Provability algebras for theories of Tarskian truthpredicates

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- Joint work with Lev Beklemishev.
- Work is still in progress.

General problem

Extend provability algebra based proof-theoretic analysis to theories stronger than **PA**. Predicative analysis theories are of particular interest.

Truthpredicates

- Truthpredicates are tightly related to reflection principles and are convenient in our framework.
- Theories of iterated truth are mutually interpretable with various standard theories of predicative strength (ramified analysis, iterated Π^0_1-comprehension).

Model case: One truthpredicate

- $T(x)$ "x is the Gödel number of a true arithmetical sentence".
- Let $\mathcal{L}(T)$ be the extension of the language of **PA** by T.

Tarski principles for truth:

- $\forall\phi(At[\phi] \to (T[\phi] \leftrightarrow Tr_0[\phi]))$;
- $\forall\phi,\psi(T[\phi \wedge \psi] \leftrightarrow (T[\phi] \wedge T[\psi]))$;
- $\forall\phi(T[\neg\phi] \leftrightarrow \neg T[\phi])$;
- $\forall\phi(\cdot)(T[\forall\phi(x)] \leftrightarrow \forall x T\phi[\underline{x}])$.

Theories:
BT = **PRA** + Tarskian truth; **PA(T)** = **BT** + full induction scheme for $\mathcal{L}(T)$.

Reflection principles

Extend arithmetical hierarchy to $\mathcal{L}(T)$:

- $\Pi_\omega = \cup_{n<\omega}\Pi_n$;
- for $n \geq 1$, $\Sigma_{\omega+n}$ are $\Sigma_n(T)$-formulas.

For $n \geq 1$, let Tr_n and $Tr_{\omega+n}$ be natural truthdefinitions for Σ_n and $\Sigma_n(T)$ sentences respectively.

For a r. e. extension S of **BT** in $\mathcal{L}(T)$, consider reflection schemes:

- $R_0(S) \rightleftharpoons \mathrm{Con}(S)$;
- $R_n(S) \rightleftharpoons \forall \sigma \in \Sigma_n(\Box_S \sigma \to Tr_n(\sigma))$ for $n \geq 1$;

Reflection principles

- $R_\omega(S) \rightleftharpoons \{R_n(S) \mid n < \omega\}$ (uniform reflection);
- $R_{\omega+1}(S) \rightleftharpoons \forall\phi \in \Pi_\omega(\Box_S(\phi) \to T(\phi))$ (global arithmetical reflection);
- $R_{\omega+1+n}(S) \rightleftharpoons \forall\sigma \in \Sigma_n(T)(\Box_S\sigma \to Tr_{\omega+n}(\sigma))$, for $n \geq 1$;
- $R_{\omega+\omega}(S) \rightleftharpoons \{R_\alpha(S) \mid \alpha < \omega + \omega\}$

Note that $R_\omega(S)$ and $R_{\omega+\omega}(S)$ are infinite schemes, not single sentences.

Induction and Reflection

- **BT** is Π_ω-conservative over **PRA** (Kotlarski, Krajewski, Lachlan model-theoretically; Halbach syntactically);
- $\mathbf{BT} + R_\omega(\mathbf{BT}) \equiv_{\Pi_\omega} \mathbf{PA}$;
- $\mathbf{PA}(\mathbf{T})$ and **ACA** are mutually interpretable;
- $\mathbf{BT} + R_{\omega+\omega}(\mathbf{BT}) \equiv \mathbf{PA}(T)$;
- $\mathbf{BT} + R_{\omega+1}(\mathbf{BT}) \equiv \mathbf{BT} + I\Delta_0(T)$ (Kotlarski model-theoretically).

Reduction property

Let $R^1_\alpha(S) = R_\alpha(S)$ and $R^{k+1}_\alpha(S) = R_\alpha(S + R^k_n(S))$.

