

# On the Worm Principle

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# Outline

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- I present some elementary observations relating the Beklemishev's Worm Principle to Kirby-Paris' Hydra Game and to Friedman's gap-embedding.
- Prompted by these observations Beklemishev proved much more interesting characterization results!

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# The Worm Principle

- Beklemishev extracted a finite **PA**-independence result from his provability-logic ordinal analysis, the Worm Principle.
- A worm is a function  $f : [0, n] \rightarrow \mathbf{N}$ .
- $next(w, m)$ : If  $f(n) = 0$  then  $next(w, m) = (f(0), \dots, f(n-1))$ . Else, let  $k = \max(i < n : f(i) < f(n))$ . Let  $r = (f(0), \dots, f(k))$  and  $s = (f(k+1), \dots, f(n-1), f(n) - 1)$ . Then  $next(w, m) = rs \dots s$  with  $m+1$  copies of  $s$ .
- Given  $w$ , let  $w_0 = w$  and  $w_{n+1} = next(w_n, n+1)$ .
- EWD is the following  $\Pi_2^0$  sentence in the language of **PA**:  
 $\forall w \exists n (w_n = \emptyset)$ .
- Beklemishev: EWD is equivalent (over **PRA**) to the 1-consistency of **PA**, and therefore unprovable in **PA**.

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# Hydra Game

- The Hydra Game is defined on finite rooted trees.
- Rule: See picture!
- Kirby and Paris: Termination of the Hydra Game is unprovable in Peano Arithmetic.
- The Hydra Game can be also presented as the iteration of fundamental sequences on ordinal notations:

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# WP reduces to HG

- The Worm Principle can be reduced to Kirby-Paris' Hydra Game.
- The reduction can be done using an explicit combinatorial map  $* : Hydras \rightarrow Worms$ .
- $red(T, n)$  = tree obtained from  $T$  by one step of the Hydra Game starting at stage  $n$  using the strategy  $R$ : *always cut the rightmost head*.
- Then:  $\forall n (red(T, n)^* \sqsubset next(T^*, n))$ .
- $T^*$  rewrites to  $red(T, n)^*$  for all  $n$  by the rules of WP.
- Termination of WP implies termination of HG. Hence **PA**  $\not\equiv$  **EWD**.
- Other approaches (giving more information) based on ordinal notations were given by Lee and Weiermann.
- In fact, the WP turned out to be an inessential variant of a game introduced by Hamano and Okada.

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# Gap-embedding

- Friedman's gap-embedding: Let  $S_{n+1}$  be the set of all finite sequences of natural numbers smaller than  $n + 1$ . If  $s_1 = (a_0, \dots, a_k)$ ,  $s_2 = (b_0, \dots, b_m)$  are in  $S_{n+1}$  then a strictly monotone function  $f : \{0, \dots, k\} \rightarrow \{0, \dots, m\}$  is a gap-embedding of  $s_1$  into  $s_2$  if the following holds:
  - ①  $a_i = b_{f(i)}$  for all  $i < k$ , and
  - ② if  $f(i) < j < f(i + 1)$  then  $b_j \geq f(i + 1)$  for all  $i < k$  and  $j < m$ .
- We say that  $s_1$  is gap-embeddable in  $s_2$  and write  $s_1 \leq_{ge} s_2$ .
- Friedman-Simpson: For each  $n$ ,  $\leq_{ge}$  is a wqo of  $S_{n+1}$ .
- The just above wqo theorem is unprovable in  $\mathbf{ACA}_0$ .
- Finite **PA**-independence results follow: long normed finite sequences are wqo under  $\leq_{ge}$ .

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- We say that  $s_1$  is gap-embeddable in  $s_2$  and write  $s_1 \leq_{ge} s_2$ .
- Friedman-Simpson: For each  $n$ ,  $\leq_{ge}$  is a wqo of  $S_{n+1}$ .
- The just above wqo theorem is unprovable in **ACA**<sub>0</sub>.
- Finite **PA**-independence results follow: long normed finite sequences are wqo under  $\leq_{ge}$ .

# Worms and gaps

- Sequences of worms are bad with respect to Friedman's gap-embeddability relation.
- The following holds (elementary comb. proof).

$$(S_1 \hookrightarrow_W S_2) \Rightarrow (S_1 \not\leq_{ge} S_2).$$

- The following holds.

$$\forall W (m > n \Rightarrow W_n \not\leq_{ge} W_m).$$

- Long sequences of worms give long bad sequences wrt  $\leq_{ge}$ . This implies *slowly well-orderedness* **PA**-independence results of Simpson and Schütte.

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# Worms and gaps, Beklemishev's results

- (Strong) gap-embeddability is *characterized* by reverse provable implication in  $GLP^-$ .
- (Strong gap-embeddability is just gap-embeddability with the extra condition that  $\forall j < f(1)(b_j \geq b_{f(1)})$ ).
- Characterization:

$$w \leq_{sge} w' \Leftrightarrow GLP^- \vdash w' \rightarrow w.$$

- For every  $n$ , the extension of  $GLP^-$  by  $n$  closed modalities corresponding to negations of worms is finitely axiomatizable. (From wqo of  $\leq_{sge}$  we get a finite base).
- The just above theorem is unprovable in  $\mathbf{ACA}_0$ . (It implies that  $\leq_{sge}$  is wqo).



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