Large cardinals and topological completeness of polymodal provability logics

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The Logic **GLP**

Recall: For any ordinal $\xi \geq 2$, we consider the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The corresponding dual operators $\neg[\alpha]\neg$ are denoted by $\langle\alpha\rangle$. The logic system $\mathbf{GLP}_\xi$ has the following axioms and rules:

Axioms:

1. Boolean tautologies.
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, for all $\alpha < \xi$.
3. $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$, for all $\alpha < \xi$.
4. $[\beta]\varphi \to [\alpha]\varphi$, for all $\beta < \alpha < \xi$.
5. $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for all $\beta < \alpha < \xi$.

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [\alpha]\varphi$, for all $\alpha < \xi$ (Necessitation)

Ordinal semantics

We are interested in topological semantics for $\mathbf{GLP}_\xi$, in the case the spaces are ordinal numbers with the canonical interval topology.

Recall: For δ an ordinal, the interval topology on δ is the topology generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β). This is the canonical topology on δ, i.e., the limit points are precisely the limit ordinals.

Note: This is a Hausdorff scattered topology in which 0 and all successor ordinals less than δ are isolated points. Hence if $\delta \leq \omega$, then the interval topology on δ is discrete.

Ordinal semantics

Thus, we consider polytopological spaces $(\delta, (\tau_\alpha)_{\alpha<\xi})$, where δ is an ordinal and the τ_α are topologies on δ that contain the interval topology.

For the **GLP**$_\xi$ axioms to be valid in $(\delta, (\tau_\alpha)_{\alpha<\xi})$, the topologies τ_α have to satisfy:

1. τ_α is scattered, all $\alpha < \xi$.
2. $\tau_\beta \subseteq \tau_\alpha$, for all $\beta \leq \alpha < \xi$.
3. If $d_\alpha : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ is the derived set operator for τ_α (i.e., $d_\alpha(A)$ is the set of limit points of A in the τ_α topology), then $d_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq \delta$.

Ordinal semantics

Recall: A valuation on δ is a map $v : Form \to \mathcal{P}(\delta)$ of formulas of **GLP**$_\xi$ to subsets of δ such that:

1. $v(\neg\varphi) = \delta - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle\alpha\rangle\varphi) = d_\alpha(v(\varphi))$, for all $\alpha < \xi$. (Hence, $v([\alpha]\varphi) = \delta - d_\alpha(\delta - v(\varphi))$, for all $\alpha < \xi$.)

A formula is valid in δ if $v(\varphi) = \delta$, for every valuation v on δ.

Topologies on ordinals

Since we want $(\delta, (\tau_\alpha)_{\alpha<\xi})$ to satisfy all $\mathbf{GLP}_\xi$ axioms, by the previous remarks we don't have much choice on how to define the τ_α topologies.

First of all, the topologies must form an increasing chain $\tau_0 \subseteq \tau_1 \subseteq \ldots \tau_\alpha \subseteq \ldots$ on δ, with τ_0 being the interval topology.

The other topologies are determined by the d_α operators.

Topologies on ordinals

Given τ_α, let $d_\alpha : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator, defined by:

$$d_\alpha(A) = \{\beta < \delta : \beta \text{ is a limit point of } A \text{ in the } \tau_\alpha \text{ topology}\}.$$

Then let $\tau_{\alpha+1}$ be the topology generated by

$$\mathcal{B}_{\alpha+1} := \mathcal{B}_\alpha \cup \{d_\alpha(A) : A \subseteq \delta\}.$$

If α is a limit ordinal, then we let $\tau_\alpha := \bigcup_{\beta<\alpha}$ and $\mathcal{B}_\alpha := \bigcup_{\beta<\alpha} \mathcal{B}_\beta$.

Topologies on ordinals

Notice that $d_0(A)$ is the set of limit points of A in the ordinal ordering. Thus, if the cofinality of α is uncountable and $\alpha \in d_0(A)$, then $d_0(A) \cap \alpha$ is a club (closed and unbounded) subset of α.

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{d_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on δ, known as the club topology.
Note that every $\alpha < \delta$ of countable cofinality is an isolated point of τ_1. So, if $\delta \leq \omega_1$, then τ_1 is discrete.

What is $d_1(A)$?

Recall that for α of uncountable cofinality, $S \subseteq \alpha$ is stationary in α if $S \cap C \neq \emptyset$, for all club $C \subseteq \alpha$.

For every $A \subseteq \delta$,

$$d_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

Topologies on ordinals

As a warm-up for the general case, let us look at the conditions under which the topology τ_2, generated by $\mathcal{B}_2 := \mathcal{B}_1 \cup \{d_1(A) : A \subseteq \delta\}$, is non-discrete.

