Recursion Theory

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- First simple question: how many degrees are there?
- Answer: there are uncountably many degrees

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- However, a countable union of countable sets is again countable via coding

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- Aside: also in $\mathcal{D}$ there is a minimal element $\mathbf{0}$

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- $\Phi_{e,s}^A$ and $W_{e,s}$
- Likewise, we define the notions Σ_1^A and Π_1^A

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- As A-c.e. coincides with Σ_1^A

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- We can iterate jumps

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- So we have proved once more that there is no maximal element in $\mathcal{D}$

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- In particular:

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' < \mathbf{0}'''' \ldots$$

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- To prove Post's Theorem we need the following lemma

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- By Jump-Theorem: $\overline{R} \leq_m \varnothing^{n+1}$
- So, $A \in \Sigma_1^{\varnothing^{n+1}}$ and by NFT c.e. in $\varnothing^{n+1}$

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  $\exists s$ and some oracle queries to $\varnothing^{(n+1)}$ and its complement such that: $x \in W_{i,a}^{\varnothing^{(n+1)}}$
- Bringing this into prenex normal form gives us $A \in \Sigma_{n+2}$.

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- Proof: By the relativized Complementation Lemma and using that A is $\Sigma^0_{n+1} \;\Leftrightarrow\; A$ is c.e. in $\varnothing^{(n)}$.

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- To go beyond ω we need hyperarithmetic sets and second order logic