Local Induction and Σ_{n+1}–Consequences of Arithmetic Theories

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Introduction

General Question: To find natural restrictions on an axiom scheme to obtain axiomatizations of its Σ_k/Π_k–consequences.

Axiom Scheme	Γ	Restriction
$I\Sigma_n, B\Sigma_n$	Π_{n+1}	Inference rule version
$I\Sigma_n, B\Sigma_n$	Σ_{n+2}	Parameter free version
$I\Sigma_n, B\Sigma_n$	Σ_{n+1}	??*

- Kaye–Paris–Dimitracopoulos [JSL'88] and Beklemishev–Visser [APAL'05] obtained axiomatizations of Σ_{n+1}–consequences of $I\Sigma_n$. But they do not correspond to a restriction of the induction scheme.
- Axiomatizations of the Σ_{n+1}–consequences of $B\Sigma_n$ were not known.

Outline

1. We introduce axiom schemes restricted up to definable elements and study their basic properties.

2. We show that this restriction captures the Σ_{n+1} consequences of $I\Sigma_n$ and $B\Sigma_n$.

3. Applications to local reflection principles.

Local axiom schemes

- Induction

$$\forall \bar{v} \in B\,[\varphi(0,\bar{v}) \wedge \forall x\,(\varphi(x,\bar{v}) \rightarrow \varphi(x+1,\bar{v})) \rightarrow \forall x \in A\,\varphi(x,\bar{v})]$$

- Collection

$$\forall \bar{v} \in B\,[\forall x\,\exists y\,\varphi(x,y,\bar{v}) \rightarrow \forall z \in A\,\exists u\,\forall x \le z\,\exists y \le u\,\varphi(x,y,\bar{v})]$$

- Minimization

$$\forall \bar{v} \in B\,\forall x \in A\,[\varphi(x,\bar{v}) \rightarrow \exists y\,(y = \mu t.\,\varphi(t,\bar{v}))]$$

Definition

1. $E(\Gamma, A, B)$ denotes the E–scheme up to elements in A restricted to Γ–formulas with parameters in B.

2. $E(\Gamma^{-}, A)$ denotes the E–scheme up to elements in A restricted to *parameter free* Γ–formulas.

Definable and minimal elements

- a is Γ–definable in $\mathfrak{A}$ with parameters in $X \subseteq \mathfrak{A}$ if there are $\varphi(x, \bar{v}) \in \Gamma$ and $\bar{b} \in X$ such that $\mathfrak{A} \models \varphi(a, \bar{b}) \wedge \exists! x\, \varphi(x, \bar{b})$.

- a is Γ–minimal in $\mathfrak{A}$ with parameters in $X \subseteq \mathfrak{A}$ if there are $\varphi(x, \bar{v}) \in \Gamma$ and $\bar{b} \in X$ such that $\mathfrak{A} \models a = \mu x.\, \varphi(x, \bar{b})$.

- $\mathcal{K}_n(\mathfrak{A}, X) = \Sigma_n$–definable elements of $\mathfrak{A}$ (parameters in X).

- $\mathcal{I}_n(\mathfrak{A}, X) =$ initial segment determined by $\mathcal{K}_n(\mathfrak{A}, X)$

$$\mathfrak{A} \quad [\!\text{———}\,)_{\omega}\text{———}\,)_{\mathcal{I}_n}\text{—————————————}\,)$$

Expressing "$\forall x \in \mathcal{K}_n$" in the language of Arithmetic

- Put $\exists! x\, \delta(x) \equiv \forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$.

$$\text{“}\forall x \in \mathcal{K}_n\, \varphi(x)\text{”}$$
$$\Updownarrow$$
$$\{\forall x\; [\delta(x) \wedge \exists! x\, \delta(x) \;\rightarrow\; \varphi(x)] \;:\; \delta \in \Sigma_n\}$$

- Fragments of Arithmetic **up to definable elements** $\rightsquigarrow$ local schemes restricted to classes of definable elements.
  - $I(\Sigma_n^-, \mathcal{K}_m)$: Σ_n^-–induction up to Σ_m–definable elements.
  - $B(\Sigma_n^-, \mathcal{K}_m)$: Σ_n^-–collection up to Σ_m–definable elements.
  - and so on...

What do fragments "up to" look like?

- Σ_n^-–induction up to Σ_m–definable elements, $I(\Sigma_n^-, \mathcal{K}_m)$, is

$$\varphi(0) \land \forall x\, (\varphi(x) \to \varphi(x+1)) \to$$
$$\forall x\, (\delta(x) \land \exists! x\, \delta(x) \to \varphi(x))$$

where $\varphi \in \Sigma_n$, $\delta \in \Sigma_m$.

