Some notes on proofs for QL2

Joost J. Joosten
University of Barcelona

February 17, 2023

Contents

0.1 Propositional logic revisited

As mentioned earlier, the word *logic* refers to the art of valid reasoning or simply to a well-defined system of reasoning. Systems of reasoning are delimited by the expressive power of the language. One of the simplest languages is that of *propositional logic*. The basic blocks of propositional logics are, as the name suggests, *propositions*.

It is actually quite difficult to pin down what exactly *is* a proposition and there is no common viewpoint. In a very broad sense, propositions are those entities to which we can attach a truth-value, a belief or, any other similar attribute. For the purpose of this chapter it is not really necessary to settle upon a precise ontological viewpoint. For the time being we can say that propositions are simply atomic pieces of information.

As such, we will use variables like p, q and r to refer to these atomic pieces of information. For example, p could stand for the piece of information that John loves Mary. Note that that *John loves Mary* contains inner structure with verbs and nouns. However, propositional logic does not aim at capturing such inner structure: we treat pieces of information as atomic and irreducible.

We will further stipulate that variables denote *independent* pieces of information so that, for example, the truth status of p and of q have nothing to do with each other.

Propositional logic aims at capturing the structural behavior of propositions with respect to the (idealized) connectives that we use in every day life: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$.

However, it turns out that this structural behavior actually very much depends on the underlying ontology of propositions and connectives. Different viewpoints yield different logics. For example, if propositions refer to measurements on quantum-systems that take Heisenberg's uncertainty principle into account, then the resulting structural behavior of the connectives would result in as of yet not entirely understood so-called *quantum logic*. In this setting, for example, the law of distributivity

$$p \wedge (q \vee r) \;\equiv\; (p \vee q) \wedge (p \vee r)$$

would fail.

In case that propositions refer to Platonic facts that are either true or false, the structural behavior will yield what is often called *classical* logic. This is one of the easier logics to work with although it may yield some contra-intuive laws like

$$(p \to q) \vee (q \to p).$$

There also exists a reading of the propositions and connectives that requires more constructive information. In this spirit, saying that p is the case would be saying that I have evidence for p. Similarly, saying that I have evidence for $A \vee B$ is only justified if I have evidence either for A or I have evidence for B and moreover, I know for which of the two it is. In such a constructive ontology, the resulting logic will be what is called *constructive logic* or synonimously *intuitionistic logic*.

For our expostion we will only be interested in the latter two logics. Please, document yourself a bit on

- The BHK interpretation (Brouwer-Heyting-Kolmogorov interpretation) of constructive logic;
- How negation is treated in constructive logic. (In class we mentioned that we treat $\neg p$ as $p \to \bot$.)

0.2 Natural Deduction for propositional logics

The name *Natural Deduction* is used since the calculus that goes by that name very naturally reflects how we make logical deductions in our day-to-day life. A typical feature of the calculus is that we can work with open assumptions that can later be closed. For example, we could reason,

> Suppose I were to go to the movies tonight. In that case I'd spend a lot of money on the entrance and drinks afterwards. Moreover, the next morning I'd be very tired and I have to wake up early. To summarize, if I go to the movies tonight, I'll spend a lot of money and I'll be very tired the next morning. In particular, I'll be very tired in the morning.

Note that at the beginning of the reasoning, one is hypothetically assuming that the person will go to the movies. So, at that stage of the reasoning, this is an open assumption. However, once the conclusion

> If I go to the movies tonight, I'll be very tired the next morning;

is made, it is no longer an open assumption since the assumption to go to the movies is incorporated in the conditional statement itself.

We will represent this structure in reasoning schematically. To this end, by $\mathcal{D}$ we will denote some piece of reasoning, also called a derivation. Next, by $\begin{matrix}\mathcal{D}\\ B\end{matrix}$ we will denote a derivation $\mathcal{D}$ which has the formula B as its conclusion. And, furthermore, by $\begin{matrix}\mathrm{A}\\ \mathcal{D}\\ B\end{matrix}$ we denote that this derivation $\mathcal{D}$ may have among its assumptions the assumption A. So, in our example above, one of the assumptions was *I go to the movies tonight* and we can denote that by A and the conclusion after the reasoning $\mathcal{D}$ would be B : *I'll be very tired in the morning.* As we observed, from the reasoning $\begin{matrix}\mathrm{A}\\ \mathcal{D}\\ B\end{matrix}$ we may obtain a new piece of reasoning $\dfrac{\begin{matrix}[\mathrm{A}]\\ \mathcal{D}\\ B\end{matrix}}{A \to B}$ where we conclude $A \to B$ (I go to the movies tonight implies I'll be very tired in the morning) by a piece of reasoning where the previous assumption A is no longer open. To flag that this assumption A is no longer open since it is incorporated in the conclusion $A \to B$, we will put square brackets around A and write $[A]$.

This piece of reasoning corresponds to what we call the *implication introduction.* We will write (an abbreviation of) the name of the rule always next to the line where the rule was applied. Moreover, in the case of implication introduction (we will write $\to$ E) we will flag which assumption is closed by writing a number next to "$\to$ E," so that this number refers to the number used to label the closed assumption. In our example, we will get $\dfrac{\begin{matrix}[A]^1\\ \mathcal{D}\\ B\end{matrix}}{A \to B} \to \mathrm{E},\ 1$.

To combine proofs using an implication elimination is much easier and it follows the following pattern. If we already have a proof $\begin{matrix}\mathcal{D}\\ A \to B\end{matrix}$ and another proof $\begin{matrix}\mathcal{D}'\\ A\end{matrix}$, then we combine those into a new proof

$$\dfrac{\begin{matrix}\mathcal{D}'\\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}\\ A \to B\end{matrix}}{B} \to \mathsf{E}\ .$$

This rule is also know as *modus ponens.* To start building proofs, we will agree that any formula A counts as a proof with conclusion A from assumption A. We say that a formula φ is provable when we can find a proof with conclusion φ which has no assumptions open. Here is an example of a proof of $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

TO INCLUDE

The rules for conjunction are also simple: $\dfrac{\begin{matrix}\mathcal{D} & & \mathcal{D}' \\ \phi & & \psi\end{matrix}}{\phi \wedge \psi}\,\wedge\mathsf{I}$ for the introduction rule and two elimination rules:

$$\dfrac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\phi}\,\wedge\mathsf{E},\mathsf{l}\; ND \text{ and } \dfrac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\psi}\,\wedge\mathsf{E},\mathsf{r}\;.$$