Recursion Theory

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- Answer: there are uncountably many degrees

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- However, a countable union of countable sets is again countable via coding

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- Aside: also in \mathcal{D} there is a minimal element 0

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- $\Phi_{e,s}^A$ and $W_{e,s}$
- Likewise, we define the notions Σ_1^A and Π_1^A

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- As A-c.e. coincides with Σ_1^A

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- We can iterate jumps

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- \checkmark So we have proved once more that there is no maximal element in $\mathcal D$

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- In particular:

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- To prove Post's Theorem we need the following lemma

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- $\textbf{SSUME } A \in \Sigma_{n+2}$

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- **9** By Jump-Theorem: $\overline{R} \leq_m \varnothing^{n+1}$
- So, $A \in \Sigma_1^{\emptyset^{n+1}}$ and by NFT c.e. in \emptyset^{n+1}

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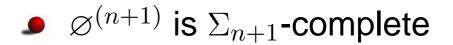
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- Bringing this into prenex normal form gives us $A \in \Sigma_{n+2}$.

• $A \in \Delta_{n+1} \Leftrightarrow A, \overline{A} \leq_T \emptyset^{(0)}$

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- Proof: By the relativized Complementation Lemma and using that A is $\Sigma_{n+1}^0 \Leftrightarrow A$ is c.e. in $\emptyset^{(n)}$.



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- To go beyond ω we need hyperarithmetic sets and second order logic