- Σ_n^-–collection up to Σ_m–definable elements, $B(\Sigma_n^-, \mathcal{K}_m)$, is

$$\forall x\, \exists y\, \varphi(x, y) \to$$
$$\forall z\, (\delta(z) \land \exists! z\, \delta(z) \to \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y))$$

where $\varphi \in \Sigma_n$, $\delta \in \Sigma_m$.

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}^-$, $Th_{\Sigma_{n+1}}(B\Sigma_n) \equiv B(\Sigma_n^-, \mathcal{K}_n)$.

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem $(n \geq 1)$

Over $I\Sigma^-_{n-1}$, $Th_{\Sigma_{n+1}}(B\Sigma_n) \equiv B(\Sigma^-_n, \mathcal{K}_n)$.

Proof: $(\vdash)$:

- $B(\Sigma^-_n, \mathcal{K}_n)$ is contained in $B\Sigma_n$.
- $B(\Sigma^-_n, \mathcal{K}_n)$ is Σ_{n+1}–axiomatizable because...

$$\text{“}\forall z \in \mathcal{K}_n \, \varphi(z)\text{”}$$

$$\Updownarrow$$

$$\{\exists z \, [\forall t \, \neg\delta(t) \,\vee\, (\delta(z) \wedge \exists! z \, \delta(z) \,\wedge\, \varphi(z))] \,:\, \delta \in \Sigma_n\}$$

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Theorem $(n \geq 1)$

Over $I\Sigma_{n-1}^-$, $Th_{\Sigma_{n+1}}(B\Sigma_n) \equiv B(\Sigma_n^-, \mathcal{K}_n)$.

Proof: $(\dashv)$:

Assume $\mathfrak{A} \models B(\Sigma_n^-, \mathcal{K}_n)$.

Case 1: $\mathcal{I}_n(\mathfrak{A}) = \mathfrak{A}$. Then, $\mathfrak{A} \models B\Sigma_n^- \vdash Th_{\Sigma_{n+1}}(B\Sigma_n)$.

Case 2: $\mathcal{I}_n(\mathfrak{A}) \neq \mathfrak{A}$.

- $\mathcal{I}_n(\mathfrak{A}) \models B\Sigma_n^-$ (end–extension properties)
- $\mathcal{I}_n(\mathfrak{A}) \models Th_{\Pi_{n+1}}(\mathfrak{A})$, by $B(\Sigma_n^-, \mathcal{K}_n)$.

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(B\Sigma_n)$ $\square$

An axiomatization of $Th_{\Sigma_{n+1}}(B\Sigma_n)$

Proposition ($n \geq 1$)

Over $I\Sigma_{n-1}^-$, $I\Pi_n^- \equiv I(\Sigma_n^-, \mathcal{K}_n)$.

Proof: ($\dashv$): Assume $\varphi(x) \in \Sigma_n$ and a definable by $\delta(v) \in \Sigma_n$. Induction up to a for $\varphi(x)$ follows from induction for $\forall v\,(\delta(v) \rightarrow \neg\varphi(v - x))$. $\square$

Corollary ($n \geq 1$)

$B\Sigma_n$ is Σ_{n+1}–conservative over $I\Pi_n^-$.

Proof: ($\dashv$): $\varphi \in \Sigma_{n+1}$ and $B\Sigma_n \vdash \varphi$

$$\begin{aligned} &\Longrightarrow\ B(\Sigma_n^-, \mathcal{K}_n) \vdash \varphi \\ &\Longrightarrow\ I(\Sigma_n^-, \mathcal{K}_n) \vdash \varphi \\ &\Longrightarrow\ I\Pi_n^- \vdash \varphi \end{aligned}$$

$\square$

What about $Th_{\Sigma_{n+1}}(I\Sigma_n)$?

- **Natural candidate**: $I(\Sigma_n^-, \mathcal{K}_n)$.

- Does $I(\Sigma_n^-, \mathcal{K}_n)$ axiomatize $Th_{\Sigma_{n+1}}(I\Sigma_n)$? NO
  Because...
  - $I(\Sigma_n^-, \mathcal{K}_n) \equiv I\Pi_n^-$.
  - $I\Pi_n^-$ is strictly weaker than $Th_{\Sigma_{n+1}}(I\Sigma_n)$.

