

The strength of Ramsey Theorem for coloring relatively large sets

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April 2012
Workshop on Proof Theory and Modal Logic
University of Barcelona

Outline

- 1 Motivation and Background
 - Ramsey Theorem for large sets
 - $\mathbf{RT}(!\omega)$ and Ramsey's Theorem
 - Known facts about Ramsey Theorems
- 2 Lower Bounds on $\mathbf{RT}(!\omega)$
 - Weak Lower Bound on $\mathbf{RT}(!\omega)$
 - Strong Lower Bounds on $\mathbf{RT}(!\omega)$
- 3 Upper Bounds on $\mathbf{RT}(!\omega)$
- 4 Conclusion and prospects

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Overview

We analyze the effective and proof-theoretic strength of an infinitary Ramsey-type theorem due to Pudlák and Rödli and independently to Farmaki.

The main result is that the theorem is equivalent over Computable Mathematics (**RCA**₀) to closure under the ω -th Turing jump.

Large Sets

- A set $S = \{s_0, s_1, \dots, s_n\}$ is
 - ▶ *large* if $n \geq s_0$, and is
 - ▶ *exactly large* if $n = s_0$.
- The notion of *large set* was introduced by Paris and Harrington and is the key notion in the Paris-Harrington Large Ramsey Theorem.
- We denote by $[X]^{! \omega}$ the set of exactly large subsets of X .

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Coloring Large Sets

The theorem we study is the following.

Theorem

For every infinite $M \subseteq \mathbf{N}$, for every coloring C of the exactly large subsets of \mathbf{N} in two colors there exists an infinite homogeneous set $H \subseteq M$.

Alternative formulations:

- 1 Pudlák-Rödl: uniform families.
- 2 Farmaki: (thin) Schreier families (Banach Space Theory).
Schreier family:

$$\{s = \{n_1, \dots, n_k\} : n_1 \geq k\}$$

Thin Schreier family: $n_1 = k$.

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A Combinatorial Proof of $\mathbf{RT}(!\omega)$

Let M be an infinite subset of \mathbf{N} , let $C : [\mathbf{N}]^{! \omega} \rightarrow 2$. We build an infinite homogeneous subset $L \subseteq M$ for C in stages. Let

$C_a : [\mathbf{N} \setminus \{1, \dots, a\}]^a \rightarrow 2$ be defined by

$$C_a(x_1, \dots, x_a) = C(a, x_1, \dots, x_a).$$

We define a sequence $\{(a_i, X_i)\}_{i \in \mathbf{N}}$ such that

- $a_0 = \min(M)$,
- $X_{i+1} \subseteq X_i \subseteq M$,
- X_i is an infinite and C_{a_i} -homogeneous and $a_i < \min(X_i)$,
- $a_{i+1} = \min X_i$.

At the i -th step of the construction use Ramsey's Theorem for coloring a_i -tuples from the infinite set X_{i-1} (where $X_{-1} = M$).

Finally apply the Infinite Pigeonhole Principle to the sequence $\{a_i\}_{i \in \mathbf{N}}$ to get an infinite C -homogeneous set.

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$\text{RT}(!\omega)$ and $\forall n \text{RT}_2^n$

Proposition

Over RCA_0 , $\text{RT}(!\omega)$ implies $\forall n \text{RT}_2^n$.

Let $n \geq 1$ and $C : [\mathbf{N}]^n \rightarrow 2$ be given. Define $C' : [\mathbf{N}]^{! \omega} \rightarrow 2$ as follows. Let $S = \{s_0, \dots, s_m\}$ be an exactly large set (then $m = s_0$).

$$C'(s) := \begin{cases} C(s_0, \dots, s_{n-1}) & \text{if } s_0 \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Let H be an infinite C' -homogeneous set of colour $i \in \{0, 1\}$.
- 2 Let $H' = H \cap [n, \infty)$, and let $S \in [H']^n$. Then $\min(S) \geq n$.
- 3 Let S' be any exactly large set extending S in H' .
- 4 Then $C(S) = C'(S') = i$. Thus H' is C -homogeneous of color i .

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Computability results

Theorem (Jockusch 1972)

- 1 For each $n \geq 2$ there exists a computable coloring $C : [\mathbf{N}]^n \rightarrow 2$ with no infinite homogeneous set in Σ_n^0 .
- 2 For each n , for each computable coloring $C : [\mathbf{N}]^n \rightarrow 2$, there exists an infinite C -homogeneous set in Π_n^0 .
- 3 For each $n \geq 2$ there exists a computable coloring $C : [\mathbf{N}]^n \rightarrow 2$ all of whose homogeneous sets compute $0^{(n-2)}$.