Th. Let U and S be r. e. theories in $\mathcal{L}(T)$ and let $S \vdash U$. Then provably in **PRA**,

$$U + R_{\alpha+1}(S) \equiv_{\Pi_{1+\alpha}} U + \{R^k_\alpha(S) \mid k < \omega\},$$

provided U is an extension of **BT** of the following complexity:

- $U \subseteq \Pi_{\alpha+2}$ if $\alpha < \omega$;
- $U \subseteq \Pi_\omega$ if $\alpha = \omega$;
- $U \subseteq \Pi_{\alpha+1}$ if $\alpha > \omega$.

Reduction property

Proof

- For $\alpha < \omega$, the standard reduction by Beklemishev;
- For $\alpha > \omega$, a relativization of the standard reduction: $\Pi_k \mapsto \Pi_k(T)$;
- Let $\alpha = \omega$. For $S = \mathbf{BT}$ this result is due to H. Kotlarski (model-theoretical proof). **PRA**-formalization is new:
  - we formalize an analogue of Kotlarski's argument in $\mathbf{WKL}_0$ and use Π_2-conservativity of $\mathbf{WKL}_0$ over **PRA** (Friedman).

Reflection calculus $\mathbf{RC}_\Omega$

Fix some ordinal Ω or let $\Omega = \mathrm{On}$. For $\mathcal{L}(T)$, putting $\Omega = \omega 2 + 1$ is enough.

- Language: $A ::= \top \mid A_1 \wedge A_2 \mid \alpha A, \qquad \alpha < \Omega$.
- Derive sequents $A \vdash B$. $A \sim B$ stands for both $A \vdash B$ and $B \vdash A$.
- Do not write $\top$ for brevity's sake.

$\mathbf{RC}_\Omega$: definition

Rules of $\mathbf{RC}_\Omega$:

1. $A \vdash A$; $A \vdash \top$; if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
2. $A \wedge B \vdash A, B$; if $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$;
3. $\alpha\alpha A \vdash \alpha A$; if $A \vdash B$ then $\alpha A \vdash \alpha B$;
4. $\alpha A \vdash \beta A$ for $\alpha > \beta$;
5. $\alpha A \wedge \beta B \vdash \alpha(A \wedge \beta B)$ for $\alpha > \beta$ and $\beta \notin Lim$.
6. $(\alpha + 1)\alpha A \sim (\alpha + 1)A$ for $\alpha \in Lim$.

Unlike $\mathbf{GLP}_\Omega$, here we have, e. g.,

$$(\omega + 1) \wedge \omega(\omega + 1) \not\sim (\omega + 1)\omega(\omega + 1) \sim (\omega + 1)(\omega + 1).$$

$\mathbf{RC}_\Omega$: arithmetical interpretation

Let S be a r. e. extension of **BT** in $\mathcal{L}L(T)$. Interpret formula A by scheme A_S:

- $\top_S = \emptyset$; $(A \wedge B)_S = (A_S \cup B_S)$;
- $(\alpha A)_S = R_\alpha(S + A_S)$.

Th. If $A \vdash B$ in $\mathbf{RC}_\Omega$, then $S + A_S \vdash B_S$.

Note that $\mathbf{RC}_\Omega$ is not arithmetically complete. E. g. the correct principle

$$0(\omega(\omega+1) \wedge \omega(\omega+2)) \vdash 0(\omega(\omega+1) \wedge (\omega+2))$$

is not derivable in $\mathbf{RC}_\Omega$.

$\mathbf{RC}_\Omega$: an ordinal notation system

- Denote $A <_\alpha B \rightleftharpoons B \vdash \alpha A$.
- Let W denote the set of all $\mathbf{RC}_\Omega$ words (i. e. formulas without $\wedge$).
- Denote by W_α the set of words in the alphabet $\{\beta \mid \alpha \leq \beta\}$.

Th. $(W/\sim, <_0)$ is a well-ordering.

