Provability algebras:

an overview and current progress

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April 16, 2012

Current progress

- Modal study of GLP and related systems
- Topological and set-theoretic semantics
- Proof-theoretic applications

Modal study of GLP

- A sound and complete class of Kripke-models (B.,07)
- PSpace-complexity of the decision problem (Shapirovsky, 08)
- PSpace-completeness of the decision problem for the closed fragment (Pakhomov, 12)
- Closed fragment: explicit presentation of the canonical model (Icard, 07)
- Positive $\{\wedge, \Diamond\}$-fragment: axiomatization, models, polynomial complexity (Dashkov, 11)
- Uniform interpolation (Shamkanov, 11)

Topological interpretation

- Ordinal-based topological models for the closed fragment (Icard, 07)
- GLP-spaces. Topological completeness of GLP_2 (B. & Bezhanishvili & Icard, 08)
- GLP and large cardinal axioms. Ordinal completeness of GLP_2 within ZFC+V=L (B., 10)
- Topological completeness of GLP within ZFC (B. & Gabelaia, 11)
- Bagaria's result on the ordinal completeness of GLP (?).

Proof-theoretic applications

- Transfinitely many modalities, GLP_Λ
- Closed fragment of GLP_Λ: normal forms, ordinal notation systems, Kripke models (B. 05, Fernandez & Joosten 11/12)
- Positive logic with limit modalities. Analysis of the theories of Tarskian truthpredicates. (B. & Dashkov, 12)

Provability algebraic view

- We view consistency assertion (along with higher reflection principles) as a function

$$\varphi \longmapsto \mathsf{Con}(S + \varphi)$$

acting on a suitable algebra of sentences. (In principle, on the whole Lindenbaum–Tarski algebra of S.)

- Minimal substructures closed under this map (and some other operations) provide suitable ordinal notations.
- Using these notations we classify consequences of theories of a specific logical complexity such as Π^0_1 or Π^0_2.

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- Using these notations we classify consequences of theories of a specific logical complexity such as Π^0_1 or Π^0_2.

Reflection principles

Notation:

$\Box_S(\varphi)$ ‘φ *is provable in* S’

$Tr_n(\sigma)$ ‘σ *is the Gödel number of a true* Σ_n*-sentence*’

Reflection principles:

$R_0(S)$ $\mathrm{Con}(S)$

$R_n(S)$ $\forall\sigma \in \Sigma_n\, (\Box_S\sigma \to Tr_n(\sigma))$, for $n \geq 1$.

$R_n(S) \iff \mathrm{Con}(S + \text{all true } \Pi_n\text{-sentences})$

Provability algebra of S

Let $\mathcal{L}_S$ be the Lindenbaum–Tarski boolean algebra of S sentences.

- Each R_n correctly defines an operator on the equivalence classes of $\mathcal{L}_S$: $\langle n \rangle : [\varphi] \longmapsto [R_n(S + \varphi)]$.
- The algebra $(\mathcal{L}_S, \langle 0 \rangle, \langle 1 \rangle, \dots)$ is the *provability algebra of* S.

This is a very big and complicated algebra, already with one modality. To study its structure is a serious task comparable to the study of degrees of computability. See the papers by Shavrukov for the case of one modality. But . . .

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We are not doing it here.

Instead, this structure allows us to easily define two things:

- An ordinal notation system up to ε_0;
- To state a key *reduction property*.

All the standard results on the proof-theoretic analysis of PA (and a bit more) follow from these two facts.

Provability logic GLP

- To calculate the terms of an algebra we need to know its identities. [1]
- Logic GLP axiomatizes the set of identities of $\mathcal{L}_S$ by a theorem of Japaridze.
- Recursiveness of the obtained ordinal notation system is a consequence of the decidability of (the positive closed fragment of) GLP. By Dashkov it is, in fact, in P.

[1] Strictly speaking, we do not need to know them all. But it is good to be sure we do not miss any substantial relation.

GLP, equational formulation

1. Boolean identities;
2. $\langle n\rangle 0 = 0$; $\langle n\rangle(x \vee y) = \langle n\rangle x \vee \langle n\rangle y$;
3. $\langle n\rangle x = \langle n\rangle(x \wedge \neg\langle n\rangle x)$ (Löb's identity)
4. $\langle n\rangle x \leq \langle m\rangle x$ if $n > m$;
5. $\langle m\rangle x \leq [n]\langle m\rangle x$ if $n > m$.

