Transfinite provability logic

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Transfinite Gödel-Löb

Λ is an arbitrary ordinal.

GLP$_\Lambda$: One modality $[\lambda]$ for each ordinal $\lambda < \Lambda$.

Axioms:

$$
\begin{array}{rlr}
[\xi](\varphi \to \psi) & \to ([\xi]\varphi \to [\xi]\psi) & (\xi < \Lambda) \\
[\xi]([\xi]\varphi \to \varphi) & \to [\xi]\varphi & (\xi < \Lambda) \\
[\xi]\varphi & \to [\zeta]\varphi & (\xi < \zeta < \Lambda) \\
\langle\xi\rangle\varphi & \to [\zeta]\langle\xi\rangle\varphi & (\xi < \zeta < \Lambda)
\end{array}
$$

Worms

Worms: Iterated consistency statements

$$\langle \xi_1 \rangle \langle \xi_2 \rangle \dots \langle \xi_n \rangle \top$$

Wrm : the class of all worms

Wrm_α : the class of all worms with entries at least α

$$\mathsf{w} <_\xi \mathsf{v} \Leftrightarrow \mathsf{GLP} \vdash \mathsf{w} \rightarrow \langle \xi \rangle \mathsf{v}$$

The relation $<_\xi$ is a well-order on Wrm_ξ (modulo equivalence).

It is still well-founded on Wrm.

Order types

Small order types:

$$o_\xi(\mathrm{w}) = \sup\{o_\xi(\mathrm{v}) + 1 : \mathrm{v} <_\xi \mathrm{w} \text{ and } \mathrm{v} \in \mathrm{Wrm}_\xi\}$$

Big order types:

$$\Omega_\xi(\mathrm{w}) = \sup\{\Omega_\xi(\mathrm{v}) + 1 : \mathrm{v} <_\xi \mathrm{w} \text{ and } \mathrm{v} \in \mathrm{Wrm}\}$$

(Note: $\sup \varnothing = 0$!)

Problem: How to compute o, Ω?

Worms recursively

- $\top$ is a worm
- if w, v are worms, w0v is a worm
- if w is a worm and α an ordinal then $\alpha \uparrow$ w is a worm

Where

- $(\langle \xi_1 \rangle \ldots \langle \xi_n \rangle \top) 0 (\langle \zeta_1 \rangle \ldots \langle \zeta_m \rangle \top)$
  $= \langle \xi_1 \rangle \ldots \langle \xi_n \rangle \langle 0 \rangle \langle \zeta_1 \rangle \ldots \langle \zeta_m \rangle \top$
- $\alpha \uparrow \langle \xi_1 \rangle \ldots \langle \xi_n \rangle \top = \langle \alpha + \xi_1 \rangle \ldots \langle \alpha + \xi_n \rangle \top$

Small order types recursively

We compute these in the paper Well-founded relations in the transfinite Japaridze algebra I

- $o_\xi(\top) = 0$
- $o_\xi(\mathrm{w}0\mathrm{v}) = o_\xi(\mathrm{v}) + 1 + o_\xi(\mathrm{w})$
- $o_\xi(\alpha \uparrow \mathrm{w}) = ??$

Fact: There is a unique function $e^\alpha : \mathrm{On} \to \mathrm{On}$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Wrm} & \xrightarrow{\alpha\uparrow} & \mathrm{Wrm} \\ \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle o} \\ \mathrm{Ord} & \xrightarrow{e^\alpha} & \mathrm{Ord} \end{array}$$

Properties of e^{ξ}

- $e^0 = \mathrm{id}$
- $e^{\xi}(0) = 0$
- $e^1(1+\alpha) = \omega^{1+\alpha}$
- $e^{\xi+\zeta} = e^{\xi} \circ e^{\zeta}$
- e^{ξ} is always normal.

Sequences with these properties are (weak) hyperations.

Hyperations

We study hyperations (and cohyperations) systematically in the paper Veblen progressions and Hyperations of ordinal functions

Definition:
The hyperation of a normal function f is the unique family of normal functions $\langle f^{\xi} \rangle_{\xi \in \mathrm{On}}$ such that

- $f^1 = f$
- $f^{\xi+\zeta} = f^{\xi} \circ f^{\zeta}$
- f^{ξ} is always normal
- f^{ξ} is pointwise minimal amnongst all such families of functions.

Fact: $\langle e^{\xi} \rangle_{\xi \in \mathrm{On}}$ is the hyperation of $\alpha \mapsto -1 + \omega^{\alpha}$ and is called the hyperexponential.

Computing hyperations

Let $\varphi(\alpha) = \omega^\alpha$ and $e(\alpha) = -1 + \omega^\alpha$.

- $\varphi^3(0) = e^2(1) = \omega^\omega$
- $\varphi^3(1) = e^3(1) = \omega^{\omega^\omega}$
- $\varphi^{\omega^\xi} = \varphi_\xi$ (Veblen functions)
- $\varphi^{\omega^{\xi_1}+\ldots+\omega^{\xi_n}} = \varphi_{\xi_1} \circ \ldots \circ \varphi_{\xi_n}$
- $\varphi^\omega(0) = e^\omega(1) = \varepsilon_0$
- $\varphi^{\Gamma_0}(0) = e^{\Gamma_0}(1) = \Gamma_0$

Computing Ω_ξ

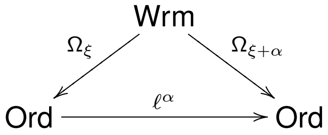
Fact: If $\xi < \zeta$ then $<_\zeta$ is a refinement of $<_\xi$

For an ordinal $\xi = \alpha + \omega^\beta$ define $\ell\xi = \beta$ ($\ell 0 = 0$)

Then,

- $\Omega_0(\mathsf{w}) = o_0(\mathsf{w})$
- $\Omega_{\xi+1}(\mathsf{w}) = \ell\Omega_\xi(\mathsf{w})$
- $\Omega_{\xi+\zeta} =$??