Reverse Mathematics results

Theorem (Simpson)

The following are equivalent over \mathbf{RCA}_0 .

- 1 \mathbf{RT}^3 ,
- 2 \mathbf{RT}^n for any $n \in \mathbf{N}$, $n \geq 3$,
- 3 $\forall X \exists Y (Y = X')$.

Thus $\mathbf{RCA}_0 + \mathbf{RT}^3 \equiv \mathbf{ACA}_0$.

Theorem (McAlloon 1985)

The following are equivalent over \mathbf{RCA}_0 .

- 1 $\forall n \mathbf{RT}^n$,
- 2 $\forall n \forall X \exists Y (Y = X^{(n)})$.

Thus $\mathbf{RCA}_0 + \forall n \mathbf{RT}^n \equiv \mathbf{ACA}'_0$.

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A weak Lower Bound

Theorem

There exists a computable coloring $C: [\mathbf{N}]^{\omega} \rightarrow 2$ such that any infinite homogeneous set for C is not Σ_i^0 , for any $i \in \mathbf{N}$.

Recall that there exists sets that are incomparable with all $0^{(i)}$, $i \geq 1$.

Proposition

There exists a computable sequence of X -computable functions $e_n^X: [\mathbf{N}]^{n+2} \rightarrow \{0, 1\}$ such that for any $n \geq 0$, for every $i \in \mathbf{N}$,

- 1 $e_n^{K_i}$ is K_i -computable, and*
- 2 $e_n^{K_i}$ computes a coloring with no homogeneous set in Σ_{i+n+2}^0 .*

Proof of Theorem from Proposition

Define $C : [\mathbf{N}]^{\omega} \rightarrow 2$ as follows.

$$C(s_0, s_1, \dots, s_{s_0}) = e_{s_0-2}^{K_0}(s_1, \dots, s_{s_0}).$$

If Y is ihs for C then for every $a \in Y$, $Y \cap \{x \in \mathbf{N} : x \geq a\}$ is ihs for $e_{a-2}^{K_0}$. Then it is not Σ_a^0 by Proposition.

Proof of Proposition

Ingredients:

- 1 Jockusch's result: There exists a computable $e : [\mathbf{N}]^2 \rightarrow 2$ without ihs in Σ_2^0 .
- 2 Relativizes to: There exists an X -computable $e^X : [\mathbf{N}]^2 \rightarrow 2$ without ihs in Σ_{i+2}^0 if X is Σ_i^0 -complete.
- 3 Schoenfield's Limit Lemma: functions computable in K are lim-computable.
- 4 Generalizes to: If B is c.e. in A and f is computable in B then f is A -lim-computable.
- 5 Uniform formulation: there exists a function $g^X(i, e, x, s)$ such that, if $B = W_i^A$ and $f = \{e\}^B$ then

$$f(x) = \lim_{s \rightarrow \infty} g^A(i, e, x, s)$$

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Proof of Proposition (continued)

- 1 Fix $\{K_i\}_{i \in \mathbf{N}}$ s.t. $K_0 = \emptyset$, K_{i+1} is Σ_{i+1}^0 -complete.
- 2 Base: result of Jockusch's, relativized.
- 3 Step: define $e_{n+1}^X : [N]^{n+3} \rightarrow \{0, 1\}$. Ensure
 - 1 If $X = K_i$ then: homset for $e_{n+1}^{K_i} \Rightarrow$ homset for $e_n^{K_{i+1}}$.
 - 2 Index for e_{n+1}^X obtained effectively from index for e_n^X .
- 4 By Shoenfield's Lemma we have that

$$\lim_{s \rightarrow \infty} g^{K_i}(\bar{e}_n, x_1, \dots, x_{n+2}, s) = e_n^{K_{i+1}}(x_1, \dots, x_{n+2}).$$

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$$\lim_{s \rightarrow \infty} g^{K_i}(\bar{e}_n, x_1, \dots, x_{n+2}, s) = e_n^{K_{i+1}}(x_1, \dots, x_{n+2}).$$

Define e_{n+1}^X as follows.

$$e_{n+1}^X(x_1, \dots, x_{n+2}, s) := g^X(\overline{e_n}, x_1, \dots, x_{n+2}, s).$$

If Y is an infinite homset for $e_{n+1}^{K_i}$ colored $b \in \{0, 1\}$, then any tuple $(x_1, \dots, x_{n+2}) \in [Y]^{n+2}$ has to be colored b by $e_n^{K_{i+1}}$

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Strong Lower Bounds

Theorem

For each set A there exists a computable in A coloring $C_\omega : [\mathbf{N}]^{!_\omega} \rightarrow 2$ such that all infinite homogeneous sets for C_ω compute $A^{(\omega)}$.

