Recursion Theory

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Hamkin’s Course

Studying informational degrees via Turing Degrees
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What is the order and how complex is it?
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- Some computable model theory: e.g., does every computable consistent theory have a computable model?
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- Some computable model theory: e.g., does every computable consistent theory have a computable model?
- Infinite time Turing Machines
Gödel’s first incompleteness theorem can be seen as a corollary from computational complexity facts.
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Incomplete: for some $\psi$ we have that $T \not\vdash \psi$ and $T \not\vdash \neg \psi$

Omega-inconsistent: for some $\psi(x)$ we have

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Proving Gödel 1

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A set $S$ is semi-representable in a theory $T$ whenever there is some formula $\varphi(x)$ in the language of $T$ such that

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  - $S$ is semi-representable in $PA$
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*SRT:* The following are equivalent (provided $PA$ is Omega-consistent)

- $S$ is c.e.
- $S$ is semi-representable in $PA$
- $S \leq_m T_{PA}$
Corollary 1: theoremhood of PA is not computable
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Corollary 2: PA is incomplete
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where \( \varphi(x) \) semi-represents \( K \)
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so, for some $x \in \overline{K}$ we have PA $\not\models \neg \varphi(x)$
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Natural incomplete sentences are hard to find
Corollary 1: theoremhood of $PA$ is not computable (undecidable)

Corollary 2: $PA$ is incomplete

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Natural incomplete sentences are hard to find

Goodstein’s sequences!
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There is an interesting link from strange attractors in chaos theory to Goodstein’s process.
PA is creative

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Pure Predicate Calculus is undecidable
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Pure Predicate Calculus is undecidable

Can be done directly by coding the halting problem, we give a shorter proof
Decidable theories

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- Applications: dense linear ordering with no begin or end-points (Algebraically closed fields of given characteristic)
PC is undecidable

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(2) If $\mathcal{T}'$ is a finite extension of $\mathcal{T}$, then

$$T_{\mathcal{T}'} \leq_m T_{\mathcal{T}}$$
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- Two main ingredients
  1. There is a finitely axiomatized creative theory (Raphael Robinson)
  2. If $\mathcal{T}'$ is a finite extension of $\mathcal{T}$, then

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- The proof of (2) is easy via the computable version of the deduction lemma
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- Two main ingredients
- (1) There is a finitely axiomatized creative theory (Raphael Robinson)
- (2) If $\mathcal{T}'$ is a finite extension of $\mathcal{T}$, then

$$\mathcal{T}_{\mathcal{T}'} \leq_m \mathcal{T}_{\mathcal{T}}$$

- The proof of (2) is easy via the computable version of the deduction lemma
- The proof of (1) is a bit more involved
Robinson’s Arithmetic

\( Q \) contains all the finite defining axioms of the symbols
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- Induction can be reduced to:

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x \neq 0 \rightarrow \exists y \ x = y'.
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- Thus, we get

  $$K \leq_m T_Q \leq_m T_{PC}.$$
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- An easy lemma teaches us that (via the embedding)

$$T_{PC^{-}} \leq_m T_{PC}$$
Robinson’s Arithmetic

- \( Q \) contains all the finite defining axioms of the symbols
- Induction can be reduced to:

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- One now can check that all the c.e. sets are semi-representable in \( Q \) (provided \( \omega \)-consistency)
- Thus, we get

\[ K \leq_m T_Q \leq_m T_{PC}^- \]

- An easy lemma teaches us that (via the embedding)

\[ T_{PC}^- \leq_m T_{PC} \]

- Thus we obtain that PC is creative!