# On the rules of intermediate logics

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**Abstract**

If the Visser rules are admissible for an intermediate logic, they form a basis for the admissible rules of the logic. How to characterize the admissible rules of intermediate logics for which not all of the Visser rules are admissible is not known. In this paper we give a brief overview of results on admisisble rules in the context of intermediate logics. We apply these results to some well-known intermediate logics. We provide natural examples of logics for which the Visser rule are derivable, admissible but nonderivable, or not admissible.

*Keywords*: Intermediate logics, admissible rules, realizability, Rieger-Nishimura formulas, Medvedev logic, Independence of Premise.

## 1 Introduction

Admissible rules, the rules under which a theory is closed, form one of the most intriguing aspects of intermediate logics. A rule $A/\ B$ is admissible for a theory if $B$ is provable in it whenever $A$ is. The rule $A/\ B$ is said to be derivable if the theory proves that $A \to B$. Classical propositional logic $\mathsf{CPC}$ does not have any non-derivable admissible rules, because in this case $A/\ B$ is admissible if and only if $A \to B$ is derivable, but for example intuitionistic propositional logic $\mathsf{IPC}$ has many admissible rules that are not derivable in the theory itself. For example, the Independence of Premise rule *IPR*

$$\neg A \to B \vee C \ / \ (\neg A \to B) \vee (\neg A \to C)$$

is not derivable as an implication within the system, but it is an admissible rule of it. Therefore, knowing that $\neg A \to B \vee C$ is provable gives you much more than just that, because it then follows that also one of the stronger $(\neg A \to B)$ or $(\neg A \to C)$ is provable. Thus the admissible rules shed light on what it means

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to be constructively derivable, in a way that is not measured by the axioms or derivability in the theory itself.

The Visser rules (given below) form an infinite collection of rules that play an important role in this context. Namely, in [10] it has been show that if for a logic Visser's rules are admissible, then they form a basis for the admisisble rules of the logic. The latter means that all the admissible rules of the logic can then be derived from the Visser rules. The paper is meant as a brief survey on the role that the Visser rules play in intermediate logic. The paper does not contain deep new results, but lists the theorems on the subject been obtained so far, and contains applications of these results to intermediate logics. This will provide a complete description of the admissible rules of some well-known intermediate logics for which Visser's rules are admissible or even derivable. In contrast to this we discuss some logics for which not all of Visser's rule are admissible. As we will see, general theorems on the admissible rules of these logics, let alone a complete description of them, are rare. The results obtained so far are mostly of the form that for a certain logic we know that this or that specific rule is admissible or not. In many cases this rule is the Independence of Premise rule given above.

The paper is build up as follows. The next section contains the intermediate logics that we will discuss. The third section coonsists of preliminaries. The fourth section lists most of the general results on admissible rules for intermediate logics that have been obtained so far. In the last section we present some new results on the admissible rules of intermediate logics given in the next section. As we will see, in case not all of the Visser rules are admissible we know not much of the admissible rules of a logic. And hence the last section contains a long list of open questions in this area.

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## 2 Intermediate logics

Below follows the list of intermediate logics that we will discuss. As the reader can see, it consists mainly of quite well-known and natural logics, whatever the word natural might exactly mean. This is not accidently so, as we are particularly interested in these kind of logics. For it might well be that for specific purposes, e.g. for showing that there exist logics for which not all Visser rules are admissible,one can cook up a logic that serves as an example, but we feel that to come up with a well-known and natural instance of such a logic is somehow much moresatisfying.

In the list below we have tried to provide references to the paper in which the logic first appears (die Uraufführung). When we do not have such a reference we refer to the book [3] or PhD thesis [5], which mention most of these logics and prove frame completeness and decidability results for them.

A point of terminology: when we say "axiomatized by ..." we mean "axiomatized over $\mathsf{IPC}$ by ...". For a class of frames $F$, $\mathsf{L}$ is called the *logic of the frames* $F$ when $\mathsf{L}$ is sound and complete with respect to $F$.

$\mathsf{Bd_n}$ The logic of frames of depth at most $n$. $\mathsf{Bd_1}$ is axiomatized by $bd_1 = A_1 \vee \neg A_1$, and $\mathsf{Bd_{n+1}}$ by $bd_{n+1} = (A_{n+1} \vee (A_{n+1} \to bd_n))$ [3].

$\mathsf{D_n}$ The Gabbay-de Jongh logics [6], axiomatized by the following scheme: $\bigwedge_{i=0}^{n+1}((A_i \to \bigvee_{j\neq i} A_j) \to \bigvee_{j\neq i} A_j) \to \bigvee_{i=0}^{n+1} A_i$. $\mathsf{D_n}$ is complete with respect to the class of finite trees in which every point has at most $(n+1)$ immediate successors.

$\mathsf{G_k}$ The Gödel logics, first introduced in [9]. They are extensions of $\mathsf{LC}$ axiomatized by $A_1 \vee (A_1 \to A_2) \vee \ldots \vee (A_1 \wedge \ldots \wedge A_{k-1} \to A_k)$. $\mathsf{G_k}$ is the logic of the linearly ordered Kripke frames with at most $k-1$ nodes [1].

$\mathsf{KC}$ De Morgan logic (also called Jankov logic), axiomatized by $\neg A \vee \neg\neg A$. The logic of the frames with one maximal node.

$\mathsf{KP}$ The logic axiomatized by *IP*, i.e. by $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$. The logic is called Kreisel-Putnam logic. It constituted the first counterexample to Łukasiewicz conjecture that $\mathsf{IPC}$ is the greatest intermediate logic with the disjunction property [13].

$\mathsf{LC}$ Gödel-Dummett logic [4], the logic of the linear frames. It is axiomatized by the scheme $(A \to B) \vee (B \to A)$.

$\mathsf{ML}$ Medvedev logic [14]. The logic of the frames $F_1, F_2, \ldots$, where the nodes of $F_n$ are the nonempty subsets of $\{1, \ldots, n\}$ and $\preccurlyeq$ is $\supseteq$.

$\mathsf{M_n}$ The logic of frames with at most $n$ maximal nodes. Note that $\mathsf{M_1} = \mathsf{KC}$.

$\mathsf{ND_n}$ The logic of frames with at most $n$ nodes.

$\mathsf{NL_n}$ The logics axiomatized by formulas in one propositional variable (so-called Nishimura formulas $nf_n$ [16]). $\mathsf{NL_n}$ is axiomatized by $nf_n$, where

| | |
|---|---|
| $nf_0 = \bot$ | $\mathsf{NL_0}$ is inconsistent |
| $nf_1 = p$ | $\mathsf{NL_1}$ is inconsistent |
| $nf_2 = \neg p$ | $\mathsf{NL_2}$ is inconsistent |
| $nf_{2n+1} = nf_{2n} \vee nf_{2n-1}$ | $nf_{2n+2} = nf_{2n} \to nf_{2n-1}$. |

Note that

| | |
|---|---|
| $nf_3 = p \vee \neg p$ | $\mathsf{NL_3} = \mathsf{CPC}$ |
| $nf_4 = \neg p \to p \equiv \neg\neg p$ | $\mathsf{NL_4}$ is inconsistent |
| $nf_5 = (\neg p \to p) \vee \neg p \equiv \neg\neg p \vee \neg p$ | $\mathsf{NL_5} = \mathsf{KC}$ |
| $nf_6 = (\neg p \to p) \to p \vee \neg p \equiv \neg\neg p \to p$ | $\mathsf{NL_6} = \mathsf{CPC}$ |
| $nf_7 = nf_6 \vee nf_5 \equiv (\neg\neg p \to p) \vee \neg\neg p$ | |
| $nf_8 = nf_6 \to nf_5 \equiv (\neg\neg p \to p) \to p \vee \neg p$ | $\mathsf{NL_8} = \mathsf{KC}$. |

($\mathsf{NL_8} = \mathsf{KC}$ follows by substituting $\neg\neg p$ for $p$.)