Proposition

There exists computable in A colorings $C_n : [\mathbf{N}]^{n+1} \rightarrow \{0, 1\}$, for $n \in \mathbf{N}$ and $n \geq 2$, and TMs $M_n(x, y)$ s.t., for any $n \geq 2$, the following three points hold.

- 1 All infinite homogeneous sets for C_n have color 1.*
- 2 If X is an ihs for C_n then for any $a_1 < \dots < a_{n+1} \in X$ then $M_n(x, (a_1, \dots, a_{n+1}))$ decides $A^{(n-1)}$ for machines with indices less than or equal to a_1 .*
- 3 Machines M_n are total. (If their inputs are not from an ihs for C_n then we have no guarantee on the correctness of their output).*

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Proposition

There exists computable in A colorings $C_n : [\mathbf{N}]^{n+1} \rightarrow \{0, 1\}$, for $n \in \mathbf{N}$ and $n \geq 2$, and TMs $M_n(x, y)$ s.t., for any $n \geq 2$, the following three points hold.

- 1 All infinite homogeneous sets for C_n have color 1.
- 2 If X is an ihs for C_n then for any $a_1 < \dots < a_{n+1} \in X$ then $M_n(x, (a_1, \dots, a_{n+1}))$ decides $A^{(n-1)}$ for machines with indices less than or equal to a_1 .
- 3 Machines M_n are total. (If their inputs are not from an ihs for C_n then we have no guarantee on the correctness of their output).

Proof of Theorem from Proposition

Define

$$C_\omega(a_1, \dots, a_{a_1+1}) = C_{a_1-1}(a_2, \dots, a_{a_1+1}).$$

Claim: Any ihs H for C_ω computes $A^\omega = \{(i, j) : j \in A^i\}$.

- For any $h \in H$, C_ω coincides with C_{h-1} on large sets with minimum h and rest in $H^- = H - \{1, \dots, h\}$.
- Thus, $M_{h-1}(e, \sigma)$ decides $A^{(h-2)}$ up to $\min(\sigma)$ if σ is picked in H^- .
- Given (i, j) find $a_1 \in H$ such that $i, j \leq (a_1 - 2)$.
- $(j \in A^i)$ iff $f_{i, (a_1-2)}(j) \in A^{(a_1-2)}$ ($f_{i, (a_1-2)}$ reducing A^i to A^{a_1-2}).
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Proof of Proposition

For this proof let

$$X' = \{e : \{e\}^X(0) \downarrow\}.$$

We define C_2 as

$$C_2(k, y, z) = \begin{cases} 1 & \text{if } \forall e \leq k (\{e\}_y^A(0) \downarrow \Leftrightarrow \{e\}_z^A(0) \downarrow) \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Any infinite homset has color 1.
- 2 We can build machine $M_2(e, (k, b, b'))$ deciding A' up to k if $b' > b > k$ are from an infinite homset for C_2 .

Proof of Proposition, continued

We set $C_{n+1}(a_1, \dots, a_{n+2}) =$

$$\begin{cases} 1 & \text{if } \{a_1, \dots, a_{n+2}\} \text{ is } C_n\text{-homogeneous and} \\ & \forall e \leq a_1 (\{e\}_{a_2}^Y(0) \downarrow \Leftrightarrow \{e\}_{a_3}^Y(0) \downarrow), \text{ where} \\ & Y = \{i \leq a_2 : M_n(i, (a_2, \dots, a_{n+2})) \text{ accepts,}\} \\ 0 & \text{otherwise.} \end{cases}$$

We are approximating the condition

$$\forall e \leq a_1 (\{e\}_{a_2}^{A^{(n-1)}}(0) \downarrow \Leftrightarrow \{e\}_{a_3}^{A^{(n-1)}}(0) \downarrow)$$

Using homogeneity and infinity we can get

$$\forall e \leq a_1 (\{e\}_{a_2}^Y(0) \downarrow \Leftrightarrow \{e\}^{A^{(n-1)}}(0) \downarrow),$$

for suitable a_1, a_2 .

Reverse Mathematics Corollary

Theorem

RT(ω) *implies* $\forall X \exists Y (Y = X^{(\omega)})$ over **RCA**₀.

Alternative Proof of Theorem

We use the following Proposition, which can be proved by adapting an argument from a recent paper by Dzhafarov and Hirst (2011).

Proposition

For every set X there exists a computable in X coloring $C^X : [\mathbf{N}]^{\omega} \rightarrow 2$ such that if $H \subseteq 2\mathbf{N}$ is an infinite homogeneous set for C then H computes $X^{(n-1)}$ for every $2n \in H$.