- **Question**: How can we extend $I(\Sigma_n^-, \mathcal{K}_n)$ to capture all the Σ_{n+1}–consequences of $I\Sigma_n$?

Iterating Σ_n–definability: $\mathcal{I}_n^\infty$

Definition

- $\mathcal{I}_n^0(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A})$
- For each k, $\mathcal{I}_n^{k+1}(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A}))$
- $\mathcal{I}_n^\infty(\mathfrak{A}) = \bigcup_{k \geq 0} \mathcal{I}_n^k(\mathfrak{A})$

$$\mathfrak{A} \quad [\text{———})_{\mathcal{I}_n^0} \text{———})_{\mathcal{I}_n^1} \text{———} \;)_{\mathcal{I}_n^2} \text{————————})_{\mathcal{I}_n^\infty} \text{————————})$$

Lemma

1. *If* $\mathfrak{A} \models I\Sigma_{n-1}$ *then* $\mathcal{I}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$.
2. *If* $\mathfrak{A} \models I\Sigma_n$ *with nonstandard* Σ_n*–definable elements,* $\{\mathcal{I}_n^k(\mathfrak{A}) \, : \, k \geq 0\}$ *form a proper hierarchy.*

Expressing "$\forall x \in \mathcal{I}_n^\infty$" in the language of Arithmetic

- Suppose $\mathfrak{A} \models I\Sigma_{n-1}$. For each $a \in \mathcal{K}_n(\mathfrak{A}, X)$ there is b Π_{n-1}–minimal (with parameters in X) such that $a = (b)_0$.

$$\text{“}\forall x \in \mathcal{I}_n^k\, \Phi(x, \bar{v})\text{”}$$

$$\Updownarrow$$

$$\forall \bar{a}, \bar{b}\, \left(\left\{ \begin{array}{lcl} a_0 = \mu x.\, \delta_0(x) & \wedge & b_0 \leq a_0 \\ a_1 = \mu x.\, \delta_1(x, b_0) & \wedge & b_1 \leq a_1 \\ & \vdots & \\ a_k = \mu x.\, \delta_k(x, b_{k-1}) & \wedge & b_k \leq a_k \end{array} \right\} \rightarrow \Phi(b_k, \bar{v}) \right)$$

where $\delta_0, \ldots, \delta_k$ run over Π_{n-1}.

An axiomatization of $Th_{\Sigma_{n+1}}(I\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}$ the following theories are equivalent:

1. $Th_{\Sigma_{n+1}}(I\Sigma_n)$
2. $I(\Sigma_n^-, \mathcal{I}_n^\infty)$
3. $I(\Sigma_n, \mathcal{I}_n^\infty, \mathcal{I}_n^\infty)$

An axiomatization of $Th_{\Sigma_{n+1}}(I\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}$ the following theories are equivalent:

1. $Th_{\Sigma_{n+1}}(I\Sigma_n)$
2. $I(\Sigma_n^-, \mathcal{I}_n^\infty)$
3. $I(\Sigma_n, \mathcal{I}_n^\infty, \mathcal{I}_n^\infty)$

Proof: (1$\Longrightarrow$2):

- For each k, $\mathcal{I}_n^k(\mathfrak{A})$ is not cofinal in $\mathfrak{A}$.
- So, $\exists y\, \theta(x, y)$ is equivalent to $\exists y \leq b\, \theta(x, y)$ for $x \leq a \in \mathcal{I}_n^k(\mathfrak{A})$.

(2$\Longrightarrow$3): It follows from a general property:

$$\left.\begin{array}{c} \mathfrak{A} \models I(\Sigma_n, \{a\}, \{b\}) \\ 2^{\langle b,b \rangle} \leq a \end{array}\right\} \Longrightarrow \mathfrak{A} \models I(\Sigma_n, (\leq a), (\leq b))$$

An axiomatization of $Th_{\Sigma_{n+1}}(I\Sigma_n)$

Theorem ($n \geq 1$)

Over $I\Sigma_{n-1}$ the following theories are equivalent:

1. $Th_{\Sigma_{n+1}}(I\Sigma_n)$
2. $I(\Sigma_n^-, \mathcal{I}_n^\infty)$
3. $I(\Sigma_n, \mathcal{I}_n^\infty, \mathcal{I}_n^\infty)$

Proof: (3$\Longrightarrow$1): Assume $\mathfrak{A} \models I(\Sigma_n, \mathcal{I}_n^\infty, \mathcal{I}_n^\infty)$.