$\mathsf{ER}$ The logic of effectively realizable formulas: the logic consisting of formulas $A(p_1,\ldots,p_n)$ for which there exists a recursive function $f$ such that for any substitution of the $p_i$ by arithmetical formulas $\varphi_i$ with Gödel numbers $m_i$, $f(m_1,\ldots,m_n)$ realizes the result, i.e. $\mathbb{N} \models$"$f(m_1,\ldots,m_n)\mathtt{r}A(\varphi_1,\ldots,\varphi_n)$". There is no r.e. axiomatization known for this logic, but it is known that it is a proper extension of $\mathsf{IPC}$ [19].

$\mathsf{UR}$ The logic of formulas that are effectively realizable by a constant function, i.e. the logic consisting of formulas $A(p_1,\ldots,p_n)$ such that there exists a number $e$ such that for any substitution of the $p_i$ by arithmetical formulas $\varphi_i$, $e$ realizes the result, i.e. $\mathbb{N} \models$"$e\mathtt{r}A(\varphi_1,\ldots,\varphi_n)$". There is no r.e. axiomatization known for this logic, but it was shown in [19] that it is a proper extension of $\mathsf{IPC}$.

$\mathsf{Sm}$ The greatest intermediate logic properly included in classical logic. It is axiomatized by $\big((A \to B) \vee (B \to A)\big) \wedge (A \vee (A \to B \vee \neg B))$ and it is complete with respect to frames of at most 2 nodes [3].

$\mathsf{V}$ The logic axiomatized by $V_1^{\to}$, i.e. by the implication corresponding to the rule $V_1^-$: $\big((A_1 \to B) \to A_2 \vee A_3\big) \to \bigvee_{i=1}^{3} \big((A_1 \to B) \to A_i\big)$.

# 3 Preliminaries

This section contains the preliminaries needed to understand the proofs in Section 5. For most of the next section, which contains an overview of the main results in the area, these preliminaries are not needed.

As mentioned above, we will only be concerned with intermediate logics $\mathsf{L}$, i.e. logics between (possibly equal to) $\mathsf{IPC}$ and $\mathsf{CPC}$. We write $\vdash_{\mathsf{L}}$ for derivability in $\mathsf{L}$. The letters $A, B, C, D, E, F, H$ range over formulas, the letters $p, q, r, s, t$, range over propositional variables. We assume $\top$ and $\bot$ to be present in the language. $\neg A$ is defined as $(A \to \bot)$. We omit parentheses when possible; $\wedge$ binds stronger than $\vee$, which in turn binds stronger than $\to$. The class of *Harrop formulas* $\mathcal{H}$ is the class of formulas in which every disjunction occurs in the negative scope of an implication.

## 3.1 Admissible rules

A *substitution* $\sigma$ in this paper will always be a map from propositional formulas to propositional formulas that commutes with the connectives. A *(propositional) admissible rule* of a logic $\mathsf{L}$ is a rule $A/B$ such that adding the rule to the logic does not change the theorems of $\mathsf{L}$, i.e.

$$\forall \sigma : \vdash_{\mathsf{L}} \sigma A \text{ implies } \vdash_{\mathsf{L}} \sigma B.$$

We write $A \mathrel{|\!\sim}_{\mathsf{L}} B$ if $A/B$ is an admissible rule of $\mathsf{L}$. The rule is called *derivable* if $A \vdash_{\mathsf{L}} B$ and *non-derivable* if $A \nvdash_{\mathsf{L}} B$. When $R$ is the rule $A/B$, we write $R^{\to}$ for the implication $A \to B$. We say that a collection $R$ of rules, e.g. $V$, is admissible for $\mathsf{L}$ if all rules in $R$ are admissible for $\mathsf{L}$. $R$ is derivable for $\mathsf{L}$ if all rules in $R$ are derivable for $\mathsf{L}$. We write $A \vdash^R_{\mathsf{L}} B$ if $B$ is derivable from $A$ in the logic consisting of $\mathsf{L}$ extended with the rules $R$, i.e. there are $A = A_1, \dots, A_n = B$ such that for all $i < n$, $A_i \vdash_{\mathsf{L}} A_{i+1}$ or there exists a $\sigma$ such that $\sigma B_i/\sigma B_{i+1} = A_i/A_{i+1}$ and $B_i/B_{i+1} \in R$. If $X$ and $R$ are sets of admissible rules of $\mathsf{L}$, then $R$ is a *basis for* $X$ if for every rule admissible rule $A$ / $B$ in $X$ we have $A \vdash^R_{\mathsf{L}} B$. If $X$ consists of all the admissible rules of $\mathsf{L}$, then $R$ is called a *basis for the admissible rules of* $\mathsf{L}$.

### 3.2 Kripke models

A Kripke models $K$ is a triple $(W, \preccurlyeq, \Vdash)$, where $W$ is a set (the set of *nodes*) with a unique least element that is called the *root*, $\preccurlyeq$ is a partial order on $W$ and $\Vdash$, the *forcing relation*, a binary relation on $W$ and sets of propositional variables. The pair $(W, \preccurlyeq)$ is called the *frame* of $K$. The notion of truth in a Kripke model is defined as usual. We write $K \models A$ if $A$ is forced in all nodes of $K$ and say that $A$ *holds in* $K$. We write $K_k$ for the model which domain consists of all nodes $k \preccurlyeq k'$ and which partial order and valuation are the restrictions of the corresponding relations of $K$ to this domain.

### 3.3 Bounded morphisms

A map $f : (W, \preccurlyeq, \Vdash) \to (W', \preccurlyeq', \Vdash')$ is a *bounded morphism* when the following conditions hold

1. $k$ and $f(k)$ force the same atoms,
2. $k \preccurlyeq l$ implies $f(k) \preccurlyeq' f(l)$,
3. if $f(k) \preccurlyeq l$, then there is a $k' \succcurlyeq k$ in $W$ such that $f(k') = l$.

$K'$ is a *bounded morphic image* of $K$, $K \twoheadrightarrow K'$, whenever there is a surjective bounded morphism from $K$ to $K'$. It is well-known (see e.g. [2]) that when $f$ is a bounded morphism from $K$ to $K'$, then for all $k$ in $K$, for all formulas $A$: $k \Vdash A \Leftrightarrow f(k) \Vdash' A$. Thus if $K'$ is a bounded morphic image of $K$, it validates exactly the same formulas as $K$.

