# A(nother) characterization of Intuitionistic Propositional Logic

Rosalie Iemhoff
ILLC/Faculty of Science
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
iemhoff@science.uva.nl

**Abstract**

In [Iemhoff] we gave a countable basis $\mathcal{V}$ for the admissible rules of IPC. Here we show that there is no proper intermediate logic with the disjunction property for which all rules in $\mathcal{V}$ are admissible. This shows that, relative to the disjunction property, IPC is maximal with respect to its set of admissible rules. This characterization of IPC is optimal in the sense that no finite subset of $\mathcal{V}$ suffices. In fact, it is shown that for any finite subset $X$ of $\mathcal{V}$, for one of the proper intermediate logics $D_n$ constructed by De Jongh and Gabbay (1974) all the rules in $X$ are admissible. Moreover, the logic $D_n$ in question is even characterized by $X$: it is the maximal intermediate logic containing $D_n$ with the disjunction property for which all rules in $X$ are admissible. Finally, the characterization of IPC is proved to be effective by showing that it is effectively reducible to an effective characterization of IPC in terms of the Kleene slash by De Jongh (1970).

## 1 Introduction

In contrast with classical propositional logic CPC, intermediate logics can have nonderivable admissible rules. For instance, in [Rybakov 97] it is shown that intuitionistic propositional logic IPC has countably many nonderivable admissible rules. There are several very natural (as far as we know open) questions concerning intermediate logics and their admissible rules which trivialize once all the admissible rules of the logic under consideration are derivable but which appear to be rather complicated otherwise:

Let us call a logic $T$ with the disjunction property *maximal with respect to a set of admissible rules* $\mathcal{R}$ if all the rules in $\mathcal{R}$ are admissible for $T$ and there is no intermediate logic with the disjunction property which is a proper extension of $T$ for which all rules in $\mathcal{R}$ are admissible. For now, if $\mathcal{R}$ is the set of admissible rules of $T$ we just say that $T$ is *maximal*. Clearly, if $T$ is maximal with respect to some set of admissible rules, it is maximal. Maximal logics are characterized by their admissible rules plus the disjunction property: if for a logic with the disjunction property containing the maximal logic $T$, all the admissible rules of $T$ are admissible, it can only be $T$ itself. We use the terms 'characterized by its admissible rules plus the disjunction property' and 'maximal' interchangeably.

It may appear to the reader that a better definition of maximality (in this sense) would be one without a restriction to logics with the disjunction property. However, this restriction is more an empirical than a natural one (or is empirical natural ... ): the only interesting results we encountered on maximality with respect to admissible rules, were in the sense of maximality as defined above and not in the broader sense.

Note that if all rules in $\mathcal{R}$ are derivable in $T$ then $T$ is maximal with respect to $\mathcal{R}$ once it has no proper extensions with the disjunction property. For in this case any extension of $T$ derives all rules in $\mathcal{R}$. Which intermediate logics with nonderivable admissible rules are maximal and which are not? In [Iemhoff] we gave a countable basis $\mathcal{V}$ for the admissible rules of $\mathsf{IPC}$ (there is no finite basis for the admissible rules of $\mathsf{IPC}$, see [Rybakov 97]). Here we show that the only intermediate logic with the disjunction property for which all rules in this basis are admissible, is $\mathsf{IPC}$. This shows that $\mathsf{IPC}$ is maximal. This characterization of $\mathsf{IPC}$ is simple in the sense that by using infinite conjunctions the basis can be expressed as one rule.

Of course, having this characterization of $\mathsf{IPC}$ we want to know if it is optimal. By optimal we mean that there is no proper subset $\mathcal{R}$ of $\mathcal{V}$ such that $\mathsf{IPC}$ is already maximal with respect to $\mathcal{R}$. We will see that the characterization is indeed optimal. We show that for any finite subset $X$ of $\mathcal{V}$ there is a proper intermediate logic for which $X$ is admissible. The logic in question is even maximal with respect to $X$. For this we use the countably many proper intermediate logics $D_0, D_1, D_2, \ldots$ with the disjunction property which were constructed in [Gabbay, De Jongh 74]. We show that there is a correspondence between finite subsets of $\mathcal{V}$ and these logics. Any such $D_n$ is maximal with respect to a finite subset $X$ of $\mathcal{V}$ and for any finite subset $X$ of $\mathcal{V}$ there is a number $n$ such that $D_n$ is maximal with respect to $X$. Furthermore, it will turn out to be a trivial observation that any cofinal subset of the basis is equivalent, in terms of the admissible rules which are derivable from it, to the basis itself. Therefore, there is no proper subset of $\mathcal{V}$ with respect to which $\mathsf{IPC}$ is maximal. Moreover, it shows that the Gabbay-de Jongh logics are all maximal.