If $\alpha < \delta$ and some stationary subset S of α does not reflect (i.e., $d_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. So, for τ_2 to be non-discrete we need at least that some $\alpha < \delta$ is stationary-reflecting, i.e., $d_1(S) \cap \alpha \neq \emptyset$, for all stationary $S \subseteq \alpha$.

It is well-known that the first stationary-reflecting cardinal, if it exists, must be either weakly inaccessible or the successor of a singular cardinal.

So if, e.g., $\delta \leq \aleph_{\omega+1}$, then τ_2 is discrete.

Topologies on ordinals

But for τ_2 to be non-discrete we need more than just the existence of a stationary-reflecting cardinal $\alpha < \delta$. What we need is some $\alpha < \delta$ such that every pair A, B of stationary subsets of α simultaneously reflect, that is, there exists $\beta < \alpha$ with $\beta \in d_1(A) \cap d_1(B)$. Let us call such an α simultaneoulsy stationary-reflecting, or s-reflecting for short.

Topologies on ordinals

Proposition

$\mathcal{B}_2$ is a sub-base for a topology on δ such that for every α, α is not isolated if and only if it is s-reflecting. Hence, τ_2 is a non-discrete topology on δ if and only if some $\alpha < \delta$ is s-reflecting.

On discreteness

We have seen that for the topologies τ_α on δ to be non-discrete, δ has to be large. E.g., τ_0 is non-discrete if and only if δ is greater than ω; τ_1 is non-discrete if and only if δ is greater than ω_1; and for τ_2 to be non-discrete, δ has two be greater than $\aleph_{\omega+1}$.

Question: How large must δ be for τ_2 to be non-discrete?

Before we answer this, let's pause for a second and ask:

Why do we care about the τ_α being non-discrete?

On discreteness

Fact

For the system **GLP**$_\xi$ *to be complete with respect to canonical ordinal semantics we need some* $(\delta, (\tau_\alpha)_{\alpha<\xi})$ *in which all the* τ_α *are non-discrete.*

Why? If, say, τ_α is discrete on every δ (equivalently, if no ordinal is α-s-reflecting), then the non-provable formula $[\alpha]\perp$ is valid.

Second-order indescribable cardinals

Back to our question:

Question: How large must δ be for τ_2 to be non-discrete?

What we are really asking is: How large is a s-reflecting ordinal?

Recall that a second-order formula φ of the language of set theory is Π^1_n if it is of the form

$$\forall X_1 \exists X_2 \forall X_3 \dots Q X_n \, \psi$$

where Q is $\forall$ if n is odd, and $\exists$ if n is even, and ψ is first-order.

A cardinal κ is called Π^1_n-indescribable if for every $A \subseteq V_\kappa$ and every Π^1_n-sentence $\varphi(A)$, if $\langle V_\kappa, \in, A\rangle \models \varphi(A)$, then there is $\lambda < \kappa$ such that $\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda)$.

A cardinal is weakly compact if and only if is Π^1_1-indescribable.

Second-order indescribable cardinals

Weakly-compact cardinals κ are inaccessible (i.e., regular and strong limit), hence they cannot be proved to exist in ZFC because $V_\kappa \models ZFC$.

Weakly-compact cardinals κ are also Mahlo (i.e., regular and the set of inaccessible cardinals below κ is stationary).

However, weakly-compact cardinals, and even Π^1_n-indescribable cardinals, all n, are much smaller than, say, measurable cardinals. In fact, if κ is Π^1_n-indescribable, then it is Π^1_n-indescribable in L.

Weakly compact cardinals reflect stationary sets

It is easy to see that every weakly compact cardinal is s-reflecting. Thus, in every model of set theory where there exists a weakly compact cardinal less than some limit ordinal δ, τ_2 is a non-discrete topology on δ.

And Jensen[1] showed that in the constructible universe L a cardinal κ is stationary-reflecting if and only if it is weakly compact, hence if and only if it is s-reflecting.

Thus, in L, the τ_2 topology on an ordinal δ is non-discrete if and only if there exists a weakly compact cardinal less than δ.

[1] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972).

ξ-stationary sets

Let us see next what are the general conditions under which the topologies τ_ξ are non-discrete. We begin with some definitions that generalize the notions of stationary set and stationary reflection.

Definition

Let δ be a limit ordinal. We say that $A \subseteq \delta$ is 0-stationary in α if and only if $A \cap \alpha$ is unbounded in α.

For $\xi > 0$, we say that A is ξ-stationary in $\alpha < \delta$ if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

ξ-stationary sets

Note that A is 1-stationary in α if and only if $A \cap \alpha$ is stationary in α.