### 3.4 Extension properties

For Kripke models $K_1, \dots, K_n$, $(\sum_i K_i)'$ denotes the Kripke model which is the result of attaching one new node at which no propositional variables are forced, below all nodes in $K_1, \dots, K_n$. $(\sum \cdot)'$ is called the *Smorynski operator*. Two models $K, K'$ are *variants* of each other, written $KvK'$, when they have the same set of nodes and partial order, and their forcing relations agree on all nodes except possibly the root. A class of models $U$ has the *extension property* if for every finite family of models $K_1, \dots, K_n \in U$, there is a variant of $(\sum_i K_i)'$

which belongs to $U$. $U$ has the *weak extension property* if for every model $K \in U$, and every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \big( (\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 \twoheadrightarrow M) \big).$$

$U$ has the *offspring* property if for every model $K \in U$, and for every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \exists M_0 \big( (\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 + K)' v M_0 \ \wedge \ (M_0 \twoheadrightarrow M) \big).$$

A logic $\mathsf{L}$ has the extension (weak extension, offspring) property if it is sound and complete with respect to some class of models that has the extension (weak extension, offspring) property. Note that for all three properties the class of models involved does not have to be the class of *all* models of $\mathsf{L}$. However, we might as well require that, because in [10] it has been shown that if a logic has the offspring property, then so does the class of all its models. Since the class of all models of a logic is closed under submodels and bounded morphic images, this also implies that for logics

extension property $\Rightarrow$ offspring property $\Rightarrow$ weak extension property.

The reason that we have chosen the definition of offspring property as given above, not the most elegant one, is that it will turn out particularly useful for the application to various frame complete logics discussed in the last section. There are quite natural classes of models that satisfy the offspring property, e.g. the class of linear models, as the reader may wish to verify for himself.

If we would not restrict our models to rooted ones, the extension property and the weak extension property would be equivalent, at least for logics. Since we require our Kripke models to be rooted, there is a subtle difference between the two:

**Fact 4** If a logic $\mathsf{L}$ has the extension property, it has the disjunction property.

As there are logics that do not have the disjunction property, but that have the weak extension property, the latter is indeed stronger. We will see examples of such logics in Section 5.

## 4 Overview of general results

In this section we state the general results on the Visser rules and intermediate logics known so far. In the next section we'll discuss results on specific intermediate logics, which will often be applications of general theorems in this scetion. We will only be concerned with intermediate propositional logics, i.e. logics between (possibly equal to) $\mathsf{IPC}$ and $\mathsf{CPC}$.

## 4.1 Computability

The first results on admissible rules were by Rybakov and Ghilardi. In [20, 21] Rybakov showed that admissible derivability for IPC, $\mathrel{|\!\sim}$, is decidable. And in two beautiful papers [7] [8] Ghilardi presented a transparent algorithm for $\mathrel{|\!\sim}$ and established a connection between admissibility and unification. A description of these results falls outside the scope of this paper, we refer the reader to the cited literature instead.

## 4.2 The situation for IPC

First, let us briefly recall the situation for IPC. As said, this logic has many non derivable rules. In [11] it has been shown that the following collection of rules, the so-called Visser rules, forms a basis for the admissible rules of IPC. This means that all admissible rules can be derived from Visser's rules and the theorems of IPC. The Visser rules are the rules

$$V_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \vee C \;/\; \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j) \vee C.$$

$V$ denotes the collection $\{V_n \mid \ldots n = 1, 2, 3, \ldots\}$ of Visser's rules. The mentioned result is a syntactical characterization of the admissible rules of IPC. Based on the algorithm for admissibility given in [8] we constructed a proof system for admissibilty. This system is still very close to the algorithm, and at the moment Ghilardi's algorithm is by far the best method to check the admissibility of a given rule.

## 4.3 Remarks on Visser's rules

Visser's rules are an infinite collection of rules, that is, there is no $n$ for which $V_{(n+1)}$ is derivable in IPC extended by the rule $V_n$ [12]. Note that on the other hand $V_n$ is derivable from $V_{(n+1)}$ for all $n$. In particular, if $V_1$ is not admissible for a logic, then none of Visser's rules are admissible.

The independence of premise rule *IPR*

$$\neg A \to B \vee C \;/\; (\neg A \to B) \vee (\neg A \to C)$$

is a special instance of $V_1$. Having *IPR* admissible is strictly weaker than the admissibility of $V_1$; below we will see examples of logics for which the first one is admissible while the latter is not.

Note than when Visser's rules are admissible, then so are the rules

$$V_{nm} \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to \bigvee_{j=n+1}^{m} A_j) \vee C / \bigvee_{h=1}^{m}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_h) \vee C.$$

As an example we will show that $V_{13}$ is admissible for any logic for which $V_1$ is admissible. For simplicity of notation we take $C$ empty. Assume that $\vdash_{\mathsf{L}} (A_1 \to B) \to A_2 \vee A_3 \vee A_4$. Then by $V_1$, reading $A_2 \vee A_3 \vee A_4$ as $A_2 \vee (A_3 \vee A_4)$,

$$\vdash_{\mathsf{L}} \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big) \vee \big((A_1 \to B) \to A_3 \vee A_4\big).$$

A second application of $V_1$, with $C = \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big)$, gives

$$\vdash_{\mathsf{L}} \bigvee_{i=1}^{2} \big((A_1 \to B) \to A_i\big) \vee \bigvee_{i=1,3,4} \big((A_1 \to B) \to A_i\big).$$

Therefore, $\vdash_{\mathsf{L}} \bigvee_{i=1}^{4} \big((A_1 \to B) \to A_i\big)$.
In a similar way one can see that when $V_1$ is derivable for a logic, then so are all the Visser rules.

### 4.4 When Visser's rules are admissible

As we will see in this section, the Visser rules play an important role for other intermediate logics too.

**Theorem 5** [10] If $V$ is admissible for $\mathsf{L}$ then $V$ is a basis for the admissible rules of $\mathsf{L}$.

Thus, once Visser's rules are admissible we have a characterization of all admissible rules of the logic. In Section 5 it will be shown that there are some well-known intermediate logics to which this result applies. e.g. the Gabbay-de Jongh logics $\mathsf{D_n}$, De Morgan logic $\mathsf{KC}$, the Gödel logics $\mathsf{G_n}$, and Gödel-Dummett logic $\mathsf{LC}$. For all these logics Visser's rules are admissible, and whence form a basis for their admissible rules.
Note that Theorem 5 in particular provides a condition for having no nonderivable admissible rules.

**Corollary 6** If $V$ is derivable for $\mathsf{L}$ then $\mathsf{L}$ has no nonderivable admissible rules.

The Gödel logics and Gödel-Dummett logic are in fact examples of this, as for these logics Visser's rules are not only admissible but also derivable. For the Gabbay-de Jongh logics and De Morgan logic one can show that this is not the case (Section 5).

### 4.5 When are Visser's rules admissible?

For logics for which we have some knowledge about their Kripke models, a necessary condition for having the Visser rules admissible exist (for definitions see Section 3.4).