Still, there are a lot more open questions concerning maximality of logics than solved ones. To name a few: Are there any logics which are not maximal with respect to their admissible rules? If so, can any such logic be extended to an intermediate logic which is maximal with respect to its admissible rules? Given a set of rules $\mathcal{R}$ which are derivable in $\mathsf{CPC}$ there is, by definition, an intermediate logic for which all rules in $\mathcal{R}$ are admissible. But is there a intermediate logic which is maximal with respect to $\mathcal{R}$?

With the characterization of $\mathsf{IPC}$ we do not claim a completely new result since a similar result, a characterization of $\mathsf{IPC}$ in terms of the Kleene slash, was already obtained by De Jongh in 1970 (see Section 5). However, not only is the reduction of the one characterization to the other not trivial, but the connection with the admissible rules is new and interesting. We show that these characterizations are effectively reducible to each other. Hence the effectiveness of the characterization in terms of the Kleene slash [De Jongh 70] implies the effectiveness of the characterization in terms of the admissible rules.

I thank Dick de Jongh for introducing me to the subject, and for posing to me the question whether the basis characterizes $\mathsf{IPC}$. I thank both him, Lex Hendriks and Albert Visser for useful comments.

## 2 Preliminaries

Unless stated otherwise, formulas are meant to be formulas in a (fixed) language for intuitionistic propositional logic. The letters $A, B, C, D, E, F$ will always range over formulas and $p, q, r, s, t$ over propositional variables. We write $\vdash$ for derivability in $\mathsf{IPC}$.

An $\mathfrak{L}$-*substitution* $\sigma$ is a map which assigns to every propositional variable a formula in the language $\mathfrak{L}$. For a propositional formula $A$, we write $\sigma(A)$ for the result of applying $\sigma$ to $A$, i.e. for the result of substituting $\sigma(p)$ for any propositional variable $p$ in $A$. When $\mathfrak{L}$ is our fixed language of propositional logic mentioned above, we say 'substitution' instead of '$\mathfrak{L}$-substitution'.

An intermediate logic is a consistent theory in the language of propositional logic, containing $\mathsf{IPC}$, which is closed under substitution. For intermediate logics $T$ we will write $\vdash_T$ for derivations in $T$. If we only know that $T$ is a theory we write $T \vdash$ instead.

### 2.1 Admissible rules

A *rule* is an expression of the form

$$\frac{A_1 \ \ldots \ A_n}{B}.$$

We sometimes write $A_1, \ldots, A_n/B$ for this expression. We say that an expression

$$\frac{A'_1 \ldots A'_n}{B'},$$

is a substitution instance of such a rule when there is a substitution $\sigma$ such that $\sigma(A_i) = A'_i$ and $\sigma(B) = B'$.

Let $T$ be some theory in a language $\mathcal{L}$ . We say that a rule $A/B$ is an *admissible rule of* $T$, and write $A \mathrel{|\!\!\sim}_T B$, if

$$\text{for all } \mathcal{L}\text{-substitutions } \sigma\text{: if } T \vdash \sigma(A) \text{ then } T \vdash \sigma(B).$$

#### 2.1.1 Bases

For a set of rules $\mathcal{R}$ and a set of formulas $\mathcal{A}$, we say that $B$ is *derivable in* $T$ *by the set of rules* $\mathcal{R}$ *from assumptions* $\mathcal{A}$ when there is a sequence of formulas $(B_1, \ldots, B_n)$, where $B_n = B$, such that for every $i \leq n$ either $B_i \in \mathcal{A}$ or there are $B_{i_1}, \ldots, B_{i_m}$ with $i_j < i$ such that either

$$\vdash_T (B_{i_1} \wedge \ldots \wedge B_{i_m}) \rightarrow B$$

or

$$\frac{B_{i_1} \ldots B_{i_m}}{B_i}$$

is a substitution instance of some rule in $\mathcal{R}$.

We call a set of rules $\mathcal{R}$ a *basis* (in $T$) [Rybakov 97] for some other set of rules $\mathcal{R}' \supseteq \mathcal{R}$ if for every rule

$$\frac{A_1 \ldots A_n}{B}$$

in $\mathcal{R}'$, $B$ is derivable in $T$ by the rules $\mathcal{R}$ from the assumptions $A_1, \ldots, A_n$. We say that a set $\mathcal{R}$ of admissible rules of $T$ is a *basis for the admissible rules of* $T$ when $\mathcal{R}$ is a basis for the set of admissible rules of $T$.

In the setting of theories with the disjunction property

$DP$ if $T \vdash A \vee B$ then $T \vdash A$ or $T \vdash B$

the notion of a *subbasis* seems more natural. A set $\mathcal{R}$ of admissible rules of $T$ is a *subbasis for the admissible rules of* $T$ if the following collection of rules is a basis for the admissible rules of $T$;

$$\frac{A \vee p}{B \vee p}$$

where the rule $A/B$ is in $\mathcal{R}$ and $p$ does not occur in $A$ or $B$.