Case 1: $\mathcal{I}_n^\infty(\mathfrak{A}) = \mathfrak{A}$. Then, $\mathfrak{A} \models I\Sigma_n$.

Case 2: $\mathcal{I}_n^\infty(\mathfrak{A}) \neq \mathfrak{A}$.

- $\mathcal{I}_n^\infty(\mathfrak{A}) \prec_n^e \mathfrak{A}$ proper.
- $\mathcal{I}_n^\infty(\mathfrak{A}) \models B\Sigma_{n+1} \vdash I\Sigma_n$ (end–extension properties)

So, $\mathfrak{A} \models Th_{\Sigma_{n+1}}(I\Sigma_n)$. □

Kaye–Paris–Dimitracopoulos' theories [JSL'88]

For each $k \geq 1$, $L\Sigma_n^{(k),-}$ denotes

$$\exists x_1, \dots, x_k \, \varphi(x_1, \dots, x_k)$$
$$\Downarrow$$
$$\exists x_1, \dots, x_k \left\{ \begin{array}{cc} x_1 = \mu t. \, \exists x_2, \dots, x_k \, \varphi(t, x_2, \dots, x_k) & \wedge \\ x_2 = \mu t. \, \exists x_3, \dots, x_k \, \varphi(x_1, t, \dots, x_k) & \wedge \\ \vdots & \\ x_k = \mu t. \, \varphi(x_1, x_2, \dots, t) & \end{array} \right\}$$

where $\varphi(x_1, \dots, x_k)$ runs over Σ_n.

- **Theorem:** $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv \bigcup_{k \geq 1} L\Sigma_n^{(k),-}$

Beklemishev–Visser's theories [APAL'05]

- The Σ_n^-–LIMR is given by:

$$\frac{\exists u \, \forall x > u \, (f(x+1) \leq f(x))}{\exists u \, \forall x > u \, (f(x) = f(u))},$$

where f runs over the Σ_n^-–functions provably total in $I\Sigma_{n-1}$.

- $[I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_0 \equiv I\Sigma_{n-1}$
$[I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_{k+1} = [[I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k, \Sigma_n^-\text{–LIMR}]$

- Theorem: $Th_{\Sigma_{n+1}}(I\Sigma_n) \equiv \bigcup_{k \geq 1} [I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$

The equivalence theorem

Theorem ($k \geq 0$)

Over $I\Sigma_{n-1}$, the following theories are equivalent:

1. $I(\Sigma_n^-, \mathcal{I}_n^k)$
2. $[I\Sigma_{n-1}, \Sigma_n^-\text{–}LIMR]_{k+1}$
3. $L\Sigma_n^{(k+1),-}$

- We prove a hierarchy theorem for induction "up to":

$$\mathcal{K}_n(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A})) \models I(\Sigma_n^-, \mathcal{I}_n^k) + \neg I(\Sigma_n^-, \mathcal{I}_n^{k+1})$$

- Kaye–Paris–Dimitracopoulos also obtained a hierarchy theorem for their theories but needed involved arguments.
- Beklemishev–Visser posed the question of characterizing the theories $[I\Sigma_{n-1}, \Sigma_n^-\text{–LIMR}]_k$ for $k > 1$ and left pending the corresponding hierarchy theorem.

Applications to Reflection Principles

- Local Reflection for T, $\mathrm{Rfn}_\Gamma(T)$, is the scheme:

$$\Box_T(\ulcorner\varphi\urcorner) \to \varphi,$$

for all sentences $\varphi \in \Gamma$.

- A number of results for local reflection are only known over EA^+ (superexponentiation) because of the use of Cut–elimination theorem. For example,
  - Over EA^+, $\mathrm{Rfn}_{\Sigma_2}(EA) \equiv I\Pi_1^-$.
  - $T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T)) + \cdots \equiv T + \Pi_1\text{–IR}$ for finite Π_2–extensions of EA^+

- However, the use of superexponentiation can be avoided.

Applications to Reflection Principles

- Cut–elimination: $EA^+ \vdash \forall x\,(\Box_{PC}(x) \rightarrow \Box^{cf}_{PC}(x))$

Proposition

If $\mathfrak{A} \models EA + I\Pi^-_1$ *then* $\mathcal{K}_1(\mathfrak{A}) \models EA^+$.