Clearly, if A is ξ-stationary in α, then A is also ζ-stationary in α, for all $\zeta < \xi$.

We have that for every ξ,

$$d_\xi(A) = \{\alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha\}.$$

ξ-stationary reflection

Definition

We say that a limit ordinal α is ξ-stationary-reflecting (ξ-reflecting, for short) if and only if $d_\zeta(S)$ is ζ-stationary in α, for every $\zeta < \xi$ and every $S \subseteq \alpha$ that is ζ-stationary in α.

It is easy to see that α is 0-reflecting if and only if it is a limit ordinal; it is 1-reflecting if and only if it has uncountable cofinality; and it is 2-reflecting if and only if it is stationary-reflecting.

ξ-stationary reflection

Definition

We say that an ordinal α is ξ-simultaneously-stationary-reflecting (ξ-s-reflecting, for short) if and only for every $\zeta < \xi$, every pair of ζ-stationary subsets $A, B \subseteq \alpha$ simultaneously ζ-reflect at some $\beta < \alpha$, i.e., $A \cap \beta$ and $B \cap \beta$ are ζ-stationary in β.

Note that α is 1-s-reflecting if and only if it has uncountable cofinality; and it is 2-s-reflecting if and only if it is s-reflecting.

One can show that α is ξ-s-reflecting if and only if $d_\zeta(A) \cap d_\zeta(B)$ is ζ-stationary in α, for every $\zeta < \xi$ and every ξ-stationary $A, B \subseteq \alpha$.

Characterizing non-discreteness

Tthe following theorem characterizes the conditions under which τ_ξ is non-discrete.

Theorem

For every ξ, $\mathcal{B}_\xi$ is a sub-base for a topology on δ such that for every $\alpha < \delta$, α is not isolated if and only if it is ξ-s-reflecting. Hence, τ_ξ is a non-discrete topology on δ if and only if some $\alpha < \delta$ is ξ-s-reflecting.

Π^1_n-Indescribable cardinals

Π^1_n-indescribable cardinals give an upper bound on the largeness of δ for the topologies τ_α on δ to be non-discrete.

Proposition

Every Π^1_n-indescribable cardinal is (n+1)-s-reflecting.

Thus, if there exists a Π^1_n-indescribable cardinal below some ordinal δ, then τ_{n+1} is a non-discrete topology on δ.

Π^1_n-indescribable cardinals in L

It is possible (or even likely) that, as shown by Jensen[2] in the case of Π^1_1-indescribable cardinals and stationary-reflection, in the constructible universe L a cardinal is $(n+1)$-reflecting if and only if it is Π^1_n-indescribable, and therefore if and only if it is $(n+1)$-s-reflecting.

If this turns out to be the case, then in L the Π^1_n-indescribable cardinals would be precisely the non-isolated points of the τ_{n+1} topology.

This is still an open conjecture.

The so-called ξ-indescribable cardinals, introduced by Jensen, may be used for the general case.

[2] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

ξ-indescribable cardinals

For $\xi > 0$, a cardinal κ is called ξ-indescribable if for every formula $\varphi(x)$ of the first-order language of set theory, and any subset $A \subseteq V_\kappa$, if

$$\langle V_{\kappa+\xi}, \in, A\rangle \models \varphi(A)$$

then for some $\lambda < \kappa$,

$$\langle V_{\lambda+\xi}, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda).$$

Observe that κ is 1-indescribable if and only if it is Π^1_n-indescribable for every n.

Jensen showed that if κ is the ω-Erdös cardinal, then there are cardinals λ below κ that are λ-indescribable. Further, if κ is ξ-indescribable, then $L \models$ "κ is ξ-indescribable".

ξ-indescribable cardinals

Theorem

For $\xi > 0$, every ξ-indescribable cardinal κ is ξ-s-reflecting.

So, if there exists a ξ-indescribable cardinal below some ordinal δ, then the topology τ_ξ on δ is non-discrete.

Theorem

CON($\exists\kappa < \lambda\,(\kappa$ is ξ-indescribable $\wedge\ \lambda$ is inaccessible)) implies CON(τ_ξ is non-discrete $\wedge\ \tau_{\xi+1}$ is discrete).

The ideal of non-ξ-stationary sets

For each limit ordinal α and each ξ, let $\mathcal{I}^{\xi}_{\alpha}$ be the set of non-ξ-stationary subsets of α, and let

$$\mathcal{F}^{\xi}_{\alpha} = (\mathcal{I}^{\xi}_{\alpha})^{*} := \{A \subseteq \alpha : \alpha - A \in \mathcal{I}^{\xi}_{\alpha}\}.$$

Thus, if α has uncountable cofinality, then $\mathcal{I}^{1}_{\alpha}$ is the ideal of non-stationary subsets of α and $\mathcal{F}^{1}_{\alpha}$ is the club filter over α.