### 2.2 Kripke models.

A *frame* is a set with a partial order $\preccurlyeq$. We say that $x$ *is below* $y$ or $y$ *is above* $x$ when $x \preccurlyeq y$. A node $y$ is called *an immediate successor* of $x$ if $x \prec y$ and, besides $y$, no node $z$ which is above $x$ is below $z$. A *maximal node* is a node which has no nodes above it except itself.

With a *model* we always mean a Kripke model [Troelstra, Van Dalen 88]. We say that $A$ is *valid* in a Kripke model $K$ ($K \models A$) if it is valid at all nodes in the model. We use $\Vdash$ for the forcing relation of a Kripke model. We write $K_x$ for the model whose domain consists of all nodes $y \succcurlyeq x$ and whose partial order and forcing relation are the restrictions of the corresponding relations of $K$ to this domain. We write $K, x \Vdash A$ if we want to stress that $x \Vdash A$ holds *in the model* $K$.

For Kripke models $K_1, \ldots, K_n$ we let $(\sum_i K_i)'$ denote the Kripke model which is the result of attaching one new node at which no propositional variables are forced, below all nodes in $K_1, \ldots, K_n$ [Smoryński 73]. A *rooted* Kripke model is a Kripke model which contains one node which is below all other nodes in the model. We say that two rooted Kripke models are *variants* of each other when they have the same domain and partial order, and their forcing relations either do not differ or they only differ at the roots.

A theory $T$ has the *extension property up to* $n$ if for every family of *rooted* models $K_1, \ldots, K_n$ of $T$, there is a variant of $(\sum_i K_i)'$ which is a model of $T$ as well. A theory $T$ has the *extension property* if it has the extension property up to $n$, for all $n$.

A *modified Jaskowski frame* [Smoryński 73] is one of the frames $J_1, J_2, \ldots$ defined via:

$J_1$ consists of one node
$J_{n+1}$ is the result of attaching one node below $(n+1)$ copies of $J_n$.

(In [Smoryński 73] $J_i$ is denoted with $J_i^*$.) A *Jaskowski model* is a model based on a modified Jaskowski frame.

A *basic model* is a model for which the following holds:

- the only nodes that force propositional variables are maximal nodes,
- every maximal node forces exactly one propositional variable and no two maximal nodes force the same propositional variable.

For example, if $1, \dots, n$ are the maximal nodes of a frame $F$, then the model given by the valuation ($x \Vdash p_i$ iff $x = i$) is a basic model on $F$. A *basic Jaskowski model* is a basic model based on a modified Jaskowski frame. It is easy to see that the following fact about basic models holds.

**Fact 2.3** Let $F$ be a frame in which no two nodes have exactly the same maximal nodes above them. Consider the basic model on $F$. There are formulas $A_x$ such that $y \Vdash A_x$ iff $x \preccurlyeq y$. Namely, if $1, \dots, n$ are the maximal nodes above $x$ and $i \Vdash p_i$, then the formula $A_x = \neg\neg(p_1 \vee \dots \vee p_n)$ has the desired properties.

## 3 The characterization of IPC.

In [Iemhoff] we gave a subbasis for the admissible rules of IPC. To keep the definition of the rules of this subbasis readable, we will use the following abbreviation,

$$(A)(B_1, \dots, B_m) \equiv_{def} (A \to B_1) \vee \dots \vee (A \to B_m).$$

Furthermore, we adhere to some reading conventions as to omit parentheses when possible. The negation binds stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\to$.

**Definition 1** Let $\mathcal{V}$ be the collection of rules $\{V_n \mid n = 1, 2, \dots\}$, where we define rules $V_n$ as

$$V_n \quad (\textstyle\bigwedge_{i=1}^n (p_i \to q_i) \to r \vee s) \;/\; (\textstyle\bigwedge_{i=1}^n (p_i \to q_i))(r, s, p_1, \dots, p_n).$$

**Theorem 3.1** [Iemhoff] The set of rules $\mathcal{V}$ is a subbasis for the admissible rules of IPC.

The rest of the paper is devoted to the proof that these admissible rules together with the disjunction property characterize IPC. That is, we will show that for any intermediate logic which is not equal to IPC either the disjunction property does not hold or one of the rules $V_1, V_2, \ldots$ is not admissible. It is convenient to have the disjunction property built-in into the admissible rules. Therefore, we define the following.

**Definition 2** A theory $T$ has the property $P_n$ if for all substitutions $\sigma$,

$$\begin{aligned}&\text{if } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^{n}(p_i \to q_i) \to r \vee s) \text{ then}\\ &\quad \vdash_T \sigma(\textstyle\bigwedge_{i=1}^{n}(p_i \to q_i) \to r) \text{ or } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^{n}(p_i \to q_i) \to s) \text{ or}\\ &\quad \vdash_T \sigma(\textstyle\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_1) \text{ or } \ldots \text{ or } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_n)\end{aligned}$$

We will show that an intermediate logic is equal to IPC iff it has the property $P_n$, for all $n \geq 0$. The characterization mentioned above is an immediate corollary of this.