Proof:

$$\begin{aligned} \mathfrak{A} \models EA + I\Pi^-_1 &\implies \mathfrak{A} \models EA + I(\Sigma^-_1, \mathcal{K}_1) \\ &\implies \mathcal{K}_1(\mathfrak{A}) \models [EA, \Sigma_1\text{–IR}] \\ &\implies \mathcal{K}_1(\mathfrak{A}) \models EA^+ \end{aligned}$$

Corollary

$EA + I\Pi^-_1 \vdash \Box_{PC}(\ulcorner\varphi\urcorner) \rightarrow \Box^{cf}_{PC}(\ulcorner\varphi\urcorner)$

Proof:

$$\begin{aligned} \mathfrak{A} \models \Box_{PC}(\ulcorner\varphi\urcorner) &\implies \mathcal{K}_1(\mathfrak{A}) \models \Box_{PC}(\ulcorner\varphi\urcorner) \\ &\implies \mathcal{K}_1(\mathfrak{A}) \models \Box^{cf}_{PC}(\ulcorner\varphi\urcorner) \\ &\implies \mathfrak{A} \models \Box^{cf}_{PC}(\ulcorner\varphi\urcorner) \end{aligned}$$

Applications to Reflection Principles

Theorem

Over EA, $\mathrm{Rfn}_{\Sigma_2}(EA) \equiv I\Pi_1^-$.

Proof: Assume $\mathfrak{A} \models EA + I\Pi_1^-$ and $\varphi \in \Sigma_2$.

$$\begin{aligned}
\mathfrak{A} \models \Box_{EA}(\ulcorner\varphi\urcorner) &\Longrightarrow \mathcal{K}_1(\mathfrak{A}) \models \Box_{EA}(\ulcorner\varphi\urcorner) \\
&\Longrightarrow \mathcal{K}_1(\mathfrak{A}) \models \Box^{cf}_{EA}(\ulcorner\varphi\urcorner) \\
&\Longrightarrow \mathfrak{A} \models \Box^{cf}_{EA}(\ulcorner\varphi\urcorner) \\
&\Longrightarrow \mathfrak{A} \models \varphi
\end{aligned}$$

Corollary

1. *For* $n \geq 1$, $\mathrm{Rfn}_{\Sigma_n}(EA) \equiv \mathrm{Rfn}^{cf}_{\Sigma_n}(EA)$.
2. *For* $n \geq 2$, $\mathrm{Rfn}_{\Pi_n}(EA) \equiv \mathrm{Rfn}^{cf}_{\Pi_n}(EA)$.

Applications to Reflection Principles

Theorem

$T_\omega = T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T)) + \cdots \equiv T + \Pi_1$–IR
for finite Π_2–extensions of EA.

Proof: Assume $\mathfrak{A} \models T + \Pi_1$–IR and let $\mathfrak{A} \prec_0 \mathfrak{B}$ with $\mathfrak{B}$ an existentially closed model w.r.t. $T + \Pi_1$–IR.

$$
\begin{aligned}
\mathfrak{B} \models T + \Pi_1\text{–IR exist. closed} \quad &\Longrightarrow \quad \mathcal{K}_1(\mathfrak{B}) \models [T, \Sigma_1\text{–IR}] \\
&\Longrightarrow \quad \mathcal{K}_1(\mathfrak{B}) \models \mathrm{RFN}_{\Sigma_1}(T) \\
&\Longrightarrow \quad \mathcal{K}_1(\mathfrak{B}) \models T_\omega \\
&\Longrightarrow \quad \mathfrak{A} \models T_\omega
\end{aligned}
$$

Corollary

$EA + \mathrm{Con}(EA) + \mathrm{Con}(EA + \mathrm{Con}(EA)) + \ldots \equiv EA + \Pi_1$–IR.

Final Remarks

- We have presented applications of the model theory of fragments of arithmetic to Proof Theory.
- We think that the study of fragments "up to" could find other applications in this context, especially for local reflection.
  - (LC2011) We applied these ideas to prove that $\mathrm{Rfn}_{\Sigma_2}(EA)$ is not Π_2–conservative over $\mathrm{Rfn}_{\Sigma_1}(EA)$, solving a problem of L.D. Beklemishev.
  - (Work in progress) Fragments "up to" seem to provide us with a Kreisel–Lévy theorem for all the levels in the local reflection hierarchy, namely

$$\mathrm{Rfn}_{\Sigma_{n+1}}(EA) \equiv I(\Sigma_n^-, \mathcal{K}_1)$$

$$\mathrm{Rfn}_{\Pi_{n+1}}(EA) \equiv EA + (\Pi_{n+1}, \mathcal{K}_1)\text{–IR}$$