Mid-term exam
Recursion Theory

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October 23, 2006

1. Give the input/output convention for URM's. Next,write a URM that calculates $x + 2y$.

2. A set $A \subseteq \mathbb{N}$ is called *creative* if the following two requirements hold.

   - A is c.e.;
   - There is a computable function f such that for each e

     $$W_e \subseteq \overline{A} \;\Rightarrow\; f(e) \in \overline{A} \setminus W_e.$$

     This f is called the *creative function* of A.

   Prove the following facts about creative sets.

   (a) A creative set has non-empty complement. (Hint: consider W_e for some TM e that loops on any input.)

   (b) Prove that the set K is creative[1] and that its creative function is the identity function.

   (c) Prove that a creative set cannot be computable.

   (d) Prove that the complement of a creative set contains an infinite computably enumerable subset.

   (e) Prove that the complement of a creative set contains an infinite computable subset.

3. Prove that there exists a TM that, on any input, outputs its own code. (Hint: use the fixed point theorem.)

4. We define the notion of a c.é. set as follows. A set A is c.é. if there is a partial computable characteristic function χ_A such that

   $$x \in A \;\Leftrightarrow\; \chi_A(x) =\downarrow 1.$$

[1] Daisuke convinced me that I should repeat the definition of K here. As this is not the final exam but the midterm, I shall do so: $K := \{x \mid x \in W_x\}$.

Note that this definition is very similar to the notion of a computable set, where we demand a computable (and hence total) function χ_A such that

$$x \in A \;\Leftrightarrow\; \chi_A(x) = 1,$$

and

$$x \notin A \;\Leftrightarrow\; \chi_A(x) = 0.$$

Prove the following theorem.

Theorem: A is c.e. $\iff$ A is c.é.

5. Let f be a (possibly partially defined) map, $f : \mathbb{N} \to \mathbb{N}$. Consider the following implications and say if they are true or false. If the implication is true, give a short (at most a couple of lines) proof. If it is false, give a counter-example, briefly outlining that, indeed, it is a counter-example.

   (a) f is computable $\Rightarrow$ $\mathsf{Graph}(f)$ is Δ^0_1

   (b) f is computable $\Leftarrow$ $\mathsf{Graph}(f)$ is Δ^0_1

   (c) f is partially computable $\Rightarrow$ $\mathsf{Graph}(f)$ is Σ^0_1

   (d) f is partially computable $\Leftarrow$ $\mathsf{Graph}(f)$ is Σ^0_1

   (e) f is partially computable $\Leftarrow$ $\mathsf{Graph}(f)$ is Δ^0_1

   (f) f is partially computable $\Rightarrow$ $\mathsf{Graph}(f)$ is Δ^0_1