Note that having the property $P_0$ is equivalent to having the disjunction property. Moreover, having $P_n$ for all $n \geq 0$ is the same as having the disjunction property and the rules $V_n$ for $n \geq 1$ admissible.

We need a Fact by Smoryński:

**Fact 3.2** [Smoryński 73] IPC is complete with respect to Jaskowski models.

**Lemma 3.3** If an intermediate logic has the extension property it is the logic IPC.

**Proof** The lemma follows from the following two claims.

*Claim* If $T$ is an intermediate logic with the extension property, then every basic Jaskowski model is a model of $T$.

*Proof of the Claim* Let $T$ be an intermediate logic with the extension property, and let $K$ be a basic Jaskowski model (Section 2.2). We show that $K_x$ is a model of $T$ by induction to the depth of the node $x$. The maximal nodes of $K$ clearly are models of $T$ since every classical model is a model of $T$. Suppose $x$ is another node in $K$ and let $x_1, \ldots, x_n$ be the immediate successors of $x$, i.e. the nodes $y$ such that $x \prec y$ and such that there is no node $x \prec z \prec y$. By induction the models $K_{x_1}, \ldots, K_{x_n}$ are models of $T$. Observe that $K_x$ is the model $(\sum K_{x_i})'$ (Section 2.2). Because every propositional variable is valid at just one node in $K$ there is no other variant of $(\sum K_{x_i})'$ then the model itself. Since $T$ has the extension property this implies that $K_x$ is a model of $T$. This proves the Claim.

*Claim* If $T$ is an intermediate logic such that every basic Jaskowski model is a model of $T$, then $T = \mathsf{IPC}$.

*Proof of the Claim* We show that $T \subseteq \mathsf{IPC}$ by proving that if $\nvdash_{\mathsf{IPC}} A$ then $\nvdash_T A$. If $\nvdash_{\mathsf{IPC}} A$ then there is a Jaskowski model $K$ on which $A$ is not valid (Fact 3.2). Let $K'$ be a basic model based on the frame of $K$. By assumption $K'$ is a model of $T$.

Now we define a substitution $\sigma$ via $\sigma(p) = \bigvee_{K,x \Vdash p} A_x$, where the formulas $A_x$ are given by Fact 2.3. To see that $\sigma(A)$ is not valid at $K'$, observe that for every node $x$ and for every formula $B$ we have that $K, x \Vdash B$ iff $K', x \Vdash \sigma(B)$. Therefore, $\nvdash_T \sigma(A)$. Hence $\nvdash_T A$. **QED**

In the following lemma we need the notion of a *saturated set.* A $T$-saturated set $x$ is a set of formulas such that $A \in x$ or $B \in x$ whenever $x \vdash_T A \vee B$. In particular, a $T$-saturated set is closed under deduction in $T$.

**Lemma 3.4** If an intermediate logic has the property $P_n$ for every $n \geq 0$, then it has the extension property.

**Proof** Let $T$ be an intermediate logic with the disjunction property, for which, for all $n$, $V_n$ is admissible. Consider models $K_1, \ldots, K_n$ of $T$ with roots $x_1, \ldots, x_n$ respectively. From now on we confuse a node with the set of formulas it forces.

*Claim* There exists a $T$-saturated set $x \subseteq x_1 \cap \ldots \cap x_n$ such that for all $T$-saturated sets $x \subset y$ there is some $i \leq n$ such that $x_i \subseteq y$.

*Proof of the Claim* Consider

$$\Delta = \{(E \to F) \mid E \notin x_1 \cap \ldots \cap x_n \text{ and } F \in x_1 \cap \ldots \cap x_n\}.$$

Clearly, $\Delta \subseteq x_1 \cap \ldots \cap x_n$. Observe that the set $x_0 = \{A \mid \Delta \vdash_T A\}$ is $T$-saturated because for all $m$, the property $P_m$ holds. Now we construct a sequence of sets $x_0 = z_0 \subseteq z_1, \ldots$ as follows. Let $C_0, C_1, \ldots$ enumerate all formulas, with infinite repetition. Define the property $*(\cdot)$ on sets via

$*(y)$ iff for all $m$, for all $A_1, \ldots, A_m$: if $y \vdash_T A_1 \vee \ldots \vee A_m$, then $A_i \in x_1 \cap \ldots \cap x_n$ for some $i = 1, \ldots, m$.