Here $x \leq y$ means $x \wedge y = y$, and $[n]x := \neg\langle n\rangle\neg x$.

All principles are easily seen to be valid in $\mathcal{L}_S$.

Rem. Modulo the rest, identity 5 is equivalent to

- $\langle n\rangle x \wedge \langle m\rangle y = \langle n\rangle(x \wedge \langle m\rangle y)$, for $n > m$.

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GLP: a Hilbert-style calculus

Basic symbols are now $[n]$, for each $n \in \omega$, and $\langle n \rangle$ is treated as an abbreviation: $\langle n \rangle \varphi = \neg [n] \neg \varphi$.

1. Tautologies;
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$;
4. $[n]\varphi \to [n+1]\varphi$;
5. $\langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$

Rules: modus ponens, $\varphi \vdash [n]\varphi$.

Th. (Japaridze) $\mathsf{GLP} \vdash \varphi(\vec{x})$ iff $\mathsf{GLP} \vDash \varphi(\vec{x}) = 1$ iff $\mathcal{L}_S \vDash \forall \vec{x} \, (\varphi(\vec{x}) = 1)$ (provided S is sound).

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Positive closed fragment as an ordinal notation system

Let W denote the set of all GLP terms generated from 1 by $\wedge$ and $\langle n \rangle$, for all $n \in \omega$. For $\alpha, \beta \in W$ define:

- $\alpha \sim \beta$ if $\mathsf{GLP} \vdash (\alpha \leftrightarrow \beta)$;
- $\alpha <_n \beta$ if $\mathsf{GLP} \vdash \beta \to \langle n \rangle \alpha$.

Theorem.

1. Every $\alpha \in W$ is equivalent to a *word* (formula without $\wedge$);
2. $(W/{\sim}, <_0)$ is isomorphic to $(\varepsilon_0, <)$.

Ordinal calculation

The isomorphism $o : W \to \varepsilon_0$ is calculated as follows.

- $o(0^k) = k$.
- Otherwise, if $\alpha = \alpha_1 0 \alpha_2 0 \cdots 0 \alpha_n$, then

$$o(\alpha) = \omega^{o(\alpha_n^-)} + \cdots + \omega^{o(\alpha_1^-)},$$

where $(132)^- = 021$.

Thus, $o(0\alpha) = o(\alpha) + 1 \in Suc$ and $o(\langle n+1 \rangle \alpha) \in Lim$.

Ex. $o(1012) = \omega^{o(01)} + \omega^{o(0)} = \omega^{\omega^1 + \omega^0} + \omega = \omega^{\omega+1} + \omega$

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Reduction property

$R_n^1(U) = R_n(U), \quad R_n^{k+1}(U) = R_n(U + R_n^k(U))$

Suppose $S \subseteq \Pi_{n+2}$ and $U \vdash S$.
Th. $R_{n+1}(U) \equiv_n \{R_n^k(U) : k < \omega\}$ modulo S,
where $\equiv_n$ denotes conservativity for Π_{n+1}-formulas.

Example. Modulo elementary arithmetic EA:

$I\Sigma_1 \equiv R_2(\mathrm{EA}) \equiv_1 \{R_1^k(\mathrm{EA}) : k < \omega\} \equiv \mathrm{PRA}$ (Parsons–Mints).

Fundamental sequences

Reduction lemma provides canonical fundamental sequences.

Suppose $\alpha = \langle n+1 \rangle \beta \in W$, so $o(\alpha) \in Lim$.
Define $\alpha[\![0]\!] := \langle n \rangle \beta$, $\alpha[\![k+1]\!] := \langle n \rangle (\beta \wedge \alpha[\![k]\!])$.

Fact. $\alpha[\![0]\!] <_0 \alpha[\![1]\!] <_0 \alpha[\![2]\!] \cdots \to \alpha$.

Let α_S denote the value of α in $\mathcal{L}_S$ and let $U := S + \beta_S$.

Cor. $\alpha_S \equiv_n \{\alpha[\![k]\!]_S : k < \omega\}$, for any $\alpha \in W$.

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Consistency proof for PA

Th. Transfinite induction over $(W, <_0)$ proves $\mathsf{Con}(\mathsf{PA})$.