**Theorem 7** [10] For any intermediate logic $\mathsf{L}$, Visser's rules are admissible for $\mathsf{L}$ if and only if $\mathsf{L}$ has the offspring property.

**Theorem 8** [10] For any intermediate logic $\mathsf{L}$ with the disjunction property, Visser's rules are admissible for $\mathsf{L}$ if and only if $\mathsf{L}$ has the weak extension property.

All the results on specific intermediate logics mentioned above and proved in Section 5, use these conditions for admissibility.

### 4.6 When Visser's rule are not admissible

In the case that not all of the Visser rules are admissible we do not know of any general results that describes the admissible rules of such logics. Up till now there only exist some partial results on specific intermediate logics, stating that some Visser rule is not admissible or that the logic in question has nonderivable admissible rules (Section 5). These results at least imply that

**Fact 9** For every $n$, there are intermediate logics for which $V_n$ is admissible while $V_{n+1}$ is not, i.e. $V_1, \ldots, V_n$ are admissible and $V_{n+1}, V_{n+2} \ldots$ are not.

**Fact 10** There are intermediate logics for which none of the Visser rules are admissible, but that do have nonderivable admissible rules.

The logics of (uniform) effective realizability $\mathsf{UR}$ and $\mathsf{ER}$ are examples of logics that have nonderivable admissible rules but for which $V_1$ is not admissible respectively derivable. Interestingly, for both these logics, the same special instance of $V_1$, namely the Independence of Premise rule *IPR* is a nonderivable admissible rule. That the rule is admissible in both logics is no coincedence, as the next section shows.

### 4.7 Disjunction property

A logic $\mathsf{L}$ has the *disjunction property* if

$$\vdash_{\mathsf{L}} A \vee B \ \Rightarrow \vdash_{\mathsf{L}} A \text{ or } \vdash_{\mathsf{L}} B.$$

The disjunction property plays an interesting role in the context of admissible rules. First of all, in combination with the admissibility of Visser's rules it characterizes $\mathsf{IPC}$.

**Theorem 11** [11] The only intermediate logic with the disjunction property for which all of the Visser rules are admissible is $\mathsf{IPC}$.

This implies that if a logic has the disjunction property, not all of the Visser rules can be admissible. However, there is an instance of $V_1$ that will always be admissible in this case, namely *IPR*, see the section on Independence of Premise below.

Theorem 11 shows the implications of the disjunction property on the admissiblity of the Visser rules. The next theorem shows the implications of the disjunction property on the derivability of the Visser rules. The proof as given here contains a funny self-application of $V_1$.

**Proposition 12** If an intermediate logic $\mathsf{L}$ has the disjunction property, $V_1$ is not derivable in $\mathsf{L}$. Hence none of the Visser rules are then derivable in $\mathsf{L}$.

**Proof** Suppose $\mathsf{L}$ has the disjunction property and that $V_1$ is derivable in $\mathsf{L}$. Thus for $X = (p_1 \to q)$, $\mathsf{L}$ derives the following instance of $V_1$,

$$\mathsf{L} \vdash (X \to p_2 \vee p_3) \to \bigvee_{i=1}^{3} (X \to p_i).$$

Since $V_1$ is derivable, it is certainly admissible. Thus so is $V_{13}$ (see the Remarks on Visser's rules in the Introduction). Applying the rule (now with $A_1 = (X \to p_2 \vee p_3)$ and $A_i = (X \to p_i)$ for $i > 1$) then gives

$$\mathsf{L} \vdash \big((X \to p_2 \vee p_3) \to X\big) \vee \bigvee_{i=1}^{3} \big((X \to p_2 \vee p_3) \to (X \to p_i)\big).$$

Since $\mathsf{L}$ has the disjunction property, this would imply that at least one of $\big((X \to p_2 \vee p_3) \to (X \to p_i)\big)$, or $\big((X \to p_2 \vee p_3) \to X\big)$ is derivable in $\mathsf{L}$. However, these formulas are not even derivable in classical logic. $\Box$

#### 4.7.1 The restricted Visser rules

For logics $\mathsf{L}$ that do have the disjunction property, $A \mathrel{|\!\sim}_{\mathsf{L}} C$ and $B \mathrel{|\!\sim}_{\mathsf{L}} C$ implies $A \vee B \mathrel{|\!\sim}_{\mathsf{L}} C$. In the context of the Visser rules this implies that when the the following special instances of the Visser rules, the *restricted Visser rules*

$$V_n^- \quad (\bigwedge_{i=1}^{n} (A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \ / \ \bigvee_{j=1}^{n+2} (\bigwedge_{i=1}^{n} (A_i \to B_i) \to A_j),$$

are admissible for $\mathsf{L}$, then so are the Visser rules. Therefore, when considering only logics with the disjunction property, like e.g. $\mathsf{IPC}$, the difference between the Visser and the restricted Visser rules does not play a role. However, when considering intermediate logics in all generality, as we do in this paper, we cannot restrict ourselves to this sub-collection of the Visser rules.

### 4.8 Independence of Premise

Although we have encountered logics for which $V_1$ is not admissible, there is an instance of this rule, an instance of $V_1^-$ in fact, that is admissible for all intermediate logics: the rule *IPR*

$$\neg A \to B \vee C \;/\; (\neg A \to B) \vee (\neg A \to C).$$

**Theorem 13** (Minari and Wronski [15]) For any intermediate logic $\mathsf{L}$, we have ($\mathcal{H}$ is the class of Harrop formulas, see preliminaries):

$$\forall A \in \mathcal{H}: \; \mathsf{L} \vdash (A \to B \vee C) \;\Rightarrow\; \mathsf{L} \vdash (A \to B) \vee (A \to C).$$

Since any negation is a Harrop formula this is a strengthening of the following theorem by Prucnal from 1979.

**Theorem 14** (Prucnal [18]) In any intermediate logic the rule *IPR* is admissible.

Note that on the other hand we cannot conclude that for every Harrop formula $A$ we have $(A \to B \vee C) \mathrel{|\!\sim}_{\mathsf{L}} (A \to B) \vee (A \to C)$, as the class of Harrop formulas is not closed under substitution.

In fact, the above theorem even holds for a wider class of formulas than the Harrop formulas. In [7], Ghilardi defined the notion of *projective formulas*, which are the formulas which class of models is closed under the extension property (see preliminaries). This class of formulas contains the class of Harrop formulas (also see preliminaries, Remark **??**).