Note that $*(z_0)$ holds. If $*(z_i \cup \{C_i\})$ does not hold then put $z_{i+1} = z_i$. If $*(z_i \cup \{C_i\})$ holds do the following: if $C_i$ is no disjunction, put $z_{i+1} = z_i \cup \{C_i\}$; if $C_i = D \vee E$, let $z_{i+1}$ be $z_i \cup \{D\}$ if $*(z_i \cup \{D\})$ holds and $z_i \cup \{E\}$ otherwise. It is easy to see that at least one of $*(z_i \cup \{D\})$ and $*(z_i \cup \{E\})$

has to hold. Therefore, $*(z_i)$ holds for all $i$. Let $x = \bigcup_i z_i$. Clearly, $x$ is $T$-saturated and $x \subseteq x_1 \cap \ldots \cap x_n$.

Finally, we have to see that for all $T$-saturated sets $x \subset y$ there is some $i \leq n$ for which $x_i \subseteq y$. Arguing by contradiction assume $y \supset x$ and $x_i \not\subseteq y$ for all $i \leq n$. From the construction of $x$ it is easy to see that $y \not\subseteq x_1 \cap \ldots \cap x_n$. Thus there are formulas $E \in y$, $E \notin x_1 \cap \ldots \cap x_n$ and $A_i \in x_i$, $A_i \notin y$, for all $i \leq n$. Hence $(E \to A_1 \vee \ldots \vee A_n) \in \Delta$. Thus $A_1 \vee \ldots \vee A_n \in y$, quod non. This proves the Claim.

Now we define a variant of $(\sum K_i)'$ by putting at the root $b$ of $(\sum K_i)'$, for propositional variables $p$, $b \Vdash p$ iff $p \in x$.

*Claim* For all formulas $B$: $b \Vdash B$ iff $B \in x$.

*Proof of the Claim* We prove this by formula-induction. The case of the propositional variables and the connectives $\wedge$ and $\vee$ is trivial. Consider a formula $B = (C \to D)$. If $(C \to D) \in x$ then it is easy to see that indeed $b \Vdash (C \to D)$. We prove that $x \Vdash B$ implies $B \in x$ by contraposition. Therefore, assume $(C \to D) \notin x$. It is not difficult to see that this implies the existence of a $T$-saturated set $y \supseteq x$ such that $C \in y$ and $D \notin y$. From the construction of $x$ it follows that $x = y$ or $x_i \subseteq y$ for some $i = 1, \ldots, n$. In the first case the induction hypothesis gives $b \Vdash C$ and $b \nVdash D$, thus $b \nVdash (C \to D)$. In the other case it follows that for some $i$, $x_i \nVdash (C \to D)$. Thus again we can conclude that $b \nVdash (C \to D)$. This proves the claim.

By the last claim the defined extension is a model of $T$. This proves that $T$ has the extension property. **QED**

These two lemmas lead to the following characterization of $\mathsf{IPC}$:

**Theorem 3.5** For any intermediate logic $T$ it holds that $T = \mathsf{IPC}$ iff $T$ has the property $P_n$ for every $n \geq 0$ .

**Corollary 3.6** For any intermediate logic $T$ it holds that $T = \mathsf{IPC}$ iff $T$ has the disjunction property and all the rules $V_n$ are admissible. Thus $\mathsf{IPC}$ is maximal with respect to $\mathcal{V}$ and hence maximal.

# 4 Optimality of the characterization

We show that no finite subset of the $P_n$ already characterizes $\mathsf{IPC}$. This proves our characterization to be optimal. Note that it is not interesting to consider infinite subsets of $P_0, P_1, P_1, \ldots$, since, for any logic, having the property $P_{m+1}$ implies having the property $P_m$.

We use logics $D_n$ ($n \geq 1$) given by Gabbay and de Jongh (1974). The logic $D_n$ axiomatized by

$$\bigwedge_{i=0}^{n+1}((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) \to \bigvee_{i=0}^{n+1} p_i.$$

We quote

**Theorem 4.1** ([Gabbay, De Jongh 74]) The intermediate logic $\mathsf{D_n}$ is a proper extension of $\mathsf{IPC}$ with the disjunction property. $\mathsf{D_n}$ is complete with respect to the class of finite trees in which every point has at most $(n+1)$ immediate successors.

Knowing this, it is easy to prove the following lemma.

**Lemma 4.2** The logic $D_n$ has the property $P_{n+1}$ and it does not have the property $P_{n+2}$.

**Proof** To see that $D_n$ has the property $P_{n+1}$, suppose $D_n$ derives $(A \to D \vee E)$, where $A = \bigwedge_{i=1}^{n+1}(B_i \to C_i)$, and suppose that $D_n$ does not derive $(A)(B_1, \ldots, B_{n+1}, D, E)$. By the disjunction property and the completeness of $D_n$ this implies that there are models $K_i$, such that $K_i \models A$ and, for $i \leq n+1$, $K_i \not\models B_i$ and $K_{n+2} \not\models D$ and $K_{n+3} \not\models E$. Furthermore, the frame of every $K_i$ is a finite tree in which every node does not have more than $(n+1)$ immediate successors. Consider $(((\sum_{i=1}^{n+1} K_i)' + K_{n+2})' + K_{n+3})'$. Clearly, the frame of this model is again a finite tree in which every node does not have more than $(n+1)$ immediate successors. In this model $A$ is valid while $(D \vee E)$ is not, contradicting the assumption that $D_n$ derives $(A \to D \vee E)$.