Words and ordinals

0	$\dots$	1	$\dots$	101	$\dots$	11	$\dots$	n	$\dots$	ω
1	$\dots$	ω	$\dots$	$\omega 2$	$\dots$	ω^2	$\dots$	ω_n	$\dots$	ϵ_0
0ω	$\dots$	10ω	$\dots$	$\omega 0\omega$	$\dots$	1ω	$\dots$	$\omega\omega$	$\dots$	$(\omega+1)$
ϵ_0+1	$\dots$	$\epsilon_0+\omega$	$\dots$	$\epsilon_0\cdot 2$	$\dots$	$\epsilon_0\cdot\omega$	$\dots$	ϵ_1	$\dots$	ϵ_ω

$\omega(\omega+1)$	$\dots$	$(\omega+1)(\omega+1)$	$\dots$
$\epsilon_{\omega+1}$	$\dots$	$\epsilon_{\omega\cdot 2}$	$\dots$

Th. If $\omega^\alpha < \Omega$, then $o(W_{\omega^\alpha}) = \{0\} \cup C$, where C is an initial segment of Cr_α. In particular, if $\Omega = \mathrm{On}$, then $C = Cr_\alpha$.

Iterated reflection

- $o_\alpha(A)$ is the order type of $\{B \in W_\alpha \mid B <_\alpha A\}$;
- let $S^\alpha_\beta \equiv S + \{R_\alpha(S^\alpha_\gamma) \mid \gamma < \beta\}$.

Th. Let S be a r. e. $\Pi_{1+\alpha}$-extension of **BT**. Provably in **PRA**, for any $A \in W_\alpha$, $S + A_S \equiv_{\Pi_{1+\alpha}} S^\alpha_{o_\alpha(A)}$.

Cor. For $n < \omega$,

- $I\Delta_0(T) \equiv \mathbf{BT} + R_{\omega+1}(\mathbf{BT}) \equiv_{\Pi_\omega} \mathbf{BT}^\omega_\omega \equiv_{\Pi_{1+n}} \mathbf{BT}^n_{\epsilon_\omega}$;
- $\mathbf{PA}(T) \equiv \mathbf{BT} + R_{\omega+\omega}(\mathbf{BT}) \equiv_{\Pi_\omega} \mathbf{BT}^\omega_{\epsilon_0} \equiv_{\Pi_{1+n}} \mathbf{BT}^n_{\epsilon_{\epsilon_0}}$.

Reflection and conservativity calculus **$\mathbf{RCC}_\Omega$**

- **$\mathbf{RC}_\Omega$** is not arithmetically complete.
- A candidate for complete system is **$\mathbf{RCC}_\Omega$**:
  - Language of **$\mathbf{RC}_\Omega$**.
  - Derive sequents $A \vdash B$
    and for $\alpha < \Omega$, $A \vdash_\alpha B$ "B is $\Pi_{1+\alpha}$-conservative over A".

$\mathbf{RCC}_\Omega$: definition

$\mathbf{RCC}_\Omega$ rules:

1. Rules of $\mathbf{RC}_\Omega$;
2. if $A \vdash_\alpha B$ and $B \vdash_\alpha C$ then $A \vdash_\alpha C$; if $A \vdash_\alpha B$ then $\alpha A \vdash \alpha B$;
3. if $A \vdash B$ then $A \vdash_\alpha B$; if $A \vdash_\alpha B$ then $A \vdash_\beta B$ for $\beta < \alpha$;
4. $\alpha A \vdash_\alpha A$; if $A \vdash_\alpha \alpha B$ then $A \vdash \alpha B$;
5. $A \vdash_\alpha B$ then $\beta C \wedge A \vdash_\alpha \beta C \wedge B$ for $\beta < \alpha$; $A \vdash_\lambda B$ then $\lambda C \wedge A \vdash_\lambda \lambda C \wedge B$

$\mathbf{RCC}_\Omega$: some known facts

- $\mathbf{RCC}_\Omega$ is correct w.r.t. the intended arithmetical interpretation;
- for any formula A, αA is $\mathbf{RCC}_\Omega$-equivalent to some word B;
- the set of $\mathbf{RCC}_\Omega$-derivable equivalences $A \sim B$ is arithmetically complete.