**Theorem 15** For any intermediate logic $\mathsf{L}$, for any projective formula $A$,

$$\mathsf{L} \vdash (A \to B \vee C) \;\Rightarrow\; \mathsf{L} \vdash (A \to B) \vee (A \to C).$$

**Proof** In [7] it has been shown that for any projective formula $A$ there is a substitution $\sigma_A$ such that

$$\mathsf{IPC} \vdash \sigma_A(A) \;\text{ and } \forall B : A \vdash_{\mathsf{IPC}} B \leftrightarrow \sigma_A(B).$$

(In fact, Ghilardi defined projective formulas in this way and showed that they have the extension property, but that is not relevant here.) Given this fact, the proof is complety analoguous to the Minari-Wronski theorem. Assume $\mathsf{L} \vdash (A \to B \vee C)$ for some projective $A$. Since $\mathsf{IPC} \vdash \sigma_A(A)$, also $\mathsf{L} \vdash \sigma_A(A)$. Hence $\mathsf{L} \vdash \sigma_A(B) \vee \sigma_A(C)$. As also $\mathsf{L} \vdash \sigma_A(B) \to (A \to B)$ and $\mathsf{L} \vdash \sigma_A(C) \to (A \to C)$, the result follows. $\square$

To see that the last theorem is a strengthening of Theorem 13 we have to show the class of Harrop formulas is properly contained in the class of projective formulas. Thatthe containment is proper follows from the fact that $(p \to q \vee r)$ is projective but not a Harrop formula. That the Harrop formulas are projective follows from the fact that the class of models of a Harrop formula has the extension poperty. Here follows the argument. Note that every Harrop formula $H$ is equivalent to a conjunction $\bigwedge_{i=1}^{n}(A_i \to p_i)$, where the $p_i$ are atoms. Given models $K_1, \ldots, K_m$ of $H$, we construct a variant $K$ of $(\sum_i K_i)'$ by forcing at the root all $p_i$ that are forced in all $K_1, \ldots, K_m$. Why is this a model of $H$? If $A_i$ is not forced in the root, $(A_i \to p_i)$ is forced in $K$ because it is forced in the $K_j$. If $A_i$ is forced at the root, it follows that $A_i$ is forced in the $K_j$. Hence $p_i$ is forced in the $K_j$, and thus $p_i$ is forced at the root by definition. Therefore, $(A_i \to p_i)$ is forced in $K$ also in this case.

The principle $IPR^{\to}$ is denoted $IP$ and called Independence of Premise:

$$IP \qquad (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

Observe that Theorem 14 implies the following corollary, which we will use in the next section to show that certain logics have nonderivable admissible rules.

**Corollary 16** If *IPR* is not derivable in a logic, i.e. if the principle *IP* does not belong to the logic, then the logic has nonderivable admissible rules.

### 4.9 General remarks

For completeness sake we include the following known facts about admissibility that states which rules might come up as admissible rules for a logic.

**Fact 17** If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then $\mathsf{CPC} \vdash A \to B$.

**Proof** Suppose $A \mathrel{|\!\sim}_{\mathsf{L}} B$. This means that for all $\sigma$, $\vdash_{\mathsf{L}} \sigma A$ implies $\vdash_{\mathsf{L}} \sigma B$. Suppose the variables that occur in $A$ and $B$ are among $p_1 \ldots p_n$. Consider $\sigma \in \{\top, \bot\}^n$. Note that for such $\sigma$, $\vdash_{\mathsf{CPC}} \sigma A$ iff $\vdash_{\mathsf{IPC}} \sigma A$ iff $\vdash_{\mathsf{L}} \sigma A$. Whence for all $\sigma \in \{\top, \bot\}^n$, if $\vdash_{\mathsf{CPC}} \sigma A$ then $\vdash_{\mathsf{CPC}} \sigma B$. Thus $\vdash_{\mathsf{CPC}} A \to B$. $\Box$

**Corollary 18** If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then the logic that consists of $\mathsf{L}$ extended with the axiom scheme $(A \to B)$ is consistent.

## 5 Results

In this section we collect the results on specific intermediate logics discussed in the introduction. We present proofs of the observations that are new, and refer to the literature for the ones that have been obtained before.

### 5.1 The Visser rules are admissible

**Theorem 19** The Visser rules are derivable in $\mathsf{Bd}_1$, $\mathsf{G_k}$, $\mathsf{LC}$, $\mathsf{Sm}$ and $\mathsf{V}$. Hence these logics do not have nonderivable admissible rules.

**Proof** Proof in [10]. For the first four logics one uses the fact that these logics are complete with respect to classes of linear frames, and the fact that $V_1^{\to}$ (Section 3.1) holds on these frames, which implies (Section 4.3) that $V_n^{\to}$ holds on these frames for all $n$. $\Box$

**Theorem 20** The Visser rules form a basis for the admissible rules of the logics $\mathsf{KC}$, $\mathsf{M_n}$ and $\mathsf{ND}_m$ ($m \geq 3$). Visser's rules are not derivable in any of these logics. However, the rule *IPR* is derivable in these logics.

**Proof** We leave the second part of the lemma, showing the derivability of *IPR* in the logics, to the reader (use their frame completeness). We turn to the Visser rules. For the first two logics the statement has been proved in [10]. For $\mathsf{ND}_m$ we treat the case $m = 3$, the other cases are similar. We show that $\mathsf{ND}_3$ has the offspring property, from which it follows that the Visser rules are admissible in the logic by Theorem 7. Let $U$ be the class of models based on the frames for $\mathsf{ND}_3$: $F_1$ consists of one node, $F_2$ of two nodes $k_0 \preccurlyeq k_1$ and $F_{31} = (\{k_0, k_1, k_2\}, \{(k_0, k_1), (k_0, k_2)\}$ and $F_{32} = (\{k_0, k_1, k_2\}, \{(k_0, k_1), (k_1, k_2), (k_0, k_2)\}$. We show that for any of these frames $F$, for any model $K$ on $F$, there is a variant $M_1$ of $(\Sigma_i K_{l_i})'$ such that $K$ is a bounded morphic image of a variant $M_0$ of $(M_1 + K)'$. This will show that $\mathsf{TF}$ has the offspring property. We leave the proof of this for the linear frames $F_1$, $F_2$ and $F_{32}$ to the reader (force at the root of $M_1$ and $M_0$ the same atoms as at the root of $K$).

We treat $F_{31}$. Let $K$ be a model based on $F_{31}$. Pick nodes $l_1, \ldots, l_n$ in $K$ distinct from the root. We have to show that there is a variant $M_1$ of $(\Sigma_i K_{l_i})'$ such that a bounded morphic image $M$ of a variant $M_0$ of $(M_1 + K)'$ has at most three nodes. If $n = 1$, say $l_1 = k_1$, then $((\Sigma K_{l_1})')'$ has frame $F_{32}$. Whence $((\Sigma K_{l_1})')'$ belongs to $U$ and we are done. If $n = 2$, we can force at the roots $m_1, m_0$ of the variants $M_1, M_0$ the same atoms as at $k_0$. We leave it to the reader to verify that $K$ is a bounded morphic image of $M_0$.

To see that the Visser rules are not derivable in $\mathsf{ND}_3$, we leave it to the reader to construct appropriate countermodels to $V_1^{\rightarrow}$, i.e. to

$$((p_1 \to q) \to p_2 \vee p_3) \to \bigvee_{i=1}^{3} (p_1 \to q) \to p_i,$$

which is an instance of $V_1^{-}$. Whence none of Visser's rules can be derivable, because $V_n^{\rightarrow}$ implies $V_1^{\rightarrow}$ (Section 4.3). $\square$

Note that all the logics in the previous theorem are examples of logics which have the weak extension property, but not the extension property, as they do not have the disjunction property (see Fact 4). That they do not have the disjunction property follows from the fact that the only logic with the disjunction property for which all Visser's rules are admissible is $\mathsf{IPC}$, Theorem 11.