To see that $D_n$ does not have the property $P_{n+2}$ consider the axiomatization of $D_n$. It is easy to see, using the completeness of $D_n$ that $D_n$ does not derive

$$(\bigwedge_{i=0}^{n+1}((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j))((p_0 \to \bigvee_{j\neq 0} p_j), \ldots, (p_{n+1} \to \bigvee_{j\neq n+1} p_j)).$$

**QED**

**Corollary 4.3** No finite subset of $P_1, P_2, \ldots$ characterizes $\mathsf{IPC}$.

In fact, $D_n$ is characterized by $P_{n+1}$ in the same way as $\mathsf{IPC}$ is characterized by all the $P_0, P_1, \ldots$, see Corollary 4.7. The proof of this proposition is analogous to the one of Theorem 3.5: the next lemma is the analogue of Lemma 3.3 and the following one is the analogue of Lemma 3.4.

**Lemma 4.4** If an intermediate logic has the extension property up to $(n+1)$ it is contained in $D_n$.

**Proof** It easily follows from Theorem 4.1 that $\mathsf{D_n}$ is complete with respect to the class of the finite trees in which every point has at most $(n+1)$ immediate successors, and in which no two nodes have exactly the same maximal nodes above them. To be precise, the last property reads:

$$\forall x \forall y \exists z (x \neq y \rightarrow \neg \exists z' (z \prec z') \wedge ((x \preccurlyeq z \wedge y \not\preccurlyeq z) \vee (y \preccurlyeq z \wedge x \not\preccurlyeq z))).$$

Suppose $D_n \nvdash A$ and let $M$ be a model based on such a frame $F$ in which $A$ is not valid. Let $M'$ be a basic model on $F$ (Section 2.2). By the same reasoning as before it follows that $M'$ is a model of $T$. Define the substitution $\sigma$ via $\sigma(p) = \bigvee_{M,x \Vdash p} A_x$, where the formulas $A_x$ are given by Fact 2.3. Clearly,

$$M, x \Vdash B \text{ iff } M', x \Vdash \sigma(B).$$

Thus $\nvdash_T \sigma(A)$. Hence $\nvdash_T A$. **QED**

**Lemma 4.5** If an intermediate logic has the property $P_n$ it has the extension property up to $n$.

**Proof** Let $T$ be an intermediate logic that has the property $P_n$. The proof that $T$ has the extension property up to $n$ is completely similar to the proof of Lemma 3.4, except for one point, which we will explain. The rest of the proof we leave to the reader.

In the first Claim of Lemma 3.4 we define a set $\Delta$ and observe that, in the notation of this lemma, the set $x_0 = \{A \mid \Delta \vdash_T A\}$ is $T$-saturated because for all $m$, $P_m$ holds. In this case, having only $P_n$, this is the only place in the proof where we have to be careful. Assume $x_0 \vdash_T A \vee B$. Hence there are $E_1, \ldots, E_m \notin x_1 \cap \ldots \cap x_n$ and $F_1, \ldots, F_m \in x_1 \cap \ldots \cap x_n$ such that

$$\vdash_T \bigwedge_{i=1}^{m} (E_i \rightarrow F_i) \rightarrow A \vee B.$$

For $i \leq n$, let $G_i = \bigvee\{E_j \mid j \leq m, E_j \notin x_i\}$ and let $F = \bigwedge_{i=1}^{m} F_i$. Observe that $G_i \notin x_i$ and that $(G_i \to F) \in \Delta$. Clearly,

$$\vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to A \vee B.$$

And thus, since $T$ has $P_n$, we can conclude

$$\vdash_T (\bigwedge_{i=1}^{n}(G_i \to F))(G_1, \ldots, G_n, A, B).$$

Since $\bigwedge_{i=1}^{n}(G_i \to F) \in x_1 \cap \ldots \cap x_n$ while $G_i \notin x_1 \cap \ldots \cap x_n$, we have either

$$\vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to A \text{ or } \vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to B.$$

And because $x_0 \vdash_T \bigwedge_{i=1}^{n}(G_i \to F)$ either $x_0 \vdash_T A$ or $x_0 \vdash_T B$. And this proves that $x_0$ is $T$-saturated. **QED**

**Proposition 4.6** Any intermediate logic $T$ which has $P_{n+1}$ is contained in $D_n$.

**Corollary 4.7** For any intermediate logic $T \supseteq D_n$ it holds that $T = D_n$ iff $T$ has $P_{n+1}$. Thus $D_n$ is maximal with respect to $V_{n+1}$ and hence maximal.