Same drill: there is a unique ℓ^α : On $\to$ On making the diagram commute.



Properties of ℓ^ξ

- $\ell^0 = \mathrm{id}$
- $\ell^1 = \ell$
- $\ell^{1+\xi}(\alpha + \beta) = \beta$
- $\ell^{\xi+\zeta} = \ell^\zeta \circ \ell^\xi$
- ℓ^ξ is always initial.

Sequences with these properties are (weak) cohyperations.

Cohyperations

Definition:
The cohyperation of an initial function f is the unique family of initial functions $\langle f^\xi \rangle_{\xi \in \mathrm{On}}$ such that

- $f^1 = f$
- $f^{\xi+\zeta} = f^\zeta \circ f^\xi$
- f^ξ is always initial
- f^ξ is pointwise maximal amnongst all such families of functions.

Fact: $\langle \ell^\xi \rangle_{\xi \in \mathrm{On}}$ is the cohyperation of ℓ and is called the hyperlogarithm.

A calculus for Ω

Big order types described in detail in the paper Well-founded relations on the transfinite Japaridze algebra II

- $\Omega_0 = o_0$
- $\Omega_{\xi+\zeta} = \ell^{\zeta}\Omega_{\xi}$

Lower bounds:

$$\Omega_{\xi}(\mathsf{w}) \geq e^{\zeta}\Omega_{\xi+\zeta}(\mathsf{w}).$$

The closed fragment

Recall: GLP^0 does not allow propositional variables (only $\bot$).

Theorem (Ignatiev)

There is a Kripke model $\mathfrak{I} = \mathfrak{I}_\omega^{\varepsilon_0}$ such that GLP_ω^0 is sound and complete for $\mathfrak{I}_\omega^{\varepsilon_0}$.

Ignatiev's model

Given an ordinal $\xi = \alpha + \omega^\beta$, define $\ell\xi = \beta$ ($\ell 0 = 0$).

The model:

$$\mathfrak{I}_\omega = \langle D_\omega^{\varepsilon_0}, \langle <_n \rangle_{n<\omega} \rangle$$

- $D_\omega^{\varepsilon_0} = \{f : \omega \to \varepsilon_0 : \forall n\, f(n+1) \leq \ell f(n)\}$
- $f <_n g$ if $f(m) = g(m)$ for $m < n$ and $f(n) < g(n)$

How to go beyond ω?

Beyond ω

Goal: define $\mathfrak{I}^{\Theta}_{\Lambda} = \langle D^{\Theta}_{\Lambda}, \langle <_{\xi} \rangle_{\xi < \Lambda} \rangle$.

"Worlds":

$f \in D^{\Theta}_{\Lambda}$ iff $f : \Lambda \to \Theta$ and:

- First approximation:
  $f(\xi + \delta) \leq \ell^{\delta} f(\xi)$ holds locally

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$$f(\zeta) \leq \ell^{-\xi+\zeta} f(\xi)$$

provided $\xi < \zeta$ is large enough

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  provided $\xi < \zeta$ is large enough
- Precisely:

$$\forall \zeta \exists \vartheta < \zeta \forall \xi \in [\vartheta, \zeta),\ f(\zeta) \leq \ell^{-\xi+\zeta} f(\xi)$$

Generalized Icard topologies

We wish to define

$$\mathfrak{T}^{\Theta}_{\Lambda} = \langle \Theta, \langle \mathcal{T}_\lambda \rangle_{\lambda < \Lambda} \rangle.$$

Generlized intervals:

$$(\alpha, \beta)_\xi = \{\vartheta : \alpha < \ell^\xi \vartheta < \beta\}.$$

$\mathcal{T}_\lambda$ is generated by intervals of the form

- $(\alpha, \beta)_\xi$ for $\xi < \lambda$
- $[0, \beta)_\xi$ for $\xi \leq \lambda$

Original Icard space: $\mathfrak{T}^{\varepsilon_0}_{\omega}$

Fact: $\mathfrak{I}^{\Theta}_{\Lambda}$ and $\mathfrak{T}^{\Theta}_{\Lambda}$ satisfy the same formulas.

Completeness

Theorem (DFD, Joosten)

GLP^0_Λ *sound for both* $\mathfrak{I}^\Theta_\Lambda$ *and* $\mathfrak{T}^\Theta_\Lambda$ *independently of* Θ, Λ.

Further, it is also complete for both structures if and only if

$$\Theta > e^\Lambda(1).$$

These results are proven in Models of transfinite provability logic

Concluding remarks

- Much of the theory for GLP_ω carries over naturally to GLP_Λ
- The generalization forces us to look at ordinal function iteration and Veblen progressions in a new light
- There are still many results to generalize

  Conjecture: Topological completeness holds for GLP_Λ
- Possible application: ordinal analysis of predicative theories.

  Arithmetical interpretations?

Thank you!

Our papers may be found at

http://personal.us.es/dfduque/publications.html