Proposition

For every ξ, an ordinal α is ξ-s-reflecting if and only if $\mathcal{I}^{\xi}_{\alpha}$ is an ideal, hence if and only if $\mathcal{F}^{\xi}_{\alpha}$ is a filter.

The ideal of non-ξ-stationary sets

We say that $A \subseteq \alpha$ has positive $\mathcal{F}^{\xi}_{\alpha}$ measure if $A \cap B \neq \emptyset$ for every $B \in \mathcal{F}^{\xi}_{\alpha}$.

Let us denote by $(\mathcal{F}^{\xi}_{\alpha})^{+}$ the set of all subsets of α of positive $\mathcal{F}^{\xi}_{\alpha}$-measure, that is, the set of all ξ-stationary subsets of α.

Notice that for every valuation $v : Form \to \mathcal{P}(\delta)$ (for the language of $\mathbf{GLP}_{\xi}$),

$$v(\langle\beta\rangle\varphi) = \{\alpha < \delta : v(\varphi) \cap \alpha \in (\mathcal{F}^{\beta}_{\alpha})^{+}\}.$$

$$v([\beta]\varphi) = \{\alpha < \delta : v(\varphi) \cap \alpha \in \mathcal{F}^{\beta}_{\alpha}\}.$$

Completeness

Let us address now the question of completeness for $\mathbf{GLP}_\xi$ (under canonical ordinal semantics).

The case $\xi = 1$ was proved, independently, by M. Abashidze (1985) and A. Blass (1990).

The case $\xi = 2$ has been proved by Beklemishev[3]. The proof uses a result of Blass (1990)[4], which holds under the assumption of a set-theoretic principle known as square.

[3] Beklemishev, L., Ordinal Completeness of Bimodal Provability Logic GLB. Lecture Notes in Computer Science, 2011, Volume 6618/2011, 1-15. Springer.

[4] Blass, A. (1990) *Infinitary Combinatorics and Modal Logic*. The Journal of Symbolic Logic, Vol. 55, No. 2, 761-778.

Jensen's Square principle

For κ an uncountable cardinal, the principle $\square_\kappa$ asserts that there exists a sequence $\langle C_\alpha : \alpha \in Lim \cap \kappa^+ \rangle$ such that:

1. C_α is a club subset of α.
2. If $cof(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.
3. If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

Jensen[5] showed that $\square_\kappa$ holds in L, for every uncountable cardinal κ.

$\square_\kappa$ implies that some stationary subset of κ^+ does not reflect.

[5] Op. cit.

Blass' Embedding Theorem

Theorem (Blass' Embedding Theorem)

Assume $\square_{\aleph_n}$, for every $n < \omega$. For every finite tree $(T, <)$ of height n there is a map $S : T \to \mathcal{P}(\aleph_n) \setminus \{\emptyset\}$ such that

1. $\{S_x : x \in T\}$ *is pairwise disjoint.*
2. *If $x < y$, then $S_x \subseteq d_1(S_y)$. That is, if $\alpha \in S_x$, then $S_y \cap \alpha$ is stationary, i.e., it belongs to $(\mathcal{F}^1_\alpha)^+$.*
3. *$S_x \subseteq -d_1(-\bigcup_{x<y} S_y)$. That is, if $\alpha \in S_x$, then $\bigcup_{x<y} S_y$ contains a club, i.e., it belongs to $\mathcal{F}^1_\alpha$.*

Completeness

What about completeness for $\mathbf{GLP}_3$?
Or for $\mathbf{GLP}_\xi$, for arbitrary ξ ?

The strategy for proving completeness for $\mathbf{GLP}_\xi$, with canonical ordinal semantics, is similar to the strategy used by Beklemishev in the case $\xi = 2$. Namely, use the completeness of Beklemishev's system **J**, with respect to **J** trees, and transfer it, via a suitable Embedding Theorem, to $\mathbf{GLP}_\xi$.

To keep things reasonably simple, let's consider the case $\xi = \omega$.

The system **J**

Beklemishev[6] introduces a subsystem **J** of $\mathbf{GLP}_\omega$, obtained by weakening axiom (iv) of $\mathbf{GLP}_\omega$ to the two axioms

(vi) $[m]\varphi \to [n][m]\varphi$, for $m \leq n$

(vii) $[m]\varphi \to [m][n]\varphi$, for $m < n$

which are consequences of $\mathbf{GLP}_\omega$.

