall v < w, v \in [\![\varphi]\!]$$

Theorem

GL *is sound for* $\langle W, < \rangle$ *if and only if* $<$ *is well-founded.*

Further, GL *is complete for the class of well-founded frames and enjoys the finite model property.*

Topological semantics:

- GL-spaces: scattered topological spaces $\langle X, \mathcal{T} \rangle$
  Scattered: Every non-empty subset contains an isolated point.
- Valuations: dA is the set of limit points of A.

$$[\![\Diamond\varphi]\!] = d\,[\![\varphi]\!] \,.$$

GL is also sound and complete for this interpretation.

Some scattered spaces

- A finite partial order $\langle W, \leq \rangle$ with the downset topology
- An ordinal ξ with the initial segment topology
- An ordinal ξ with the order topology

Non-scattered:

- The real line
- The rational numbers
- The Cantor set

Polymodal Gödel-Löb

GLP: Contains one modality $[n]$ for each $n < \omega$.

Axioms:

$$
\begin{array}{rr}
[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi) & (n < \omega) \\
[n]([n]\varphi \to \varphi) \to [n]\varphi & (n < \omega) \\
[n]\varphi \to [m]\varphi & (n < m < \omega) \\
\langle n \rangle \varphi \to [m]\langle n \rangle \varphi & (n < m < \omega)
\end{array}
$$

(Possible) arithmetic interpretation:

$[n]\varphi \equiv$ "φ is provable using n instances of the ω-rule".

Introduced by Japaridze in 1988.

Kripke semantics

Frames:

$$\langle W, \langle <_n \rangle_{n<\omega} \rangle$$

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

Valid iff $<_n$ is well-founded

$[n]\varphi \to [n+1]\varphi$:

Valid iff $w <_{n+1} v \Rightarrow w <_n v$

$\langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$:

Valid iff

$$v <_n w \text{ and } u <_{n+1} w \Rightarrow v <_n u$$

Even GLP_2 has no non-trivial Kripke models.

Topological semantics

Spaces:

$$\langle X, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle$$

Write d_n for the limit point operator on $\mathcal{T}_n$.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$: Valid iff $\mathcal{T}_n$ is scattered

$[n]\varphi \to [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$: Valid iff

$$A \subseteq X \Rightarrow d_n A \in \mathcal{T}_{n+1}$$

Topological completeness

Beklemishev, Gabelaia: GLP is complete for the class of GLP-spaces

The proof uses non-constructive methods.

Blass: It is consistent with ZFC that the canonical ordinal spaces for GLP_2 are all trivial

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for its canonical ordinal spaces

Bagaria More generally, for all n it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.

The closed fragment

Written GLP^0, it does not allow propositional variables (only $\bot$).

Beklemishev: GLP^0_ω may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

Theorem (Ignatiev)

There is a Kripke frame $\mathfrak{I}$ such that GLP^0_ω is sound and complete for $\mathfrak{I}$.

Ignatiev's model of GLP^0

Given an ordinal $\xi = \alpha + \omega^\beta$, define $\ell\xi = \beta$ ($\ell 0 = 0$).

Ignatiev's model:

$$\mathfrak{I} = \langle D, \langle <_n \rangle_{n<\omega} \rangle$$

- $D = \{f : \omega \to \varepsilon_0 : \forall n\, f(n+1) \leq \ell f(n)\}$
- $f <_n g$ if $f(m) = g(m)$ for $m < n$ and $f(n) < g(n)$

Example:

$$\langle \omega^{\omega+1}, \omega, 0, \ldots \rangle <_2 \langle \omega^{\omega+1}, \omega, 1, 0, \ldots \rangle$$

Frame conditions

Ignatiev's model does not satisfy all frame conditions.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

$<_n$ is based on an ordinal and hence well-founded

$[n]\varphi \to [n+1]\varphi$:

$$\langle \omega, 1, 0, \ldots \rangle \begin{array}{c} <_1 \\ \not<_0 \end{array} \langle \omega, 0, 0, \ldots \rangle$$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$:

$$\begin{array}{ccc} \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \\ \langle \omega^\omega, \omega, 0, 0, \ldots \rangle & <_2 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \\ \hline \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 0, 0, \ldots \rangle \end{array}$$

The main axis

Definition

A sequence $f : \omega \to \varepsilon_0$ is exact if for all n,

$$f(n+1) = \ell f(n).$$

Main axis: Set of exact sequences.

Lemma

Every closed formula which is satisfied on $\mathfrak{I}$ is satisfied on the main axis.

Is GLP^0 sound for $\mathfrak{I}$?

Icard topologies

Icard defined a structure

$$\mathfrak{I} = \langle \varepsilon_0, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle.$$

Generlized intervals:

$$(\alpha, \beta)_n = \{\vartheta : \alpha < \ell^n \vartheta < \beta\}.$$

$\mathcal{T}_n$ is generated by intervals of the form

- $(\alpha, \beta)_m$ for $m < n$
- $[0, \beta)_m$ for $m \leq n$

Topological conditions

Icard's model does not satisfy all frame conditions either.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

$\mathcal{T}_n$ is scattered since $\mathcal{T}_0$ is.

$[n]\varphi \to [n+1]\varphi$: $\mathcal{T}_{n+1}$ is always a refinement of $\mathcal{T}_n$.

$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: The point

$$\omega^\omega = \lim_{n\to\omega} \omega^n$$

should be isolated in $\mathcal{T}_2$.

Ignatiev vs. Icard

Define $\vec{\ell} : \varepsilon_0 \to D$ by

$$\vec{\ell}\xi = \langle \xi, \ell\xi, \ell^2\xi, \ldots, \ell^n\xi, \ldots \rangle$$

Lemma

For every $\xi < \varepsilon_0$,

$$\langle \mathcal{T}, \xi \rangle \models \varphi \Leftrightarrow \langle \mathfrak{I}, \vec{\ell}\xi \rangle \models \varphi$$

Corollary

$\mathfrak{I}$ and $\mathfrak{T}$ satisfy the same set of formulas.

Soundness and completeness

Theorem

GLP^0 *is sound for both* $\mathfrak{I}$ *and* $\mathfrak{T}$.

Proof.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$: Valid on both $\mathfrak{I}$ and $\mathfrak{T}$.

$[n]\varphi \to [n+1]\varphi$: Valid on $\mathfrak{T}$.

$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: Valid on $\mathfrak{I}$. $\square$

Theorem (Ignatiev, Icard)

GLP^0 *is complete for both* $\mathfrak{I}$ *and* $\mathfrak{T}$.

Concluding remarks

- GL is very nice as a modal logic, but only takes us so far
- GLP is very useful! (Ordinal analysis, ordinal notation systems, unprovable statements...)
- But GLP is tougher to work with
  - No Kripke frames
  - Topological completeness is hard
- The closed fragment gives us a good middle ground
- Here we have Kripke models, simple topological models.
- Future work: Generalize to stronger theories

  Tomorrow's talk: Modalities beyond ω??

Thank you!