Proof of Theorem from Proposition

- Recall $\{0^i : i \in \mathbf{N}\}$ has no lub.
- Enderton-Putnam result: If I is infinite such that $(\forall i \in I)(X^i \leq_T Y)$, then $X^\omega \leq_T Y^2$.
- Prove Theorem as follows.
 - 1 X computable, C^X as in Proposition, H ihs for C^X .
 - 2 Then for infinitely many $i \in \mathbf{N}$, $X^i \leq_T H$.
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Proof of Proposition

$$Y = X' \Leftrightarrow \forall x \forall e (\langle x, e \rangle \in Y \Leftrightarrow \exists s (\{e\}_s^X(x) \downarrow)).$$

For any set X and integer s define

$$X'_s = \{\langle m, e \rangle : (\exists t < s) m \in W_{e,t}^X\}.$$

For integers u_1, \dots, u_n and s define $X^{(0)} = X$, and

$$X_{u_n, \dots, u_1, s}^{(n+1)} = (X_{u_n, \dots, u_1}^{(n)})'_s.$$

Let $A = \{a_0, \dots, a_{2n+1}\}$ be exactly large. Let $C^X(A) = 1$ if there exist $1 \leq i \leq n$ and $\exists (e, m) < a_{n-i}$ such that

$$\neg((m, e) \in X_{a_n, \dots, a_{n-i+1}}^{(i)} \Leftrightarrow (m, e) \in X_{a_{2n}, \dots, a_{2n-i+1}}^{(i)})$$

and $C^X(A) = 0$ otherwise.

Let $H \subseteq [2, \infty) \cap 2\mathbf{N}$ be an ihs for C^X . We claim that

- 1 The color of C^X on $[H]^{\omega}$ is 0.
- 2 For every $h \in H$, $X^{(n-1)}$ is definable by recursive comprehension from H , where $h = 2n$.

A proof in \mathbf{ACA}_0^+

Theorem

$\forall X \exists Y (Y = X^{(\omega)})$ implies $\mathbf{RT}(!\omega)$ over \mathbf{RCA}_0 .

Idea of proof:

- Turn the induction in the combinatorial proof of $\mathbf{RT}(!\omega)$ into a first-order induction.
- Replace the sets X_i by Turing machines with oracles from $C^{(a)}$, (for a an element of a model of \mathbf{RCA}_0) constructed in a uniform way and computing the sets X_i .

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We want the following first-order induction: for each n there exists a sequence $\{(a_i, f_{a_i}) : i \leq n\}$ such that:

- $a_0 = 2$,
- $rg(f_{a_{i+1}}) \subseteq rg(f_{a_i}) \subseteq M$,
- $rg(f_{a_i})$ is infinite and C_{a_i} -homogeneous,
- $a_{i+1} = \min(rg(f_{a_i}) \cap \{x \in \mathbf{N} : x > a_i\})$.

Lemma

Let $a \geq 1$. Let $C: [U]^a \rightarrow 2$. One can find effectively a machine f_a with oracle $(C \oplus U)^{(2a)}$ such that f_a computes a C -homogeneous set.

- 1 Base: f_1 . Ask the $\Pi_2^0(C \oplus U)$ oracle whether $\forall n \exists k \geq n (C(k) = 0 \wedge U(k))$. If yes, then compute the set $C(x) = 0 \wedge U(x)$, else compute the set $C(x) = 1 \wedge U(x)$.
- 2 Step: f_{a+1} .
 - 1 Build Erdős–Rado tree T_a for $C: [U]^{a+1} \rightarrow 2$ (computably in $C \oplus U$).
 - 2 Obtain index for a machine computing the leftmost infinite path P of T_a using a $\Pi_2^0(C \oplus U)$ -complete oracle.
 - 3 The color of any $(a+1)$ -tuple from P does not depend on the last element of the tuple. Induces C' coloring of a -tuples.
 - 4 Use f_a (inductive hypothesis) with oracle $(C' \oplus P)^{(2a)}$. Any infinite C' -homogeneous subset of P computed by f_a is also C -homogeneous.
 - 5 Since P is recursive in $\Pi_2^0(C \oplus U)$, the complexity of the oracle is $(C \oplus U)^{(2(a+1))}$.

Conclusion and prospects

1 Other results:

- 1 Analogous results for Regressive Ramsey Theorem on large sets.
- 2 Characterization in terms of ω truth-predicates over **PA**.
- 3 Alternative proof by reduction to well-ordering preservation principles.

2 Further research:

- 1 Prove generalization for **RT**($!\alpha$) for $\alpha \in \omega^{\text{CK}}$.
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