Beklemishev shows that $\mathbf{GLP}_\omega$ is reducible to **J** in the following sense: For each formula φ, there is a formula $M^+(\varphi)$ that is provable in $\mathbf{GLP}_\omega$, hence if $\mathbf{J} \vdash M^+(\varphi) \to \varphi$, then $\mathbf{GLP}_\omega \vdash \varphi$, and conversely: if $\mathbf{GLP} \vdash \varphi$, then $\mathbf{J} \vdash M^+(\varphi) \to \varphi$. Thus, $\mathbf{GLP}_\omega$ and **J** are equivalent.

The point is that **J** is complete with respect to a nice class of finite frames, the so-called **J**-frames.

[6] Beklemishev, L. (2010) *Kripke semantics for provability logic GLP*. Annals of Pure and Applied Logic, 161, 756-774.

$\mathbf{J_n}$-trees

The notion of **J**-tree[7] is a special case of the notion of **J**-frame[8].

Recall that a frame $\langle T, R_0, R_1, \ldots \rangle$ consists of a non-empty set T, together with binary relations $R_0, R_1, \ldots$ on T. For each m, let E_m be the reflexive, symmetric, and transitive closure of $R_m \cup R_{m+1} \cup \ldots$. Notice that E_{m+1} refines E_m.

An E_m-equivalence class is called an m-plane. Each R_m can be naturally extended to a relation R_m^+ on $m+1$-planes by: XR_m^+Y iff xR_my for some $x \in X$ and $y \in Y$.

[7]Beklemishev, L. and Gabelaia, D. *Topological completeness of the provability logic GLP*. Preprint.

[8]Beklemishev, L. (2010) *Kripke semantics for provability logic GLP*. Annals of Pure and Applied Logic, 161, 756-774.

J-trees

Definition

A finite **J**-tree is a frame $\langle T, R_0, \ldots, R_n \rangle$, where T is finite, and

1. R_m is irreflexive and transitive, all $m \leq n$.
2. All elements in an m-plane are $R_{m'}$-incomparable, for all $m' < m$.
3. If XR_m^+Y, then xR_my for all $x \in X$ and all $y \in Y$.
4. The set of $m+1$-planes contained in an m-plane is a rooted tree under R_m^+.
5. Each n-plane is a rooted tree under R_n.

Thus, a finite **J**-tree may be seen as a finite collection of 0-planes, each one of them being a finite rooted tree under R_0^+ whose nodes are 1-planes, each one of them being a finite rooted tree under R_1^+ whose nodes are 2-planes, and so on.

J-trees

The following Completeness Theorem is due to Beklemishev.[9]

Theorem

J *is complete with respect to the class of finite* **J***-trees. That is, if* φ *is valid in all finite* **J***-trees, then* φ *is provable in* **J***.*

[9] Beklemishev, L. (2010) *Kripke semantics for provability logic GLP*. Annals of Pure and Applied Logic, 161, 756-774.

Completeness

The Theorem we would like to prove is the following:

Theorem (Completeness)

Under suitable (necessary) set-theoretic assumptions (e.g., there exists an ω-s-reflecting cardinal κ, and $\square_\lambda$ holds for all $\lambda < \kappa$), every valid formula of the language of $\mathbf{GLP}_\omega$ *is provable from the* $\mathbf{GLP}_\omega$ *axioms.*

By a result of Blass, assuming only large cardinals is not enough.

This Completeness Theorem follows from the following (still unproved) Embedding Theorem.

The Embedding Theorem

Let us write $R_i(x) := \{y : xR_iy\}$ and $\bar{R}_i(x) := \bigcup_{i \leq j \leq n} R_j(x)$.

Theorem (Embedding Theorem)

Under the same set-theoretic assumptions of the Completeness Theorem, if $\langle T, R_0, \ldots, R_n \rangle$ is a finite J-tree, then there is an ordinal $\delta < \kappa$ and a map $S : T \to \mathcal{P}(\delta) \setminus \{\emptyset\}$ such that

1. *$\{S_x : x \in T\}$ is pairwise disjoint, and*

   for every $i \leq n$,

2. *If xR_iy, then $S_x \subseteq d_i(S_y)$. That is, if $\alpha \in S_x$, then $S_y \cap \alpha \in (\mathcal{F}^i_\alpha)^+$.*
3. *$S_x \subseteq -d_i(-\bigcup_{y \in \bar{R}_i(x)} S_y)$. That is, if $\alpha \in S_x$, then $\bigcup_{y \in \bar{R}_i(x)} S_y \cap \alpha \in \mathcal{F}^i_\alpha$.*