Work in $S = \mathsf{EA}$, $\Diamond$ means Con_S. We prove $\forall \alpha \, \Diamond \alpha_S$. Claim:

$$\mathsf{PRA} \vdash \forall \beta <_0 \alpha \, \Diamond \beta_S \to \Diamond \alpha_S.$$

Assume $\forall \beta <_0 \alpha \, \Diamond \beta_S$.
If $\alpha = 0\beta$, then $\Diamond \beta_S$, hence $\Diamond \Diamond \beta_S$ since $\mathsf{PRA} \vdash R_1(S)$.
If $\alpha = \langle n+1 \rangle \beta$, then $\forall k \, \Diamond \alpha[\![k]\!]_S$, because $\alpha[\![k]\!] <_0 \alpha$.
By Reduction (provably in PRA):

$$\alpha_S \equiv_n \{\alpha[\![k]\!]_S : k < \omega\}.$$

Therefore $\forall k \, \Diamond \alpha[\![k]\!]_S$ yields $\Diamond \alpha_S$.

Iterated reflection and analysis of PA

W_n is the set of words in the alphabet $\{k \in \omega : k \geq n\}$.

Let $S^n_\alpha \equiv S + \{R_n(S^n_\beta) : \beta <_n \alpha\}$ over $(W_n, <_n)$.

Let S be a Π_{n+1} extension of PRA.
Theorem. For any $\alpha \in W_n$, $S + \alpha_S \equiv_n S^n_\alpha$.

Cor. $\mathsf{PA} \equiv_n \mathsf{PRA}^n_{\varepsilon_0}$ (U. Schmerl)

1. For $n = 0$: Consistency proof for PA (Gentzen);
2. For $n = 1$: Characterizing provably recursive functions of PA (Schwichtenberg–Wainer).

Set-theoretic interpretation

Are there any other natural examples of GLP-algebras?

Let X be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of X.

Consider any operators $\delta_n : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta_0, \delta_1, \dots)$.

Question: Can $(\mathcal{P}(X), \delta_0, \delta_1, \dots)$ be a GLP-algebra and, if yes, when?

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Derived set operators

Let X be a topological space, $A \subseteq X$.
Derived set $d(A)$ of A is the set of limit points of A:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \; y \in U_x \cap A.$$

Fact. If $(X, \delta) \vDash GL$ then X bears a unique scattered topology τ for which $\delta = d_\tau$, that is, $\delta : A \longmapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, A is τ-closed iff $\delta(A) \subseteq A$.
Equivalently, $c(A) = A \cup \delta(A)$ is the closure of A.

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Scattered spaces

Definition (Cantor): X is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixon sequence:

$$X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha, \text{ if } \lambda \text{ is limit.}$$

Notice that all X_α are closed and $X_0 \supset X_1 \supset X_2 \supset \ldots$

Fact (Cantor): X is scattered $\iff \exists \alpha : X_\alpha = \varnothing$.

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Examples

- Left topology $\tau_{\prec}$ on a strict partial ordering $(X, \prec)$.
  $A \subseteq X$ is open iff $\forall x, y\ (y \prec x \in A \Rightarrow y \in A)$.

  Fact: $(X, \prec)$ is well-founded iff $(X, \tau_{\prec})$ is scattered.

- Ordinal Ω with the usual order topology generated by intervals (α, β), $[0, \beta)$, (α, Ω) such that $\alpha < \beta$.

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Löb's identity = scatteredness

Simmons 74, Esakia 81

Löb's identity: $\Diamond A = \Diamond(A \wedge \neg\Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\mathrm{iso}(A)),$$

where $\mathrm{iso}(A) = A \setminus d(A)$ is the set of isolated points of A.

Fact: The following are equivalent:

- X is scattered;
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Completeness theorems

Theorem (Esakia 81): There is a scattered X such that $Log(X, d) = \mathsf{GL}$. In fact, X is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^\omega$ with the order topology. Then $Log(\Omega, d) = \mathsf{GL}$.

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Topological models for GLP

We consider poly-topological spaces $(X; \tau_0, \tau_1, \dots)$ where modality $\langle n \rangle$ corresponds to the derived set operator d_n w.r.t. τ_n.

Definition: X is a GLP-space if

- τ_0 is scattered;
- For each $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open;
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Generated GLP-space

Any scattered space *generates* an associated GLP-space in a natural way.

Let (X, τ) be a scattered space and let τ^+ denote the topology generated (as a subbase) by τ and $\{d_\tau(A) : A \subseteq X\}$.

Thus, any (X, τ) generates a GLP-space $(X; \tau_0, \tau_1, \dots)$ with $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^+$, for each n.