Next we consider intermediate logics for which a full characterization of their admissible rules is not known. We will see

- examples of logics for which *some* but not all of the Visser rules are nonderivable admissible rules: the logics $\mathsf{D}_n$.
- an example of a logic for which *none* of the Visser rules are admissible but that has nonderivable admissible rules (the rule *IPR*): the logic $\mathsf{UR}$.

- example of a logic for which *none* of the Visser rules are admissible but in which $IPR$ is derivable: the logic $\mathsf{KP}$.

Finally, we discuss some logics for which we do not know whether the Visser rules are admissible or not:

- the logics $\mathsf{Bd_n}$ $(n \geq 2)$, in which the restricted Visser rules are nonderivable admissible rules.
- the logic $\mathsf{ML}$, in which *IPR* is derivable, and the logic $\mathsf{EU}$, in which *IPR* is a nonderivable admissible rule.

### 5.2 *IPR* is a nonderivable admissible rule

**Theorem 21** [10] The restricted Visser rules and *IPR* are admissible but not derivable for $\mathsf{Bd_n}$ for $n \geq 2$.

**Theorem 22** [11] For the logics $\mathsf{D_n}$ $(n \geq 1)$, $V_{n+1}$ is admissible, while $V_{n+2}$ is not. In none of the logics Visser's rules or *IPR* are derivable.

**Proof** The first part has been proved in [11]. For the sceond part, it suffices to construct a countermodel to the principle *IP* in $\mathsf{D_1}$. This will show that *IP* does not hold $\mathsf{D_n}$, and whence that *IPR* and $V_1$, and thus $V_m$, cannot be derivable in $\mathsf{D_n}$. We leave the construction of the countermodel to the reader. □

**Proposition 23** (with Jaap van Oosten) $V_1$ is not admissible in $\mathsf{UR}$. *IPR* is a nonderivable admissible rule of $\mathsf{UR}$ and $\mathsf{ER}$ (and thus $V_1$ is not derivable in $\mathsf{ER}$).

**Proof** It is convenient to assume that our coding of pairs and recursive functions is such that $\langle 0, 0\rangle = 0$ and $0{\cdot}x = 0$ for all $x(a{\cdot}b$ denotes the result of applying the $a$-th partial recursive function to $b$). Then 0 realizes every negation of a sentence that has no realizers.

First, we show that $V_1$ is not admissible for $\mathsf{UR}$. In [19] G.F. Rose showed that the following formula, not derivable in $\mathsf{IPC}$, belongs to $\mathsf{UR}$: for $A = \neg p \vee \neg q$,

$$\mathsf{UR} \vdash \big((\neg\neg A \to A) \to \neg\neg A \vee \neg A\big) \to \neg\neg A \vee \neg A.$$

Let $B = \big((\neg\neg A \to A) \to \neg\neg A \vee \neg A\big)$. If the 1st Visser rule $V_1$ would be admissible, this would imply that

$$\mathsf{UR} \vdash (B \to \neg\neg A) \vee (B \to \neg A) \vee (B \to (\neg\neg A \to A)).$$

The fact that $\mathsf{UR}$ has the disjunction property, plus some elementary logic, leads to

$$\mathsf{UR} \vdash (B \to \neg\neg A) \text{ or } \mathsf{UR} \vdash (B \to \neg A) \text{ or } \mathsf{UR} \vdash (\neg\neg A \to A).$$

As classical logic does not even derive $(B \to \neg\neg A)$ or $(B \to \neg A)$, certainly $\mathsf{UR} \nvdash (B \to \neg\neg A)$ and $\mathsf{UR} \nvdash (B \to \neg A)$. Also $\mathsf{UR} \nvdash (\neg\neg A \to A)$. For if not, there is a realizer $e$ of every substitution instance $\neg\neg(\neg\varphi \vee \neg\psi) \to \neg\varphi \vee \neg\psi$ of $(\neg\neg A \to A)$. From this we derive a contradiction as follows. Thus for all $x$ such that $x\mathbf{r}\neg\neg(\neg\varphi\vee\neg\psi)$, $(e\cdot x)_0 = 0$ and $(e\cdot x)_1\mathbf{r}\neg\varphi$, or $(e\cdot x)_0 = 1$ and $(e\cdot x)_1\mathbf{r}\neg\psi$. Take $\varphi = \bot$ and $\psi = \top$. Let $\chi = (\neg\varphi \vee \neg\psi)$ and $\chi' = (\neg\psi \vee \neg\varphi)$. Note that $\forall y\neg(y\mathbf{r}\neg\chi)$ and $\forall y\neg(y\mathbf{r}\neg\chi')$. Since for all $\phi$

$$x\mathbf{r}\neg\neg\phi \ \leftrightarrow \forall y\neg(y\mathbf{r}\neg\phi),$$

this implies that every number, in particular 0, is a realizer of $\neg\neg\chi$ and $\neg\neg\chi'$. Whence $(e \cdot 0)$ is a realizer of both $\chi$ and $\chi'$. If $(e \cdot 0)_0 = 0$, then $(e \cdot 0)_1\mathbf{r}\neg\psi$, and if $(e \cdot 0)_0 = 1$, then $(e \cdot 0)_1\mathbf{r}\neg\psi$ too. As $\neg\psi$ cannot have a realizer, we have reached the desired contradiction.

To show that $\neg A \to B_0 \vee B_1 \mathrel{|\!\!\sim}_{\mathsf{UR}} (\neg A \to B_0) \vee (\neg A \to B_1)$, assume that $\mathsf{UR} \vdash \neg A \to B_0 \vee B_1$, for some $A, B_0, B_1$, and suppose that the atoms that occur in $A, B_0, B_1$ are $p_1, \dots, p_n$. So there is a number $e$ such that for all $\psi_1, \dots, \psi_n$, $e$ realizes $(\neg A \to B_0 \vee B_1)(\psi_1, \dots, \psi_n)$. We write $A(\bar{\psi})$ for $A(\psi_1, \dots, \psi_n)$, and similarly for $B_0, B_1$. We have to construct a realizer that, for all $\psi_1, \dots, \psi_n$, realizes

$$(\neg A(\bar{\psi}) \to B_0(\bar{\psi})) \vee (\neg A(\bar{\psi}) \to B_1(\bar{\psi})). \tag{1}$$

Since we reason classically, as we consider uniform effective realizability, either $\exists x(x\mathbf{r}\neg A(\bar{\psi}))$ or $\forall x\neg(x\mathbf{r}\neg A(\bar{\psi}))$. Thus by the definition of realizability, $\forall x\neg(x\mathbf{r}A(\bar{\psi}))$ or $\forall x\neg(x\mathbf{r}\neg A(\bar{\psi}))$. In the first case, $e \cdot 0\mathbf{r}(B_0(\bar{\psi}) \vee B_1(\bar{\psi}))$. Thus for $i = 0, 1$, if $(e\cdot 0)_0 = i$, $(e\cdot 0)_1\mathbf{r}B_i((\bar{\psi})$, whence if $d$ is the code of the program that always outputs $(e \cdot 0)_1$, then $< i, d >$ realizes (1). In the second case, $\forall x\neg(x\mathbf{r}\neg A(\bar{\psi}))$, $< e, 0 >$ realizes (1), as $\neg A(\bar{\psi})$ has no realizers.