Since the union of the $D_n$ is equivalent to $\mathsf{IPC}$, Theorem 3.5 follows from the previous proposition. However, we preferred to give a separate proof of the theorem in advance.

## 5 Effectiveness

De Jongh (1970) proved the following characterization of $\mathsf{IPC}$ in terms of the Kleene slash $|$ [Kleene 62]: $\mathsf{IPC}$ is the only intermediate logic $T$ satisfying

if $A \,|_T\, A$ and $\vdash_T (A \to B \vee C)$, then $\vdash_T (A \to B)$ or $\vdash_T (A \to C)$.

We remind the reader that the Kleene slash is defined as (using the abbreviation $\Gamma \Vdash_T A \equiv (\Gamma|_T A \text{ and } \Gamma \vdash_T A)$)

$$\begin{array}{lcl}
\Gamma \mid_T p & \equiv & \Gamma \vdash_T p \text{ for } p \text{ a propositional variable or } \bot \\
\Gamma \mid_T A \wedge B & \equiv & \Gamma \mid_T A \text{ and } \Gamma \mid_T B \\
\Gamma \mid_T A \vee B & \equiv & \Gamma \Vdash_T A \text{ or } \Gamma \Vdash_T B \\
\Gamma \mid_T A \to B & \equiv & \Gamma \Vdash_T A \text{ implies } \Gamma \mid_T B.
\end{array}$$

Moreover, De Jongh proved in the same paper that this characterization is an effective one: given any intermediate logic $T \neq \mathsf{IPC}$ we can obtain formulae $A, B, C$ such that $A \mid_T A, \vdash_T (A \to B \vee C)$ but $\nvdash_T (A \to B), \nvdash_T (A \to C)$ in an effective way. We show that the characterization in terms of the admissibles rules treated in this paper, is effective as well, by giving an effective reduction from the characterization in terms of the Kleene slash to the one in terms of the admissible rules.

Let us call, for now, a triple of formulas $A, B, C$ a *$J$-example* or an *$I$-example* of $T \neq \mathsf{IPC}$ if

$$A \mid_T A,\ \vdash_T (A \to B \vee C),\ \nvdash_T (A \to B),\ \nvdash_T (A \to C)$$

respectively $A = \bigwedge(D_i \to E_i)$ and

$$\vdash_T (A \to B \vee C),\ \nvdash_T (A \to B),\ \nvdash_T (A \to C),\ \nvdash_T (A \to D_i).$$

The following proposition is trivial except for the effectiveness.

**Proposition 5.1** For any intermediate logic $T \neq \mathsf{IPC}$ there is an effective way of creating an $I$-example from a $J$-example, and vice versa.

**Proof** During the proof $\vdash$, $\mid$ stand for $\vdash_T$, $\mid_T$ respectively. The second part of the proposition is easy: any $I$-example $A = \bigwedge(D_i \to E_i), B, C$ of $T \neq \mathsf{IPC}$ is a $J$-example because $\nvdash (A \to D_i)$ for all $i$, implies $A \mid A$.

For the other part, suppose $A, F, G$ is an $J$-example of $T \neq \mathsf{IPC}$. We are going to construct, in an inductive way, formulas $A_1, A_2, \ldots$ which are all equivalent to $A$ in $T$. Every $A_i$ is a conjunction of propositional variables, disjunctions and implications such that for the implications $(B \to C)$ either $A_i \mid (B \to C)$ or $A_i \nvdash B$, and for the disjunctions $B$, $A_i \mid B$. Note that $A$ is such a formula. Let $A_1 = A$. During the construction we will often use, without mentioning, the fact that if $E \mid F$ and $\vdash E \leftrightarrow E'$ then $E' \mid F$.

If $A_i$ is a conjunction in which one of the conjuncts is a disjunction (note that this captures the case that $A_i$ is a disjunction), let $(B \vee C)$ be the first such reading from left to right. Thus $A_i = D \wedge (B \vee C) \wedge E$ for some $D, E$. By assumption $A_i \mid (B \vee C)$. Hence $A_i \Vdash B$ or $A_i \Vdash C$. In the first case put $A_{i+1} = D \wedge B \wedge E$, in the second case $A_{i+1} = D \wedge C \wedge E$. Now consider the case that $A_i$ is a conjunction of implications and propositional variables. If every conjunct either is a propositional variable or an implication $(B \to C)$ such that $A_i \nvdash B$, put $A_{i+1} = A_i$. If not, let $(B \to C)$ be the first implication, reading from left to right, such that $A_i \vdash B$. Thus $A_i = D \wedge (B \to C) \wedge E$ for some $D, E$. By assumption $A_i \mid (B \to C)$. We inductively define $A_{i+1}$.