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Topological completeness

GLP is complete w.r.t. (countable, hausdorff) GLP-spaces.

Th. (B., Gabelaia 10): There is a countable hausdorff GLP-space X whose logic is GLP.

In fact, X is ε_0 equipped with topologies refining the order topology.

Open question: Is GLP complete w.r.t. some naturally generated GLP-space?

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Ordinal GLP-spaces

Let τ_0 be the left topology on an ordinal Ω. It generates a GLP-space $(\Omega; \tau_0, \tau_1, \dots)$. What are these topologies?

Fact: τ_1 is the order topology on Ω.

Club filter topology

Def. Let α be a limit ordinal.

- $C \subseteq \alpha$ is a club in α if C is τ_1-closed and unbounded below α.
- The filter generated by clubs in α is called the club filter. It is improper iff α has countable cofinality.

Fact. τ_2 is the club filter topology:

- τ_2-isolated points are ordinals of countable cofinality;
- if $cf(\alpha) > \omega$ then clubs in α form a neighborhood base of α;
- the least non-isolated point is ω_1.

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Stationary sets

Def. $A \subseteq \alpha$ is stationary in α if A intersects every club in α.

We have: $d_2(A) = \{\alpha : \; cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary$\}$

Remark: Set theorists call d_2 Mahlo operation.
Ordinals in $d_2(Reg)$, where Reg is the class of regular cardinals, are called weakly Mahlo cardinals. Their existence implies consistency of ZFC.

Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

Def. Ordinal κ is *reflecting* if whenever A is stationary in κ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in α.

Def. Ordinal κ is *doubly reflecting* if whenever A, B are stationary in κ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Theorem. κ is a τ_3-limit point iff κ is doubly reflecting.

Stationary reflection

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Mahlo topology τ_3

Fact (characterizing τ_3):

- If κ is not doubly reflecting, then κ is τ_3-isolated;
- If κ is doubly reflecting, then the sets $d_2(A) \cap \kappa$, i.e.,

$$\{\alpha < \kappa : cf(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\},$$

where A is stationary in κ, form a base of τ_3-open punctured neighborhoods of κ.

Corollaries

Fact.

- If κ is weakly compact then κ is doubly reflecting.
- (Magidor) If κ is doubly reflecting then κ is weakly compact in L.

Cor. Assertion "τ_3 is non-discrete" is equiconsistent with the existence of a weakly compact cardinal.

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Summary

Let θ_n denote the first limit point of τ_n.

	name	θ_n	$d_n(A)$
τ_0	left	1	$\{\alpha : A \cap \alpha \neq \varnothing\}$
τ_1	order	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	club	ω_1	$\{\alpha : cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	... ...

θ_3 is the first doubly reflecting cardinal.

Analogies

How far do the analogies between the GLP-algebras $\mathcal{L}_S$ and $\mathcal{P}(X)$ go?

Questions about $\mathcal{P}(X)$:

- What corresponds to $<_n$?
- What corresponds to Π_n?
- What corresponds to $\equiv_n$ (and *provable* $\equiv_n$)?
- What corresponds to the reduction property?

Analogies, continued

- $A <_n B$ iff $B \subseteq d_n A$.
  - On Ω, $A <_0 B$ iff $\min(A) < \min(B)$.
  - *Note:* Jech considered $<_2$ (for the club topology). Proved $<_2$ well-founded on $\mathcal{P}(\Omega) \setminus \{\varnothing\}$.
- $\Pi_n = \{\langle n \rangle y : y \in \mathcal{L}_S\}$ in $\mathcal{L}_S$ mod $\langle n \rangle 1$.

  Fact (Friedman–Harrington–Goldfarb): In $\mathcal{L}_S$, $\exists y\, x = \langle n \rangle y$ iff ($x \in \Pi_{n+1}$ and $x \leq \langle n \rangle 1$). Hence, Π_{n+1} in $\mathcal{P}(X)$ (restricted to $d_n(X)$) is $\{d_n(A) : A \subseteq X\}$.

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Analogies, continued

- $A \equiv_n B$ iff $\forall C\,(C <_n A \iff C <_n B)$.

Fact. If (X, τ_n) is T_3, then $A \equiv_n B$ iff ($A, B \subseteq d_n X$ and $c_n A = c_n B$, or both $A, B \nsubseteq d_n X$).

This applies to the ordinal space Ω, as all the topologies τ_n for $n > 0$ are zero-dimensional, hence T_3.