Jaap van Oosten in [17] showed that *IPR* is not derivable in $\mathsf{ER}$, which implies that *IPR* is a non-derivable admissible rule of both $\mathsf{ER}$ and $\mathsf{UR}$ by Corollary 14. Thus, to finish the proof of the theorem, it remains to prove the non-derivability of *IPR* in $\mathsf{ER}$. We repeat van Oosten's proof, as given in [17]:

Let $A(f)$ be the sentence $\forall x\exists yT(f, x, y)$ and let $B(f)$ and $C(f)$ be negative sentences, expressing "there is an $x$ on which $f$ is undefined, and the least such $x$ is even" (respectively, odd). Suppose there is a total recursive function $F$ such that for every $f$, $F(f)$ realizes

$$(\neg A(f) \to B(f) \vee C(f)) \ \to \ ((\neg A(f) \to B(f)) \vee (\neg A(f) \to C(f))).$$

Choose, by the recursion theorem, an index $f$ of a partial recursive function of two variables, such that:

$f\cdot(g, x) = 0$ if there is no $w \leq x$ witnessing that $F(S^1_1(f, g))\cdot g$ is defined, or if $x$ is the least such witness, and *either* $(F(S^1_1(f, g))\cdot g)_0 = 0$ and $x$ is even, *or* $(F(S^1_1(f, g))\cdot g)_0 \neq 0$ *and* $x$ is odd;
$f\cdot(g, x)$ is undefined in all other cases.

Then for every $g$ we have:

- $F(S^1_1(f, g))\cdot g$ is defined. For otherwise, $f\cdot(g, x) = 0$ for all $x$, hence $S^1_1(f, g)$ is total, so $g$ realizes

$$\neg A(S^1_1(f, g)) \to B(S^1_1(f, g)) \vee C(S^1_1(f, g)),$$

which would imply that $F(S^1_1(f,g)) \cdot g$ is defined, a contradiction;

- If $(F(S^1_1(f,g)) \cdot g)_0 = 0$ then the first number on which $S^1_1(f,g)$ is undefined is odd, so $C(S^1_1(f,g))$ holds;
- If $(F(S^1_1(f,g)) \cdot g)_0 \neq 0$ then $B(S^1_1(f,g))$ holds.

Now let, again by the recursion theorem, $g$ be chosen such that for all $y$:

$$g \cdot y = \begin{cases} \langle 1,0 \rangle & \text{if } (F(S^1_1(f,g)) \cdot g)_0 = 0 \\ \langle 0,0 \rangle & \text{if } (F(S^1_1(f,g)) \cdot g)_0 \neq 0 \end{cases}$$

Then $g$ is a realizer for $\neg A(S^1_1(f,g)) \rightarrow [B(S^1_1(f,g)) \vee C(S^1_1(f,g))]$. However, it is easy to see that $F(S^1_1(f,g)) \cdot g$ makes the wrong choice

This finishes van Oosten's proof that *IPR* is not derivable in $\mathsf{ER}$, and thereby the proposition is proved. $\square$

### 5.3 *IPR* is derivable

**Proposition 24** *IPR* is derivable in $\mathsf{ML}$. $V_1$ is not derivable in $\mathsf{ML}$.

**Proof** That $V_1$ is not derivable in $\mathsf{ML}$ follows from Proposition 12, because the logic has the disjunction property [5]. To see that *IPR* is derivable in $\mathsf{L}$, i.e. that *IP* is a principle of $\mathsf{ML}$, we use the frame characterization of $\mathsf{ML}$ given above. The proof is left to the reader. $\square$

As mentioned above, we do not know whether the Visser rules are admissible in $\mathsf{ML}$. For the following logics we do not know whether they have nonderivable admissible rules, although we know that the Visser rules are not admissible.

**Proposition 25** [10] *IPR* is derivable in $\mathsf{KP}$. $V_1$ is not admissible for $\mathsf{KP}$.

### 5.4 The Rieger-Nishimura formulas

**Proposition 26** For the logics

| | |
|---|---|
| $\mathsf{NL_n}$ $(n \geq 9,\ n \text{ odd})$ | $V_1$ is not admissible |
| $\mathsf{NL_n}$ $(n \geq 9,\ n \text{ even})$ | $V_1^-$ is not admissible (whence $V_1$ is not admissible too) |
| $\mathsf{NL_n}$ $(n = 5, 8)$ | the Visser rules are admissible and nonderivable |
| $\mathsf{NL_n}$ $(n \leq 4, n = 6)$ | the Visser rules are derivable. |

We do not know what the situation is for $n = 7$.

**Proof** Observe that for $n = 0, 1, 2, 4$ the logic is inconsistent ($nf_4 \equiv \neg\neg p$), for $n = 5, 8$ it is equal to $\mathsf{KC}$ [16], and for $n = 3, 6$ it is $\mathsf{CPC}$ ($nf_6 \equiv \neg\neg p \rightarrow p$, substituting $A \vee \neg A$ for $p$ shows that the corresponding logic is $\mathsf{CPC}$). This treats the cases $n \leq 6$ and $n = 8$. For $n \geq 9$ we show that $V_1$ is not admissible for $\mathsf{NL_n}$. Since for even $n \geq 10$ the logics $\mathsf{NL_n}$ have the disjunction property

[22], this will imply that $V_1^-$ is not admissible for $n \geq 10$ (see the section on the disjunction property), and whence prove the theorem.

To prove that $V_1$ is not admissible, we will use the following fact.

**Fact 27** [16] $\mathsf{NL_n} \nvdash \mathsf{NL_m}$ for all $7 \leq m < n$.
For all $l$, for all $k \geq l+3$: $\mathsf{IPC} \vdash (nf_l \to nf_k)$.
For all $l$: $\mathsf{IPC} \vdash (nf_{2l+2} \vee nf_{2l} \equiv nf_{2l+3})$ (use line above).

The main ingredient of the proof is the following claim.

**Claim** For all $n$, if $V_1$ is admissible for $\mathsf{NL}_n$, then for all even $k \geq 8$, for all $A$,

$$\mathsf{NL}_n \vdash nf_k \vee A \;\Rightarrow\; \mathsf{NL}_n \vdash nf_{k-4} \vee nf_{k-6} \vee A. \tag{2}$$

**Proof of the Claim** Assume $V_1$ is admissible for $\mathsf{NL}_n$ and $\mathsf{NL}_n \vdash nf_k \vee A$ for some even $k \geq 8$. Note that the assumption that $k \geq 8$ guarantees that $nf_{k-8}, \dots, nf_k$ are all well-defined. We will use the observation that the admissibility of $V_1$ for $\mathsf{NL}_n$ implies that for even $m$:

$$\mathsf{NL}_n \vdash nf_m \vee A \;\Rightarrow\; \mathsf{NL}_n \vdash (nf_{m-2} \to nf_{m-4}) \vee (nf_{m-2} \to nf_{m-5}) \vee A. \tag{3}$$

To see that (3) holds, observe that

$$nf_m = nf_{m-2} \to nf_{m-3} = (nf_{m-4} \to nf_{m-5}) \to nf_{m-4} \vee nf_{m-5}$$

since $m$ is even. Application of $V_1$ to this formula then gives

$$(nf_{m-2} \to nf_{m-4}) \vee (nf_{m-2} \to nf_{m-4}) \vee (nf_{m-2} \to nf_{m-5}) \vee A.$$

This shows that (3) holds.