$\star$ If $B = p$, put $A_{i+1} = D \wedge C \wedge E$. Note that $A_{i+1} \mid C$ since $A_i \mid C$ which again follows from $A_i \mid (B \to C)$ and $A_i \Vdash B$.

$\star$ If $B = B_1 \wedge B_2$ observe that $A_i \vdash B$ implies $\vdash A_i \leftrightarrow D \wedge (B_j \to C) \wedge E \leftrightarrow D \wedge C \wedge E$. Hence $D \wedge (B_j \to C) \wedge E \vdash B_j$. If for some $j = 1, 2$, $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$, let $A_{i+1} = D \wedge (B_j \to C) \wedge E$. It cannot be that for no $j$, $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$. For if so, then $D \wedge (B_j \to C) \wedge E \Vdash B_j$. Hence $D \wedge (B \to C) \wedge E \Vdash B$, and so $D \wedge (B \to C) \wedge E \mid C$. Whence $D \wedge C \wedge E \mid C$ and thus $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$, a contradiction.

$\star$ If $B = B_1 \vee B_2$ observe that $\vdash A_i \leftrightarrow D \wedge (B_1 \to C) \wedge (B_2 \to C) \wedge E$ and that $A_i \mid (B_j \to C)$. Put $A_{i+1} = D \wedge (B_1 \to C) \wedge (B_2 \to C) \wedge E$.

$\star$ Finally $B = (B_1 \to B_2)$. If $A_i \nVdash B_1$ or $A_i \mid B_2$ then $A_i \Vdash B$ and therefore $A_i \mid C$. Put $A_{i+1} = D \wedge C \wedge E$. If $A_i \Vdash B_1$ and not $A_i \mid B_2$ then $\vdash A_i \leftrightarrow D \wedge B_1 \wedge (B_2 \to C) \wedge E$ and clearly $A_i \mid B_1$ and $A_i \mid (B_2 \to C)$. Put $A_{i+1} = D \wedge B_1 \wedge (B_2 \to C) \wedge E$. This ends the construction of the $A_i$.

It is easy to check that the $A_i$ have the desired properties. Moreover, the construction shows that eventually $A_i = A_{i+1}$. Hence $A_i$ is a conjunction of propositional variables and implications $\bigwedge_{i=1}^{n} p_i \wedge \bigwedge_{i=1}^{m} (B_i \to C_i)$ such that $A_i \nvdash B_i$. Let $A' = \bigwedge_{i=1}^{m} (B_i \to C_i)$ and let $\sigma$ be the substitution which is the identity on all variables except $p_1, \ldots, p_n$, on which it is $\top$. Hence $\sigma(A_i)$ is equivalent to $\sigma(A')$. Since $A_i$ is equivalent with $A$ in $T$,

$$\vdash (A_i \to F \vee G),\ \nvdash (A_i \to F),\ \nvdash (A_i \to G).$$

Clearly, we have

$$\vdash (\sigma(A') \to \sigma(F) \vee \sigma(G)),$$

In general, nonderivability is not preserved under substitution but this particular choice of $\sigma$ leads to

$$\nvdash (\sigma(A') \to \sigma(F)),\ \nvdash (\sigma(A') \to \sigma(G)),\ \nvdash (\sigma(A') \to \sigma(B_i)).$$

Hence $\sigma(A'), \sigma(F), \sigma(G)$ is an $I$-example of $T \neq \mathsf{IPC}$. **QED**

## References


[Iemhoff] R. Iemhoff. On the admissible rules of Intuitionistic Propositional Logic. *Journal of Symbolic Logic* 66, 2001.

[De Jongh 70] D.H.J. de Jongh. A characterization of the Intuitionistic Propositional Calculus. In *Intuitionism and Proof Theory*, eds. Kino, Myhill, Vesley. North-Holland, 1970.

[De Jongh 80] D.H.J. de Jongh. A class of intuitionistic connectives. *The Kleene Symposium*, eds. Barwise, Keisler, Kunen. North-Holland Publishing Company, 1980.

[Gabbay, De Jongh 74] D.M. Gabbay and D.H.J. de Jongh. A sequence of decidable finitely axiomatizable intermediate logics with the disjunction property. *Journal of Symbolic Logic* 39, 1974.

[Kleene 62] S.C. Kleene. Disjunction and existence under implication in elementary intuitionistic formalisms. *Journal of Symbolic Logic* 27, 1962.

[Rybakov 97] V.V. Rybakov. Admissibility of Logical Inference Rules. Elsevier, 1997.

[Smoryński 73] C.A.Smoryński. Applications of Kripke Models. In *Mathematical Investigations of Intuitionistic Arithmetic and Analysis*, ed. A.S. Troelstra, Springer-Verlag, 1973.

[Troelstra, Van Dalen 88] A.S. Troelstra and D. van Dalen. Constructivism in Mathematics, Vol. I, North-Holland, 1988.