Another observation we will apply is that

$$\forall k \geq l+1 (l \text{ even}) : \; \mathsf{NL}_n \vdash (nf_k \to nf_l) \to (nf_{l-2} \to nf_{l-3}) \vee A. \tag{4}$$

To see that (4) holds, observe that $(nf_k \to nf_l) = nf_k \to (nf_{l-2} \to nf_{l-3}) \equiv nf_k \wedge nf_{l-2} \to nf_{l-3}$, and then apply the second part of Fact 27.

We return to the proof of the claim. Since $k$ is even we can apply (3) and obtain

$$\mathsf{NL}_n \vdash (nf_{k-2} \to nf_{k-4}) \vee (nf_{k-2} \to nf_{k-5}) \vee A. \tag{5}$$

Using that

$$nf_{k-2} \to nf_{k-5} = (nf_{k-4} \to nf_{k-5}) \to nf_{k-6} \vee nf_{k-7},$$

we can apply $V_1$ to (5) again. This gives

$$\mathsf{NL}_n \vdash (nf_{k-2} \to nf_{k-4}) \vee (nf_{k-2} \to nf_{k-6}) \vee (nf_{k-2} \to nf_{k-7}) \vee A.$$

Applying (4) to the first disjunt and the second part of Fact 27 to the third disjunct leads to

$$\mathsf{NL}_n \vdash (nf_{k-6} \to nf_{k-7}) \vee (nf_{k-2} \to nf_{k-6}) \vee (nf_{k-6} \to nf_{k-7}) \vee A.$$

Using the definition of the $nf$'s this gives

$$\mathsf{NL}_n \vdash nf_{k-4} \vee (nf_{k-2} \to nf_{k-6}) \vee A. \tag{6}$$

Finally, we have to distinguish two cases. If $k \geq 9$, (4) shows that the second disjunct of (6) implies $nf_{k-8} \to nf_{k-9} = nf_{n-6}$. This leads to

$$\mathsf{NL}_n \vdash nf_{k-4} \vee nf_{k-6} \vee A. \tag{7}$$

If $k = 8$, the second disjunct of (6) is $nf_6 \to nf_2 = nf_6 \to \neg p \equiv nf_6 \wedge p \to \bot$, which is equivalent to $\neg p = nf_2 = nf_{k-6}$. This also leads to (7), as desired. This proves (2), and thereby the claim. $\square$

We continue with the proof of the theorem by showing that for all $n \geq 9$, the assumption that $V_1$ is admissible for $\mathsf{NL}_n$ leads to a contradiction. We treat the odd and even cases separately.

First, assume $V_1$ is admissible for $\mathsf{NL}_n$, for some even $n \geq 10$. Since $\mathsf{NL}_n \vdash nf_n$, application of the Claim (take $A$ empty) gives

$$\mathsf{NL}_n \vdash nf_{n-4} \vee nf_{n-6}. \tag{8}$$

We distinguish the cases $n = 10$ and $n \geq 12$. If $n = 10$, we have

$$nf_{n-4} \vee nf_{n-6} = nf_6 \vee nf_4 \equiv nf_7.$$

The equivalence follows from Fact 27. Together with (8) this implies $\mathsf{NL}_{10} \vdash \mathsf{NL}_7$, contradicting Fact 27. For the case of the even $n \geq 12$, a second application of the Claim, with $A = nf_{n-6}$, to (8) leads to $\mathsf{NL}_n \vdash nf_{n-8} \vee nf_{n-10} \vee nf_{n-6}$. Note that we can apply the Claim because $n \geq 12$ implies that $n - 4 \geq 8$. By the second part of Fact 27,

$$\mathsf{IPC} \vdash (nf_{n-6} \vee nf_{n-8} \vee nf_{n-10}) \to nf_{n-1}.$$

Hence $\mathsf{NL}_n \vdash nf_{n-1}$, and thus $\mathsf{NL}_n \vdash \mathsf{NL}_{n-1}$, which contradicts Fact 27.

Second, assume $V_1$ is admissible for $\mathsf{NL}_n$, for some odd $n \geq 9$. Observe that $\mathsf{NL}_n \vdash nf_{n-1} \vee nf_{n-2}$. Applying the Claim (with $A = nf_{n-2}$) gives

$$\mathsf{NL}_n \vdash nf_{n-5} \vee nf_{n-7} \vee nf_{n-2}. \tag{9}$$

Since $nf_{n-2} = nf_{n-3} \vee nf_{n-4}$ and $nf_{n-4} = nf_{n-5} \vee nf_{n-6}$ this gives

$$\mathsf{NL}_n \vdash nf_{n-3} \vee nf_{n-5} \vee nf_{n-6} \vee nf_{n-7}. \tag{10}$$

If $n = 9$, this disjunction is equal to $nf_6 \vee (nf_4 \vee nf_3) \vee nf_2 \equiv nf_6 \vee nf_5 \vee nf_2$. Using the second part of Fact 27 this is again equivalent to $nf_6 \vee nf_5 = nf_7$. Thus (10) implies $\mathsf{NL}_9 \vdash \mathsf{NL}_7$, contradicting Fact 27. For the odd $n \geq 11$, we apply the Claim again to (10), with $A = nf_{n-5} \vee nf_{n-6} \vee nf_{n-7}$. This can be done as $n \geq 11$, whence $n - 3 \geq 8$. This leads to

$$\mathsf{NL}_n \vdash nf_{n-5} \vee nf_{n-6} \vee nf_{n-7} \vee nf_{n-9}.$$

By the second part of Fact 27 this implies $\mathsf{NL}_n \vdash nf_{n-1}$, and thus $\mathsf{NL}_n \vdash \mathsf{NL}_{n-1}$, which contradicts Fact 27. $\square$

### 5.5 Some questions

There are too many questions to list them all. We list some of the most interesting ones that are related to the results discussed in this paper.

- Is the rule $V_{n+1}$ a basis for the admissible rules of $\mathsf{D_n}$?
- Which of the Visser rules (if any) are admissible for the logics $\mathsf{ER}$, $\mathsf{NL}_7$, or $\mathsf{ML}$?
- Do the logics $\mathsf{ML}$, $\mathsf{NL}_n$ ($n \geq 9$), or $\mathsf{KP}$ have nonderivable admissible rules?
- If $n$ is the largest $n$ for which $V_n$ is admissible for a logic with the disjunction property, do the rules $\{V_1, \ldots, V_n\}$, i.e. $\{V_n\}$, form a basis for the admissible rules of the logic? And a similar question for the $V_{mn}$ in case the logic does not have the disjunction property.
- Do there exist intermediate logics that have nonderivable admissible rules that are not instances of one of the Visser rules?
- Do there exist intermediate logics for which the restricted Visser rules are nonderivable admissible rules and the Visser rules are not?

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